BIOMETRIKA

A JOURNAL FOR THE STATISTICAL STUDY OF BIOLOGICAL PROBLEMS

FOUNDED BY

W. F. R. WELDON, FRANCIS GALTON AND KARL PEARSON

EDITED BY

KARL PEARSON

ASSISTED BY

EGON SHARPE PEARSON

AND

GEOFFREY M. MORANT

VOLUME XXVII

1935

ISSUED BY THE BIOMETRIKA OFFICE
UNIVERSITY COLLEGE, LONDON
AND PRINTED AT THE
UNIVERSITY PRESS, CAMBRIDGE

First printed in Great Britain at the University Press, Cambridge
Reprinted by offset-litho by Percy Lund Humphries & Co. Ltd

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Vol. XXVII. Parts I and II March, 1935

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VOLUME XXVII MARCH, 1935 PARTS I AND II

BIOMETRIKA

A SECOND PIEBALD FAMILY FROM SUFFOLK.

BY E. A. COCKAYNE, D.M., F.R.C.P.

IN 1914 I published an account of a piebald family*, which originally came from the neighbourhood of Bury St Edmunds in Suffolk. About a year ago at the Hospital for Sick Children, Great Ormond Street, I saw two piebald children, members of a family whose ancestors lived near Ipswich in Suffolk. It has been impossible to trace any relationship between these two families, nor even to find a surname common to both. In both of them there has been descent through females and neither knows the name of the member who first came to London. It is, however, most improbable that two unrelated families with so rare a condition lived in the same county. The only other piebald family recorded in England is the London one described by Bishop Harman in 1909 and this may be descended from the same stock.

I am only able to give a short and incomplete pedigree, because the members of the family, whom I saw, have lost touch with their relations and it has been impossible to communicate with them.

I, 2. Had a white frontal blaze.

II, 1. Has dark hair and is normal, and has eight or nine normal children.

II, 5. Has a white frontal blaze and no children.

II, 9. Has a white frontal blaze, but all her four children are normal and one of her daughters has a number of children and grandchildren, who are likewise normal.

II, 11. Piebald.

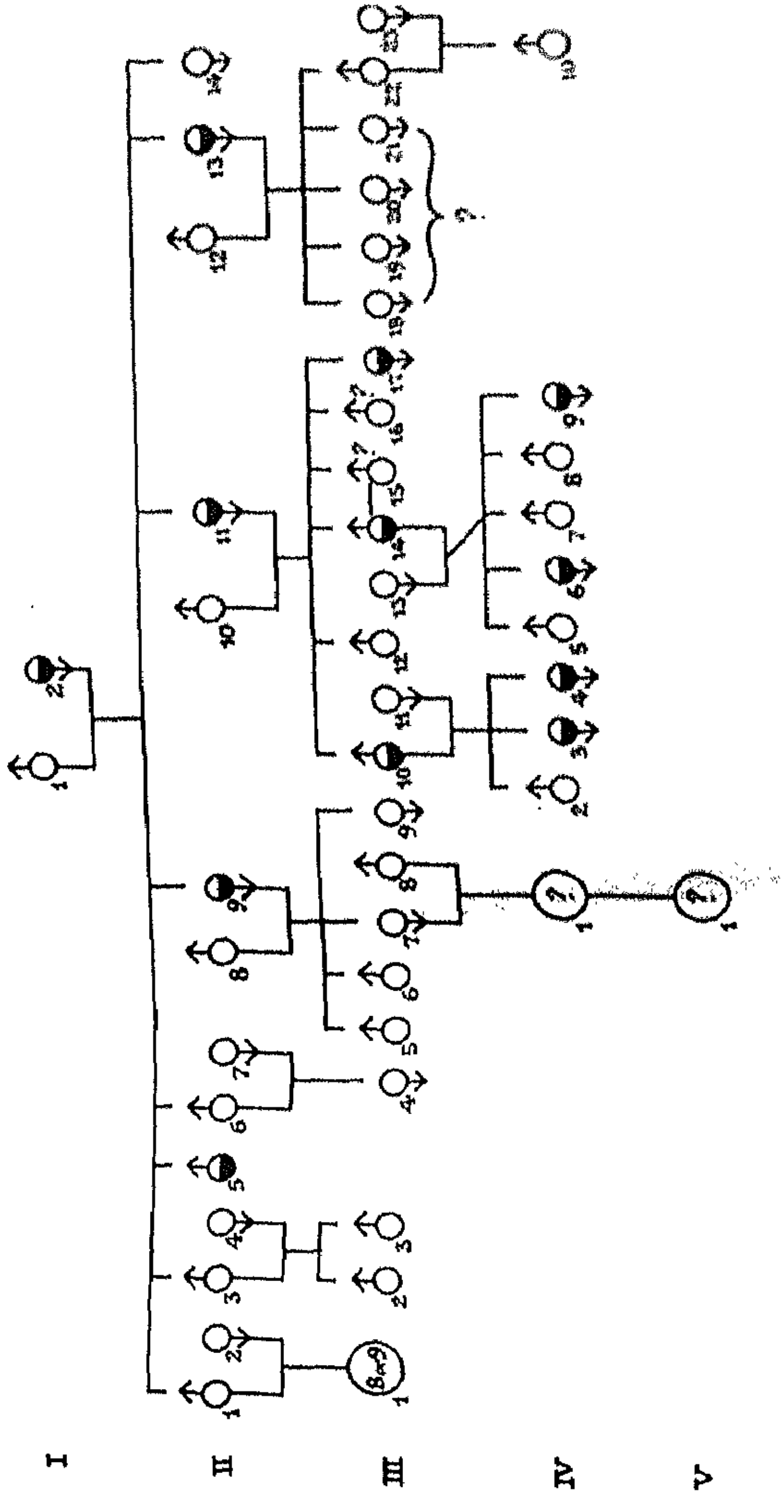
III, 10. Aged 43. Has very fair hair with distinct white frontal blaze of medium size. The inner third of the left eyebrow is white. Almost the whole of the skin on the front of the chest and the greater part of that on the abdomen is unpigmented and was severely reddened and blistered by the sun years ago when he was in Malta and Corfu. Large areas of skin on the backs of both arms above the elbows and on the legs round the knees and upper part of the calves are also unpigmented, and there are smaller patches on the forearms and backs of the legs.

III, 12. Normal with rather dark brown hair.

IV, 2. Aged 18 years. Is dark haired and normal.

IV, 3. Aged 14 years. Has blue eyes and very fair hair. She has a large white frontal blaze and a triangular area of unpigmented skin on the forehead which ends above the root of the nose. The whole of the skin of the greater

* E. A. Cockayne, Biometrika, Vol. x. 1914, p. 186.



surface of the lower two-thirds of both upper arms is unpigmented and ends sharply in a curved line with its convexity upwards. There are smaller and less clearly defined patches on the forearms. The skin over the knees and for a short distance above and a greater distance below them is also unpigmented, and forms a large oval well-defined area on the front of each leg. There are other similar patches on the upper part of the calves. The unpigmented skin becomes bright red when exposed to the sun, whereas the pigmented skin turns brown, but in winter little difference can be seen between them. A large part of the skin of the chest and abdomen appears to be unpigmented, but has never been exposed to sunlight.

IV, 4. Aged 9 years. Is very fair with blue eyes. She has a much smaller frontal blaze than her sister, but the V-shaped area of unpigmented skin on her forehead was very clearly defined, when I saw her, as it was still bright red from exposure to the sun. The unpigmented skin ends a short distance above the root of the nose and there are no hairs in her eyebrows or eyelashes. The skin over the elbows and on the backs of the arms above them is unpigmented and separated from the normal skin by a curved line, like that in her sister, but situated nearer to the elbow. The skin over the knees and above and below them is unpigmented, and these areas are similar in shape and size to those in her sister. There are other patches of unpigmented skin on the forearms and on the legs, especially on the upper part of the calves. The skin on the chest and abdomen has never been exposed to the sun, and it is impossible to be sure that any of it is unpigmented, though a large part of it appears to be.

III, 14. I have been unable to see this man, but a photograph I have seen shows a small white frontal blaze. His brother says that he has other unpigmented areas similar in distribution to his own. His wife, III, 13, has rather dark brown hair and brown eyes.

IV, 5. Aged 13 years. Is dark and normal.

IV, 6. Aged 10 years. Has grey eyes and fair hair with a very large white frontal blaze. The skin of the forehead and front of the neck is unpigmented. The greater part of the skin of the right arm over the deltoid, that of the inner side of the upper arm and along the radial side of the forearm almost as far as the wrist is unpigmented, and there is an area of unpigmented skin over the elbow. On the left arm the skin over the deltoid is unpigmented, and there is another unpigmented area along the radial side of the forearm.

IV, 7 and 8. Aged 5 and 3 years respectively. Grey eyes and light brown hair. Normal.

IV, 9. Aged 2 years. Very fair hair and grey eyes. There is a small white frontal blaze and the skin of the upper part of the forehead is unpigmented. On other parts of the body the skin is so fair that it is impossible to distinguish the unpigmented from the normal skin.

III, 15. The twin brother of III, 14 died at birth, and it is not known whether he was piebald or not.

III, 16. Died in infancy. It is not known whether he was piebald or normal.

III, 17. Piebald. Died unmarried.

II, 13. Piebald. The condition of her children, III, 18 to 22, and of her grand child, IV, 10, is not known.

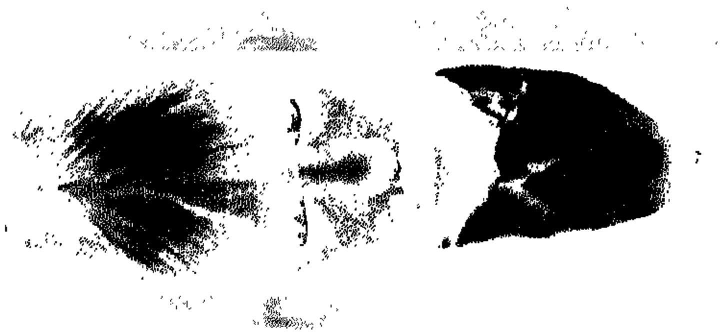
In connection with the family reported in 1914 I should like to state that another piebald boy, a younger brother of VI, 2, was born soon after the publication of the paper. Unlike his brother he had very dark brown hair and his frontal blaze was smaller. On p. 200 I said that it was probable that in some members of the family at least the areas of unpigmented skin were larger and more numerous than I had stated. Proof of this was obtained in the case of the boy VI, 2. I saw him after he had been bathing at the seaside and found that he had extensive areas of unpigmented skin on the chest, smaller ones on the abdomen and flanks, several spots of unpigmented skin on his penis, and large patches on the legs as well as on the arms, so that it is probable that other members of the family had white patches on the legs, though they said that this was not so. There is little doubt therefore that the distribution of unpigmented skin is very similar in these two families, as one would expect if they have a common origin, though none of the members of the family described in this paper has heterochromia iridis. They are like the majority of hereditary and isolated cases described by others in having the dorsal region fully pigmented and in possessing a white frontal blaze. Jenks has described a family of American whites of Welsh-Scottish origin*, in which the piebald pattern, transmitted by direct descent, differed very much from the usual one. All the piebald members had a stripe of unpigmented skin running down the mid-dorsal region and most of them had no frontal blaze in spite of having very extensive areas of unpigmented skin on the trunk and limbs.

In this family as in other published cases, with the exception of the two families recorded by Meirowsky and Spickernagel, and by Meirowsky respectively†, the condition has only been transmitted by those affected. Assuming that the condition is due to a single dominant gene, and that the piebald parents were all heterozygous, the expectation is that the piebald and normal members will be equal in number. In the family described in this paper in five complete sibships with 26 members, 11 were piebald, 13 were normal, and in 2 the condition is not known, and in the five sibships of the family described in 1914 with 28 members 15 were piebald and 13 normal. The numbers agree closely with those expected, if the piebald condition is due to a dominant gene. If, however, the full sibships in this family and the ten others I collected in my book are taken together, there is a slight excess of piebalds, 144 piebalds and 131 normal members. The two exceptional families, in which normal individuals are said to have transmitted the condition, and the one described by Jenks, are omitted from these figures. Considering the sex incidence in the same eleven families, there were 79 males and 81 females with the piebald pattern, so that both sexes are about equally affected.

* Jenks, *American Anthropologist*, Vol. xvi. 1914, p. 221.

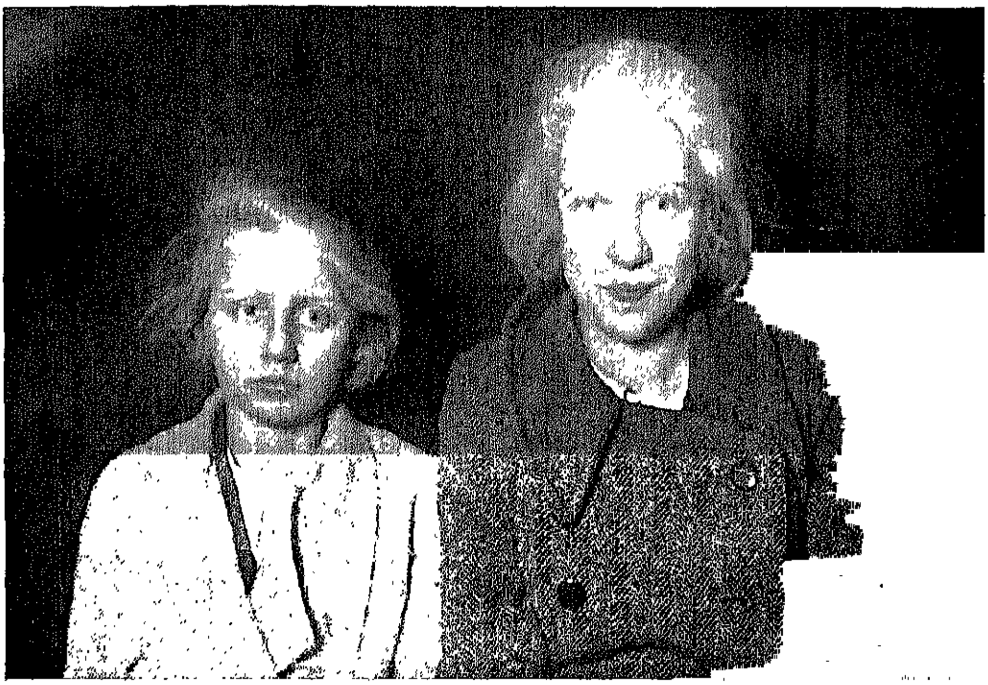
† E. A. Cockayne, *Inherited Abnormalities of the Skin and its Appendages*, 1933, p. 54. (Bibliography.)

IV, 9.



IV, 6.

III, 10.



IV, 3. IV, 4

IV, 3. IV, 4.

PEDIGREES OF HEREDITARY EPICANTHUS.

By C. H. USHER, M.B.

In 1828 Schön* described the condition now known as epicanthus. Von Ammon in 1831 named it and in 1860 he classified it. His classification is given in my paper on a pedigree of epicanthus and ptosis in the *Annals of Eugenics*†. The epicanthic fold is represented in all races during foetal life‡. Waardenburg§ found epicanthus present from the beginning of the third to the beginning of the sixth month of pregnancy. In his embryos it was never present as a fold covering the canthus medialis. Eisler‖ states that the human foetus of 28–30 cm. has an epicanthus which coalesces only with the upper lid fold to form the typical mongolian fold. In the first six months of life, according to Martin's statistics¶ for epicanthus in the Munich population, 33·1 % of males and 32·6 % of females have epicanthus, but the numbers are reduced to 3·3 % and 2·3 % respectively in the years 12 to 25, thus showing that in most cases an epicanthus disappears in early life. In consequence peculiar difficulties are encountered in the investigation of a pedigree of simple epicanthus and these are added to by the circumstance that individuals who have epicanthus and their relations are usually unaware that an abnormal condition is present. This is in marked contrast with the characteristic cases of ptosis with epicanthus which are well known to the relatives. Clinical observation sometimes shows the presence of an epicanthus at the first examination and its absence at a subsequent examination. Thus: (1) A female (*Fig.* 21, *II*, 1, p. 15) at age of 5 years had well-marked bilateral epicanthus, which had disappeared by the age of 24. (2) Another female (Fig. 12, II, 8, p. 11) child when seen at 5 years of age had a unilateral epicanthus, which was not present 21 years later. (3) A male (Fig. 17, III, 1, p. 13) with bilateral epicanthus in infancy had at the age of 7 no epicanthus on right side, but the remains of one on left side. (4) A female when seen in infancy and again at 5 years of age had bilateral epicanthus, which was not present at the age of 9.

Published pedigrees of simple epicanthus are few. I know of ten of which Meirowsky's is the only large one, though in a pedigree by Rosenstein, epicanthus occurs in four generations. The first of these ten pedigrees to appear was V. Ammon's published in 1860.

* Schön, Matth. Joh. Albrecht, *Handbuch der pathologischen Anatomie des menschlichen Auges*, Hamburg, S. 59–60, 1828.

† *Annals of Eugenics*, Vol. I. Parts I and II. pp. 128–138, 1925.

‡ Keith, *Human Embryology and Morphology*, 5th ed. p. 244, 1933.

§ Waardenburg, P. J., *Graefes Arch. f. Ophth.* S. 124, 221–229, 1930.

‖ Eisler, P., *Kurzes Handbuch der Ophthalmologie* (Schieck u. Brückner) I. S. 280, 1930.

¶ Martin, Rudolf, *Lehrbuch der Anthropologie*, Jena, S. 425, 1924.

V. AMMON, Fig. 1, observed a son and three daughters with a small tarsal epicanthus which they had inherited from their mother.

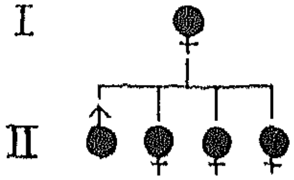


Fig. 1.

DREWS, Fig. 2, knew a family in Munich in which a mother, her son and daughter had epicanthus.

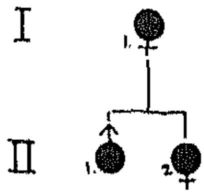


Fig. 2.

GALEZOWSKI, Fig. 3, in 1875 wrote that epicanthus is most frequently double, and it is congenital. He had seen it hereditary in a family; the father and the two children presented the same deformity in an excessive degree.

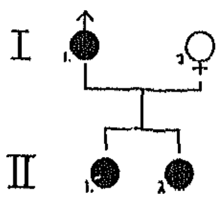


Fig. 3.

MANZ, Fig. 4, saw five children in a family of ten with epicanthus. The other five children did not have the deformity. Sex and order of birth not given. No mention of eye movements.

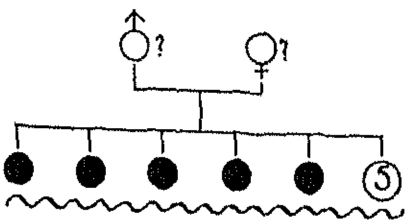


Fig. 4.

SCHEIDT, Fig. 5, observed partial lid fold and epicanthus in four of six children of a family of Alpine descent, whose parents, grandparents and brother and sister were not affected.

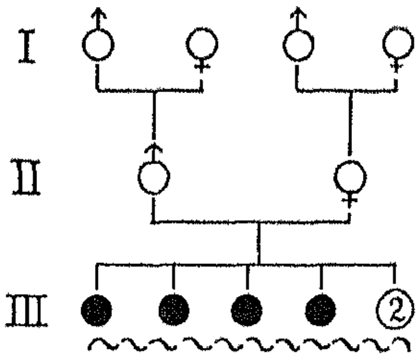


Fig. 5.

ROSENSTEIN, 1st family, Fig. 6, IV, 2, a girl, age 9, with bilateral epicanthus and strabismus, H. 6 D. in right and 4 D. in left eye and 1 D. of astigmatism in each. Asymmetry of face. IV, 1, her brother, age 13, with bilateral epicanthus and concomitant convergent strabismus. H. R. 3 D., L. 4 D. with 1 D. of astigmatism in each. No facial asymmetry. The mother, III, 1, age 43, has bilateral epicanthus,

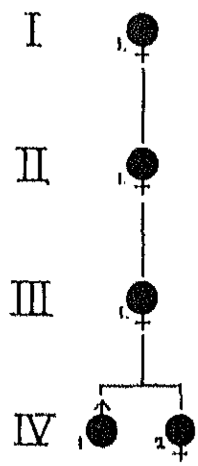


Fig. 6.

H, 6 to 7 D., astigmatism 1·5 D. in each. Very marked facial asymmetry. II, 1, the maternal grandmother, always had bad vision, bridge of nose was flat with vertical skin fold on both sides. I, 1, the great grandmother, had always poor vision, also epicanthus and facial asymmetry. There is direct transmission of epicanthus in four generations.

MEIROWSKY, Fig. 7 (p. 8). In generation I a female married twice. From the first marriage she had a son and daughter both with epicanthus. The son had nine children, all males, of whom six had the hereditary character. All affected members

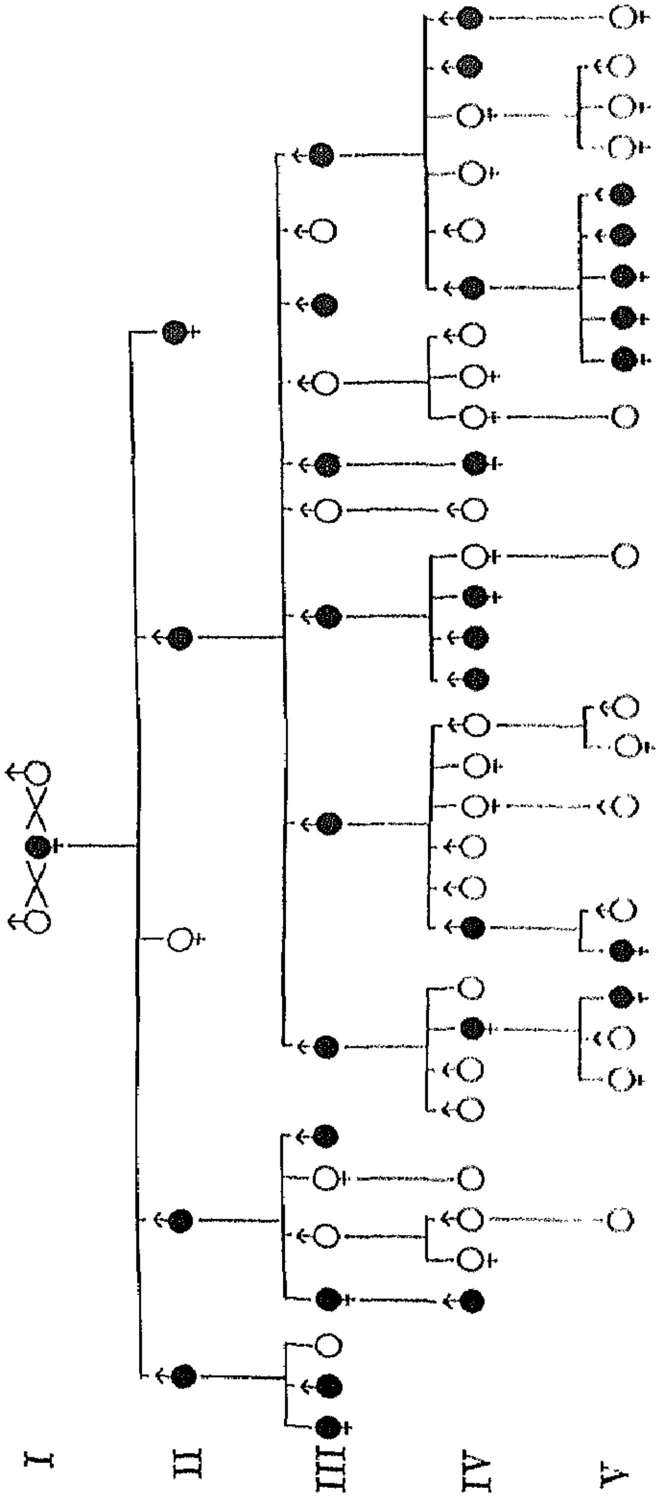


Fig. 1.

who married transmitted epicanthus to some of their descendants, whilst the unaffected members had children without epicanthus. In the same manner the three children of the second marriage of whom two were affected, both males, transmitted the affection to two further generations. Of the 87 descendants of the female in generation I, 31 have epicanthus.

Rosenstein, 2nd family, Fig. 8, III, 1, a male, age 20, with bilateral epicanthus and a high degree of myopia, R. 13 and L. 16, astigmatism of 1 D. in each and blepharochalasis of left. III, 2, his twin brother, with a less degree of myopia, had the same condition. III, 3, the youngest brother, a 14-year-old boy, who died of inflammation of the lungs, was short-sighted and had exceptionally well-marked bilateral epicanthus. III, 4, two unaffected brothers. II, 1, the father, died. He had a high degree of short sight and bilateral epicanthus. It is not known whether any of the grandparents (I, 1–4) had a similar anomaly. There is direct inheritance of epicanthus with other anomalies of the eye from father to three sons.

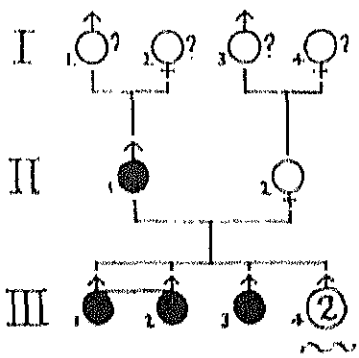


Fig. 8.

Rosenstein, 3rd family, Fig. 9, III, 1, a 15-year-old boy with bilateral epicanthus and myopia, R. 5 D., L. 7 D., with astigmatism of 1 D. in each. He is an only child. II, 1, his father, died. He had bilateral epicanthus and very poor vision both far and near. It is not known whether the boy's grandparents (I, 1 and 2) or near relations had a similar anomaly. There is direct inheritance of the deformity from father to son.

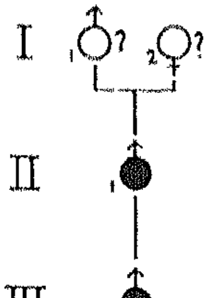


Fig. 9.

WAARDENBURG, Fig. 10, observed epicanthus in five brothers. Their father did not have the character, but their mother had it in a slight degree.

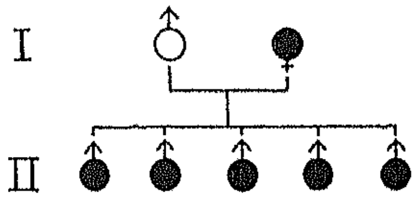


Fig. 10.

In these 10 pedigrees are 69 persons with epicanthus; 35 are males, 23 are females, and in 11 the sex is not recorded. In 8 pedigrees (Figs. 1–3, 6–10) the inheritance is continuous with transmission through the male thirteen times and through the female nine times. In 2 pedigrees (Figs. 4 and 5) epicanthus is confined to a single sibship. Consanguinity is not recorded in any of the pedigrees.

In submitting the results of an investigation of some new pedigrees of simple epicanthus—24 in all—I am fully aware how unsatisfactory these are in some of the pedigrees, owing to the probable disappearance of the epicanthus in certain cases, but I venture to record them in the hope that when considered collectively some light may be thrown on such pedigrees*.

My own material consists of epicanthus pedigrees which show heredity. In some of these, investigation originated from clinical notes made in some instances many years ago, and in others from cases which Professor Low† had seen from one to three days after birth in the years 1924—27. With few exceptions, only examined cases of epicanthus occurring in the pedigrees have been accepted. None of the pedigrees is large. Examples have already been given of the disappearance of epicanthus. In addition to these proved cases, in some pedigrees individuals without an epicanthus are suspected of formerly having had this character. The three youngest members of a sibship, in Fig. 11, have bilateral epicanthus, but

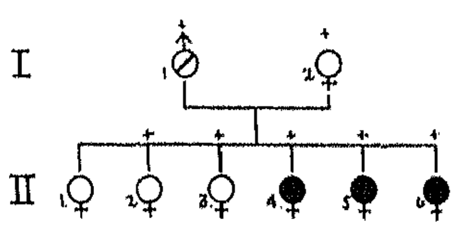


Fig. 11.

two older ones when they were examined had no epicanthus. The first-born, not seen, had died in infancy. The distribution of the abnormality is suggestive that

* Some of these pedigrees are briefly described on pp. 10—18, and fuller particulars of each individual pedigree are provided on pp. 18—25.

† Professor Alexander Low, University of Aberdeen.

one or more of the three first born sibs may have had epicanthus in infancy. Their father has congenital coloboma of iris and microcornea.

(The + sign above individuals in the pedigree charts indicates that they have been examined.)

In Fig. 12 are two sisters with affected offspring. One of them, II, 8, had unilateral epicanthus at the age of 5 years, whereas there was no epicanthus

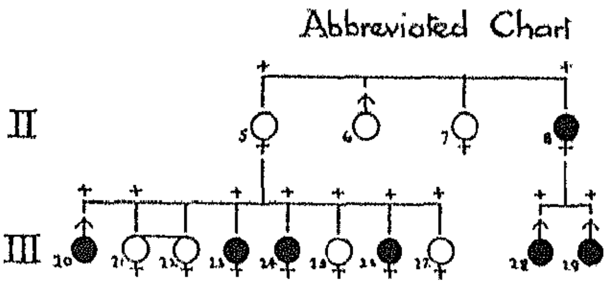


Fig. 12.

when she was 26 years of age. An older sister, II, 5, with four affected children, has no epicanthus. She had not been seen in childhood, and similarly with her mother (not shown in abbreviated chart), I, 2. It may also be of significance that *only the youngest of the four affected children of II, 5 has bilateral epicanthus, in the others it is unilateral.*

In Fig. 13, a mother with two children, all showing well-marked bilateral epicanthus, said that her mother, I, 1, had the same condition. Her medical man,

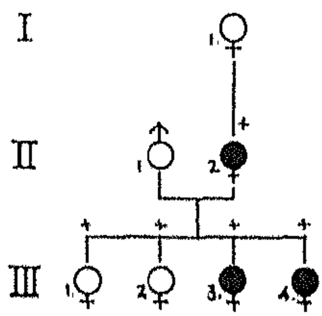


Fig. 13.

however, reported that she has no traces of epicanthus. Had it possibly disappeared?

In Fig. 14 are two sisters with affected offspring. One of them, II, 5, has a definite epicanthus on left side only. Her sister, II, 4, who is older than she is, has no epicanthus. Their parents, I, 1 and 2, have no epicanthus. The questions arise: first, had II, 5 bilateral epicanthus at an earlier age and, secondly, did II, 4 and I, 1 or I, 2 formerly have epicanthus?

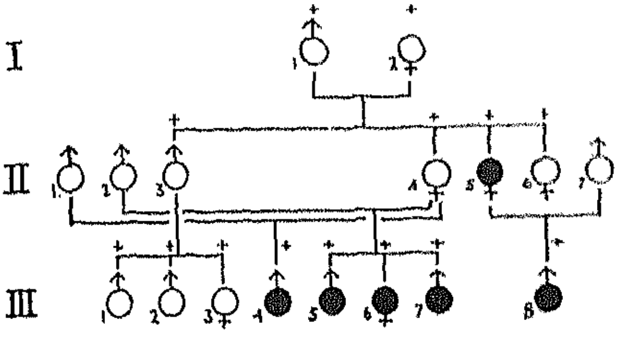


Fig. 14.

In Fig. 15 a male, IV, 3, with bilateral epicanthus in infancy and at age of 5 has now, at the age of 9, what appears to be only the upper part of each epicanthus remaining. The boy's maternal aunt, III, 13, age 12, has a bilateral fold resembling

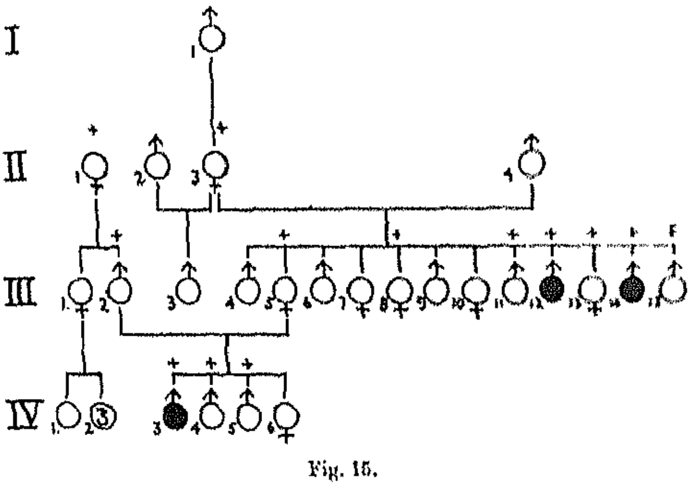


Fig. 15.

the upper part of an epicanthus, which does not extend down so far as the internal tarsal ligament. She is marked in the chart as having no epicanthus, though possibly, as in the case of IV, 3, a definite bilateral epicanthus was present earlier. These cases (IV, 3 and III, 13) and also the case of a male, II, 3 in Fig. 29, suggest that the epicanthus disappears from below upwards. In the mother's sibship, epicanthus occurs only amongst the younger members (Fig. 15, III, 12 and 14), so that the mother at their age may have had the same condition.

The next four pedigrees of epicanthus in three generations illustrate direct inheritance of epicanthus. Fig. 16, a male, I, 2, is dead, but his photograph shows epicanthus. He married twice and transmitted the condition to a member of both his families. His daughter, II, 2, and her son, III, 3, have epicanthus. It is unknown whether II, 1, 3 and 4, three unaffected parents who have children with epicanthus, had the same condition in childhood.

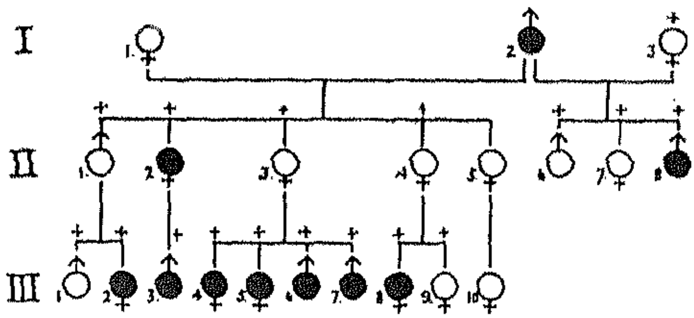


Fig. 16.

Fig. 17. Albert S., III, 1, age 7, with remains of an epicanthus on left side only, in infancy had bilateral epicanthus. His brother, III, 3, and sisters, III, 2 and 4, have bilateral epicanthus, which is most conspicuous in the youngest sister,

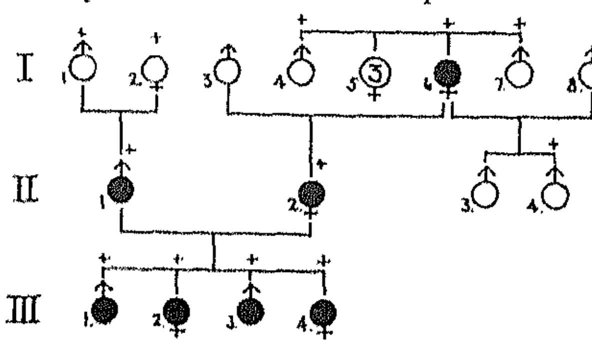


Fig. 17.

aged 2. The father, II, 1, has unilateral epicanthus on left side. The mother, II, 2, illegitimate, has distinct bilateral epicanthus. The paternal grandparents, I, 1 and 2, have no epicanthus. Maternal grandfather, I, 3, is dead. Maternal grandmother, I, 6, has feebly marked bilateral epicanthus; I, 5, three females, are dead. I, 4 and 7 and II, 4 have no epicanthus. All those affected in this pedigree have epicanthus palpebralis.

Fig. 18 shows a pedigree with epicanthus in three generations and direct transmission of the affection.

Abbreviated Chart.

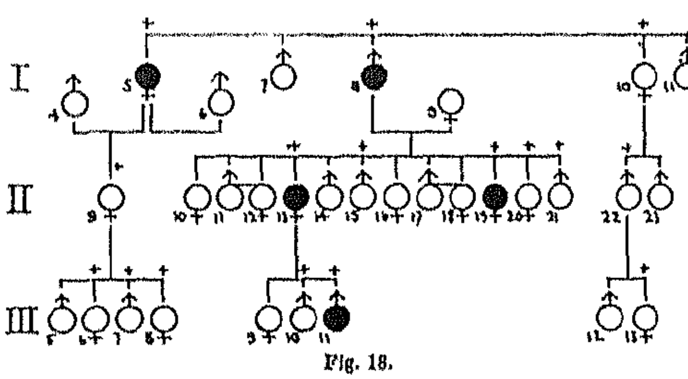


Fig. 18.

Fig. 19. Epicanthus in three generations. Of 11 members who were examined in a sibship of 11, ten have epicanthus. In three of these, II, 1, 8 and 10, the

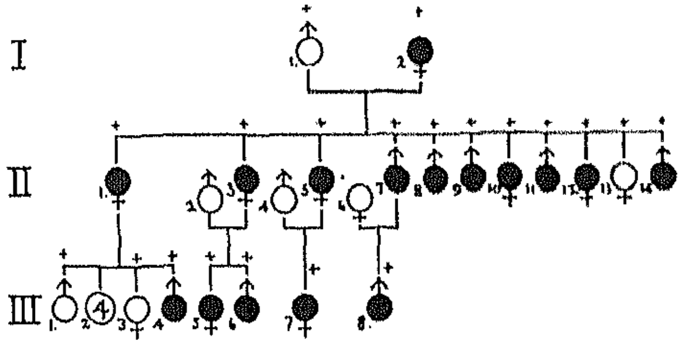


Fig. 19.

epicanthus is unilateral, it has an attenuated appearance and disappears at lower edge of the internal tarsal ligament. Their mother, I, 2, has bilateral epicanthus. III, 4 has left unilateral epicanthus; III, 5 and 6 bilateral epicanthus palpebralis, and III, 7 and 8 bilateral epicanthus tarsalis.

An example of what appears to be discontinuous inheritance is seen in Fig 20. I, 1, now deceased, is said to have had no epicanthus, but was blind from inflamed eyes in infancy. II, 4 when examined had no epicanthus, which may however have been present in childhood. Other examples are seen in Figs. 14 and 16 on pp. 12, 13. It is important to recollect that if certain parents in these pedigrees had been examined in childhood, epicanthus might have been found, in which case the inheritance would be continuous.

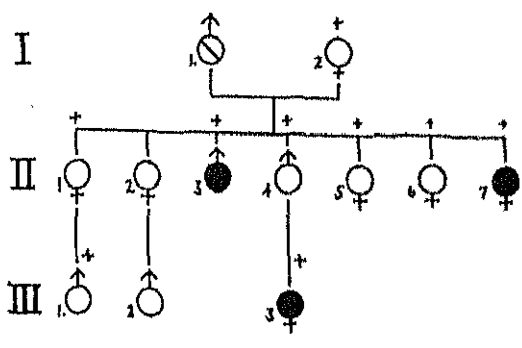


Fig. 20.

The next three pedigrees show the abnormality limited to members of a single sibship.

Fig. 21 shows two normal parents with six children, of whom two males and two females have epicanthus.

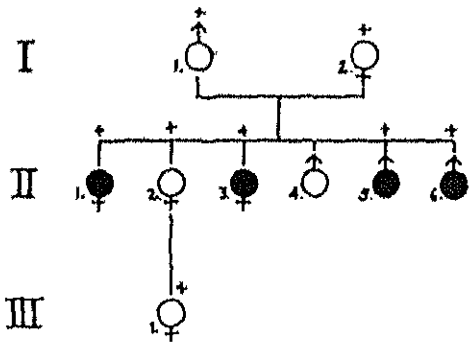


Fig. 21.

Fig. 22. A normal woman, I, 2, had a normal child by her first husband and nine children by her second husband, of whom seven have epicanthus.

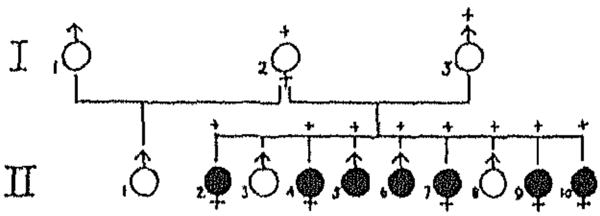


Fig. 22.

Fig. 23. Four affected sons of normal parents. A fifth son, II, 1, the oldest, be it noted, has no epicanthus. Other five sibs, II, 2, 3, 6, 7, 10, died in infancy.

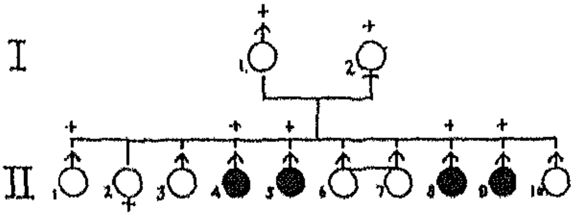


Fig. 23.

Unilateral epicanthus may result from the natural disappearance of an epicanthus from the opposite side. The artificial disappearance of an epicanthus may be the result of operation. If the epicanthus be removed at an early age from both sides it is conceivable that in adult life such an individual would be unaware of his previous condition, which might never have been reported to the investigator of the pedigrees.

Examples of persistence of an epicanthus for a considerable number of years are seen in the next four pedigrees, two of which do not show heredity.

Fig. 24. Dorothy B. T., II, 4, at age of 3 had a bilateral epicanthus extending from tarsal fold and curving downwards to inner end of lower lid, which was still present at age of 28.

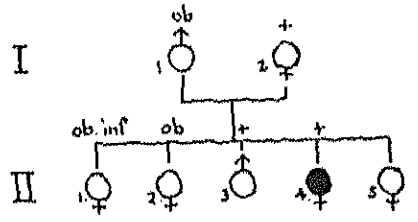


Fig. 24.

Fig. 25. Margaret S., II, 7, a twin, at age of 9 had a unilateral epicanthus which was still present when she was 19 years of age.

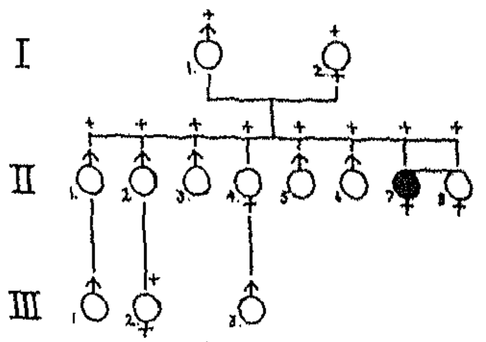


Fig. 25.

Fig. 26. II, 4, a male, the first born of triplets, examined at 5 and 28 years of age. Bilateral epicanthus was present on both occasions.

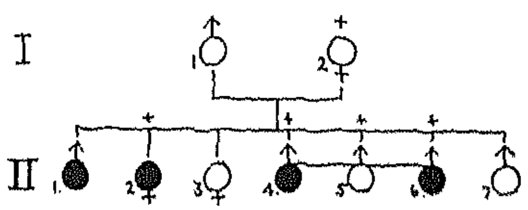


Fig. 26.

In pedigree, Fig. 27, five of six sisters with epicanthus were re-examined 24 years later. In four of them epicanthus was still present. The oldest of these was 33.

The following results have been obtained from the 22 new pedigrees, of which a number of charts have been shown. Consanguinity occurs in one pedigree. The number of individuals with epicanthus is 111, of whom 53 are males and 58 females. Transmission from parent to child occurs in 13 pedigrees, by the father in 6 sibships and by the mother in 14. Discontinuous inheritance occurs in 8 pedigrees, including 5 of those that also show continuous inheritance. Epicanthus is confined to members of a single sibship in 6 pedigrees. Though epicanthus is transmitted

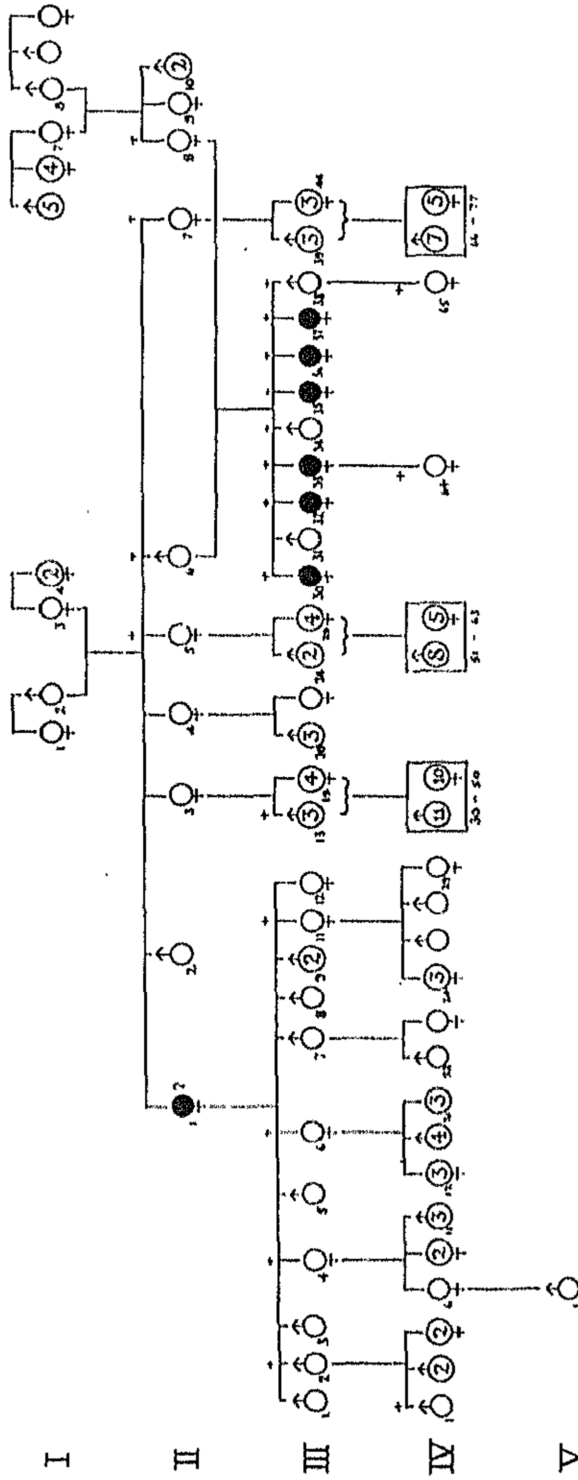


Fig. 27.

from an affected father, or mother, to the children in only a small majority of the pedigrees, transmission is in accordance with what is seen so markedly in Meirowsky's large pedigree. Furthermore, as epicanthus is known to disappear as age advances in a considerable number of people—see Martin's table—there can be little doubt that some of the remaining 9 pedigrees would have shown direct inheritance had all the examinations been made at an early age. When these results are added to those from published pedigrees, the following figures are obtained: Number of affected persons 180; of these 88 are males, 81 females, and in 11 the sex is unrecorded. There are 32 pedigrees in all*, in 21 of these inheritance is continuous, with transmission through males 19 times, and through females 24 times. Discontinuous inheritance alone occurs in 3 pedigrees, and both continuous and discontinuous inheritance in 7. When it is considered that 21 of the 32 pedigrees show transmission from parent to child and that a proportion of normal parents of affected sibships may in childhood have had the abnormality, there is good reason to believe that usually in inherited epicanthus the transmission is continuous.

Changes must occur in the appearance of an epicanthus, which, though well marked at first, finally disappears. No doubt ophthalmologists see the different stages of retrogression, but are the appearances presented recognised as those of an epicanthus which will later disappear? The ill-defined skin folds which in some instances are difficult to diagnose as epicanthus are possibly late stages of a formerly well-defined epicanthus. Cases have been recorded in this memoir of well marked epicanthus, which after some years appeared to be represented by merely short thin skin folds at the site of the upper part of the vanished epicanthus. Forster† had doubt as to whether an obliquely lying skin fold which began at the internal tarsal ligament and extended upwards and outwards above the tarsal fold was an epicanthus, though he regarded the fold as probably an epicanthus. He saw this oblique fold in five cases, aged respectively 23 (2), 20, 37, 19 years. It may be suggested that these were possibly late stages in the disappearance of an epicanthus palpebralis. The age of the cases supports such a view. It is obvious that a thorough investigation of the inheritance of epicanthus in a pedigree requires not only an examination of each individual, but that this should be made at an early age.

Simple Epicanthus.

A short Account of 22 *unpublished Pedigrees which show Heredity.*

Fig. 12 (p. 11), Forbes family. Epicanthus in three sibships of two generations. Wm., III, 20, age 15, Isobel, III, 23, and Catherine, III, 24 have left unilateral epicanthus; Ann, III, 26, has bilateral epicanthus more marked on left side. Their mother, II, 5, has no epicanthus. One of the twins, III, 21 and 22, died at birth, the other and Patricia, III, 25, and Alice, III, 27, have no epicanthus. Douglas Y.,

* Figs. 24 and 25 are not included in this statistical summary as the author considers that they do not show heredity.

† Forster, A., *Anatomischer Anzeiger*, Bd. 52, No. 3/4, S. 49—63, 1919.

III, 28, age 4, and his brother William, III, 29, age 11 weeks, have bilateral epicanthus. Their mother, II, 8, has no epicanthus (1934), yet when seen (1913) at the age of 5, epicanthus on left side was recorded, right side was normal. I, 1, father of II, 8, and II, 7, deceased. Besides those marked in the chart, 20 other members were examined, including four females the oldest in sibship II, 1—8 and their children, and also I, 2, the mother of II, 8. None of these had epicanthus. No consanguinity.

Fig. 21 (p. 15), Gerrie family. Four with epicanthus in a sibship of six. Mary G., II, 1, in 1914, at age of 5, had marked bilateral epicanthus which was absent in 1934. Mrs L., II, 2, and her child, III, 1, have no epicanthus. Alice G., II, 3, age 21 years, has epicanthus on right side, none on left, extending from tarsal fold above to just below inner end of lower lid. Richard G., II, 5, age 15, with bilateral epicanthus; distance from one epicanthus edge to that of the other is 38 mm. and from outer canthus to edge of epicanthus is 25 mm. James G., II, 6, age 13, has bilateral epicanthus; distance from edge of one epicanthus to that of the other is 38 mm. and from outer canthus to edge of epicanthus is 23 mm. I, 1 has no epicanthus. It is not stated whether I, 2, now dead, who came with a daughter in 1914, was examined for epicanthus, though this is likely. No consanguinity.

Fig. 28, Mack family. III, 4, male, age 3 (1905) was brought for a definite but not very marked bilateral epicanthus. In 1911, R. and L. fundus normal,

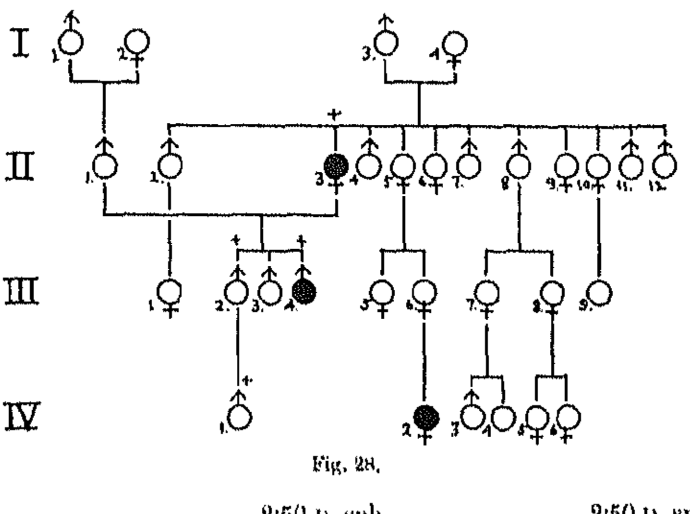


Fig. 28.

refraction H. in each R.V. with $\frac{2{\cdot}50 \text{ D. sph.}}{1{\cdot}50 \text{ D. cyl.}} = \frac{6}{12}$, L.V., with $\frac{2{\cdot}50 \text{ D. sph.}}{1 \text{ D. cyl.}} = \frac{6}{18}$; is mentally deficient. His mother, II, 3, now deceased, had an epicanthus on right side only. She stated that it was a family complaint and that her sister's children, III, 5, and 6, have it. When the mother, II, 5, of these children was written to in 1934 she replied that it does not exist in herself or her daughters, nor in any of her brothers and sisters, nor in any member of her brother L.'s, II, 8, family, but in her granddaughter, IV, 2, 16 months old, she could see a faint indication of it.

"I don't suppose anyone else would notice anything unusual in her left eye, but I have a practised eye so to speak, having been used to seeing it in my sister and her son. I don't understand Mrs Mack, II, 3, saying the same affection was apparent in her sister's children. She must have been mistaken." I, 3 and I, 4 had no epicanthus and none of the photographs or paintings of the earlier generations indicates the presence of an epicanthus. No consanguinity.

Fig. 16 (p. 13), Evans family. Pedigree showing both continuous and discontinuous descent. I, 2, whose photograph shows epicanthus, married twice and had offspring with epicanthus from both marriages. He and his first wife are dead. His second wife, I, 3, has no epicanthus. The following have bilateral epicanthus: Robert E., II, 8, age 10, form palpebralis, has internal concomitant strabismus; III, 2, female, has a minor degree; II, 2, and her son, III, 3, form tarsalis; III, 6, male, age 4, and III, 7, male, age 2 years, form tarsalis; III, 4 and 5, females, ages 8 and 7, in a minor degree; III, 8, female, age 4, form palpebralis. II, 6, male, age 22; II, 7, female, age 14; III, 9, female, age 2; II, 1, 3, and 4 have no epicanthus. No consanguinity.

Fig. 29, McGuffie family. Three brothers, III, 1, 2, 3, in a sibship of three with bilateral epicanthus, and a paternal uncle and possibly their father with the

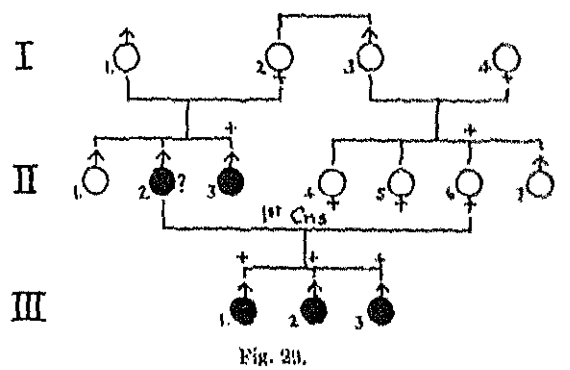


Fig. 29.

same condition. The brothers' ages are from 10 to 2 years. Their father, II, 2, deceased, his wife says had folds of skin similar to those present in her boys. She, II, 6, has no epicanthus. The uncle, II, 3, has attenuated bilateral epicanthus. The boys' parents were first cousins, their paternal grandmother, I, 2, and maternal grandfather, I, 3, being sister and brother. I, 1, 2, 4 and II, 1 are deceased. Consanguinity.

Fig. 17 (p. 13), Stewart pedigree. Epicanthus in three generations; disappearance of epicanthus in one of the members. III, 1, Albert S., age 7, has remains of an epicanthus on left side only. In infancy and also at age of 5 years when seen by Professor Low he had bilateral epicanthus. His brother, III, 3, and sisters, III, 2 and 4, have bilateral epicanthus best *marked in the* younger sister, age 2 years. Father, II, 1, has left-sided unilateral epicanthus. Mother, II, 2, illegitimate, has distinct bilateral epicanthus. Paternal grandparents I, 1 and 2, *normal*.

Maternal grandfather, I, 3, deceased. Maternal grandmother, I, 6, has feebly marked bilateral epicanthus. I, 5 are deceased. I, 4 and 7 and II, 4 are normal. Epicanthus palpebralis was the form in each affected person. No consanguinity.

Fig. 11 (p. 10), Findlay pedigree. Epicanthus in the three youngest of six sisters. II, 5, Daisy F., bilateral epicanthus seen at 6 months and 3½ years of age. II, 4, Jessie F., and II, 6, Rose F., age 1 year, have bilateral epicanthus. II, 1 died in infancy; Marjory F., II, 2, age 12, and Dorothy F., II, 3, are normal. Father, I, 1, and mother, I, 2, have no epicanthus. The former has congenital coloboma at nasal side of right iris and microcornea of same eye. R.V. with high power + lens = reading small newspaper print at 10". Seen in bed with cardiac disease. No consanguinity.

Fig. 13 (p. 11), Cleveland pedigree. Shows direct transmission from mother to children. The family lives in Aberdeen. Sheila C., III, 1, firstborn, age 11, and Jill C., III, 2, have no epicanthus. Jaqueline C., III, 3, age 6, and III, 4, an infant, have marked bilateral epicanthus. Mother, II, 2, has definite bilateral epicanthus. She stated that her mother, I, 1, who lives in England, has the same condition, but this was not confirmed by the family doctor. No consanguinity.

Fig. 14 (p. 12), Brown pedigree. James B., III, 4, age 7, illegitimate, and III, 5, 6, 7 have bilateral epicanthus tarsalis. Mother, II, 4, has no epicanthus. Father, II, 1, is abroad, II, 2 not seen. III, 8, Adam B., age 7, illegitimate, has bilateral epicanthus tarsalis. His mother, II, 5, has it on left side only. II, 6, II, 3, and III, 1, 2, 3 have no epicanthus. III, 1 is aged 6. No consanguinity. I, 1 and I, 2 have no epicanthus. Had II, 5 bilateral epicanthus at an earlier age and did II, 4 formerly have epicanthus?

Fig. 15 (p. 12), Polson pedigree. A boy and two maternal uncles with epicanthus. Parents, III, 2 and 5, and maternal grandmother, II, 3, have no epicanthus. Gordon II., III, 12, age 16, has a minor degree of epicanthus bilateralis. Neil II., III, 14, also has bilateral epicanthus better defined on right side. Lizzie II., III, 8, has broad root of nose, but no fold at either inner canthus. Paternal grandmother, II, 1, very broad from one inner canthus to the other, no epicanthus, skin wrinkled. IV, 4 and 5, no epicanthus. IV, 6, III, 6, and IV, 1 died in infancy. III, 9 married, no offspring.

Fig. 30, Calder pedigree. Showing both direct and indirect inheritance. Nan C.,

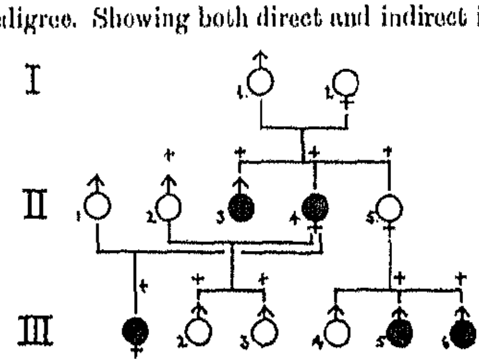


Fig. 30.

III, 1, age 9, illegitimate; her mother, II, 4, and uncle, II, 3, unmarried, have bilateral epicanthus tarsalis. II, 4, not consanguineous with II, 1, or her husband, II, 2, by whom she has two normal sons, III, 2, 3. II, 5, with no epicanthus, has three sons, one (III, 4) of whom was killed. The other two have epicanthus tarsalis, bilateral in III, 5, age 7, and unilateral on right side in III, 6, age 3. I, 1 and 2, deceased. It is of course open to doubt whether II, 5, at an earlier age, had epicanthus. No consanguinity.

Fig. 20 (p. 14), Jamieson pedigree. Epicanthus in a child, her paternal uncle and a paternal aunt. Disappearance of an epicanthus from operation. Jas. J., II, 3, age 28, unmarried, at age of 3 had marked bilateral epicanthus which covered the caruncles; distance from edge of one epicanthus to that of the other was 26 mm Excellent result after Wicherkiewicz operation on each. Annie J., II, 7, age 12, with unilateral left-sided epicanthus, has internal concomitant strabismus. Irene J., III, 3, age 2, with unilateral left-sided epicanthus. Father, II, 4, has no epicanthus. I, 1, deceased, his wife states had no skin fold, worked at blind asylum all his life, eyes inflamed in infancy. No consanguinity.

Fig. 26 (p. 16), Nicol pedigree. Epicanthus in four members of a sibship of seven. II, 4, 5, 6 are triplets, age 28; of these, the first born, II, 4, and the last born, II, 6, have bilateral epicanthus, which was first seen in II, 4 when he was examined at 5 years of age. He was brought for divergent squint which passed off; refraction of each eye was +1 D. Their mother says that the two triplets, II, 4 and II, 6, were remarkably alike. II, 5 has no trace of epicanthus. II, 2, female, age 30, has marked epicanthus on left side; on right side is merely a trace. In all these cases the epicanthus arises from the upper lid tarsal fold. II, 1, male, age 31, not seen, but, to judge by the account of his brother, II, 4, and a photograph, he has bilateral epicanthus best marked on left side. Father, I, 1, deceased. Mother, I, 2, has no epicanthus. II, 3 died in infancy; II, 7 is abroad, age 26; photograph shows no epicanthus. No consanguinity.

Fig. 22 (p. 15), Maitland pedigree. Normal parents with seven children in a sibship of nine with epicanthus. Mary M., II, 2, age 13, with bilateral epicanthus. II, 3 died in infancy. II, 4, age 9, has feebly marked bilateral epicanthus, nose well developed. Father says the skin folds were formerly more marked in her than in any of the others. John M., II, 5, Edward M., II, 6, and Daisy M., II, 7, have bilateral epicanthus. James M., II, 8, is normal. Margaret M., II, 9, and Frances M., II, 10, age 7 months, have bilateral epicanthus. These are Mrs M.'s, I, 2, children by her second marriage with M., I, 3. By her first marriage with W., I, 1, she had a son, T. W., II, 1, an only child without epicanthus. Neither he nor his parents have epicanthus. Both I, 2 and I, 3 are normal. No consanguinity.

Fig. 18 (p. 13). Lemmon pedigree. Showing continuous transmission in three generations. I, 8 and his daughter, II, 13, have bilateral epicanthus, and his grandson, III, 11, right-sided unilateral epicanthus. II, 19, when brought for internal concomitant strabismus when 5 years of age (1916), had well-marked bilateral epicanthus, which is apparently unchanged (1934). It covers the plica

semilunaris and caruncle, arises from tarsal fold of skin on upper lid, and is attached below to the skin down and in from inner end of lower lid. I, 5, Mrs G., with bilateral epicanthus, had an illegitimate daughter, II, 9, by a half-caste, I, 4; she married I, 6, but had no more children. II, 9 and her children, III, 6, 7, 8, have no epicanthus. II, 21, the youngest in the sibship, is 19 years of age. Abroad: I, 7; II, 2, 4, 10. Married without issue: I, 6; II, 5, 7, 10. Deceased: I, 1, 2, 11; II, 6, 11, 12, 14, 16, 17, 18; III, 5, 9. No consanguinity. Some of those referred to are not shown in the abbreviated chart

Fig. 31, Dunbar pedigree. III, 4, Susan D., age 14 months, with well-marked bilateral epicanthus and alternating internal strabismus, is the child of a man,

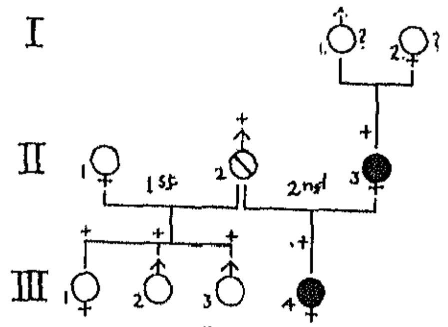


Fig. 31.

II, 2, age 49, with congenital club foot, and a woman, II, 3, with bilateral epicanthus. By his first wife, II, 1, he, II, 2, had three normal children, III, 1, 2, 3, ages 16, 14, 10 years. I, 1 and 2 are dead; no information regarding the presence of epicanthus in them is obtainable. No consanguinity.

Fig. 23 (p. 15), Stewart family. Epicanthus in four brothers in a sibship of ten. II, 9, George S., age 6 (1932), with marked bilateral epicanthus and right convergent internal strabismus. When seen again two years later, it was noted that the epicanthus covered the plica semilunaris and caruncle. It extended from tarsal fold of upper lid to below inner end of lower lid. II, 8, Gordon S., age 11, bilateral epicanthus more marked on left side. II, 5, Robert S., age 14, bilateral epicanthus better marked on left side. II, 4, Wm. S., poorly marked bilateral epicanthus. Died in infancy: II, 2, 3, 6, 7, 10. Parents, I, 1 and 2, normal. No consanguinity.

Fig. 32, Massie pedigree. A mother and two sons with bilateral epicanthus.

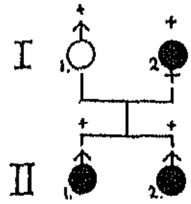


Fig. 32.

II, 1, Alexander M., age 10, and his brother Jim, II, 2, age 8, to a less extent, have bilateral epicanthus. The tarsal fold is continued downwards to level of lower

lid margin, where it ends on side of nose. Their mother, I, 2, presents the same appearances, caruncle and plica normally developed. Father, I, 1, has no epicanthus. No consanguinity.

Fig. 33. Rey pedigree. Bilateral congenital epicanthus in a female, her niece, and paternal grandmother. III, 9, A. R., female, age 22, has well-marked bilateral

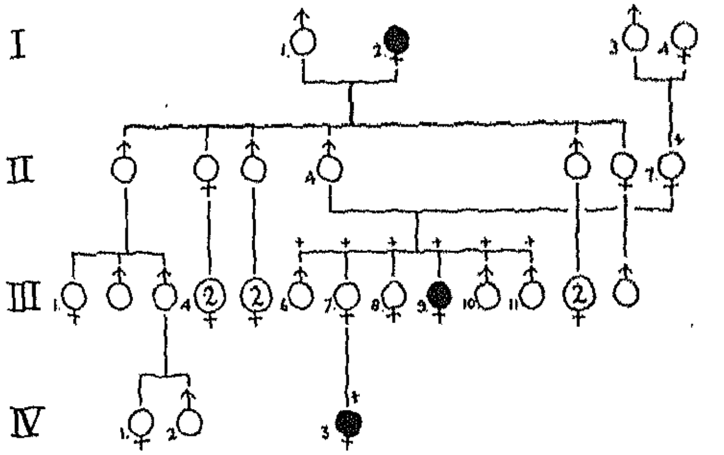


Fig. 33.

epicanthus continuous with tarsal fold of upper lid, no ptosis, caruncle visible, lower punctum not displaced outwards. None of her three brothers, III, 6, 10, 11, or two sisters, III, 7, 8, shows the anomaly, but her niece, IV, 3, has definite bilateral epicanthus continuous with tarsal fold of upper lid. I, 2, her paternal grandmother (not seen), now dead, had exactly the same condition. When A. R., III, 9, was born, the then family doctor (deceased) reminded her mother, II, 7, of the condition in I, 2. II, 7 says that her mother-in-law was able to stretch the epicanthus by raising her brows. She, II, 7, and her husband (deceased) had no epicanthus. She was fifth born in a sibship of eight; neither of her parents nor any progeny of her brothers and sisters (not shown in chart) had epicanthus. III, 1 and the elder of the two sisters, III, 4, are married, without issue. No consanguinity.

Fig. 34, Donald pedigree. Father and two sons with bilateral epicanthus. II, 1, age 3½, has also left internal concomitant strabismus. In the father, the epicanthus

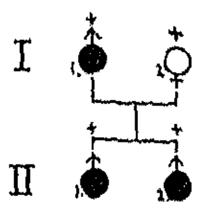


Fig. 34.

begins below the tarsal fold of upper lid and curves downwards to a point below the inner canthus. It has a more solid appearance than usual.

Fig. 27 (p. 17), Gray pedigree. Six sisters with epicanthus. Re-examination of five of them after an interval of 24 years. III, 30, Mary G., age 15 (1912), had a minor degree of epicanthus more marked on left side, which had disappeared by 1934; bridge of nose well developed. III, 32, Joan G., age 9 (1910), brought for blepharo-conjunctivitis, had well-marked bilateral epicanthus which had not altered by 1934. III, 33, Jeannie G., age 8 (1912), with bilateral epicanthus which had not altered by 1934. Her daughter has no epicanthus. III, 35, Daisy G., age 6 (1912), with bilateral epicanthus, which by 1934 had disappeared from left side and was fully marked on right side. III, 36, Annie G., age 4 (1912), with bilateral epicanthus which was still present in 1934. III, 37, Williamina G., age 2 (1912), with unilateral left-sided epicanthus, in 1934 was abroad. III, 31, male, died age 2 (not seen), had no epicanthus. III, 34 and 38 have no epicanthus. None of those affected in the sibship is married except III, 33. Both parents, II, 6 and II, 8, are normal. The mother, II, 8, now dead, said in 1912 that her husband's sister, II, 1, deceased, and one of her daughters had the same condition as her own daughters. This could not be confirmed in 1934 as the three daughters, III, 4, 6, 11, had then no epicanthus. A fourth daughter, III, 12, died young. I, 1, female, age 94 (1912), knew nothing of the condition in any of her relatives when interviewed by II, 8. No consanguinity.

Fig. 19 (p. 14), McLean pedigree. Sixteen affected members in three generations. Transmission continuous through male or female. In a sibship of 11, epicanthus bilateralis was present in seven; in one there was no epicanthus, and in three (II, 1, 8, 10) only feebly marked unilateral epicanthus was found. I, 1, age 56, is normal; I, 2 has bilateral epicanthus palpebralis which has a flattened appearance, skin is thin. III, 2, one died in infancy, three still-born. III, 1, age 10. III, 3, age 3. III, 4, age 10 months, has left unilateral epicanthus. III, 5, age 3 years, and III, 6, age 4 months, have bilateral epicanthus palpebralis. III, 7 and III, 8 have bilateral epicanthus tarsalis, and so have II, 5 and II, 7, but not so markedly. No consanguinity.

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GROWTH CHANGES IN BODILY SIZE AND PROPORTIONS DURING THE FIRST THREE YEARS: A DEVELOPMENTAL STUDY OF SIXTY-ONE CHILDREN BY REPEATED MEASUREMENTS.

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I. INTRODUCTION.

INVESTIGATORS of the problems of growth changes in bodily size and proportion have frequently pointed out that seriatim measures, in a group large enough for statistical treatment, are greatly to be desired. Such measures over the entire growth period would, of course, require many years; in consequence, most seriatim studies cover only a few years of the growth cycle, or are limited to few cases. These studies have usually been limited further by restriction to measurements of height and weight only.

The present study presents cumulative data for the first three years of life. During these early years growth is, of course, much more rapid than at any later age, and consequently a greater portion of it is included for a given unit of time.

The use of a constant sample through a three-year period presents many advantages. With a constant sample we need not make allowances for sampling fluctuations from one age level to the next, and so may work with a smaller number of cases than would otherwise be necessary for securing growth and variability trends. From repeated measures, we can determine the stability of growth rates in individuals. We can study the relationship between growth rates of different parts of the body, for the group as a whole, and for individual members in the group. We can study the consistency in types of build between the earlier and later ages measured. Finally, as a result of these, we may learn, in part, the extent to which one may predict adult size and body build in infancy.

If, as a number of students of body build have claimed, build and certain mental characteristics are related, findings of inconstant body characteristics would suggest a search for concomitant variations in personality traits.

Anthropometric studies of infants are attended with certain serious difficulties. One of these is the problem of securing normal unselected samples. This difficulty is decreasing in recent years, with the prevalence of Well-baby clinics, and the direction of attention to the importance of infant health and welfare. Another difficulty lies in the comparative unreliability of anthropometric measures of infants. Infant measures are less reliable than those of adults because they are smaller; the

importance of chance variations is thus comparatively greater, even though the absolute value of these variations may be the same. But an even more important factor is that of the emotional reactions of the infants to the procedures of measuring. Before they reach an age where they can understand what is wanted of them and are no longer afraid of the measuring instruments, many infants protest, more or less vehemently, against being held in position for the measures and against the application of the instruments. Besides this is the fact that the examiner is usually not well known to the child, and that children between the ages of six months and two years or so of age usually dislike being handled by strangers. The result is that the measures must at times be made on a tense, struggling, crying child. This situation adds greatly to the difficulty of making accurate, reliable measures. Often the measure must be caught in a fleeting moment when the child is in the correct position, and comparatively quiescent. As Baldwin (5) has pointed out, however, much can be done by the examiner, in the manner of his approach, to put the child at his ease and so secure more accurate measures*.

Since normal samples of healthy infants are now available, and the examiner may learn a method of approach which diminishes the child's resistance to being measured, the only remaining question is that of the reliability of the measures, as such. Bakwin and Bakwin (1), who recently made a large number of measures on infants under one year, repeated a series of measurements on the same children and found what they felt was a high degree of reliability—at least as reliable as Todd (38) and others have found for adults. For ten repeated measurements of from 11 to 20 infants on total body length, sitting height, diameter of face, bicristal diameter, biacromial diameter and circumference of thorax at nipples, the coefficients of variation ranged from ·04 per cent. for length to 1·3 per cent. for bicristal diameter†. It seems possible, then, that the disturbing factors can be overcome, and that a study based on the anthropometric measurements of infants may be of some significance.

We are quite familiar, now, with the difference between infants and adults in body proportion, and some of the changes they undergo in the process of maturation. The relatively greater maturity of the upper end at birth, and the cephalo-caudal trend of development is true, in general, of body proportions (Godin (19), Bardeen (7), Bean (8)), as it is of neural and motor development (Coghill, Shirley). However, there is still much to learn about the relative growth rates of the different parts, especially in the same individuals. Seriatim measures have not hitherto been made on a sufficiently large number of infants to allow adequate statistical treatment. Valuable studies of proportional changes with growth (based, for the most part, on samples which are made up of different individuals at successive ages) have been made by Scammon and Calkins (35) for foetal growth, Scammon and others (24) for

* This does not necessarily mean carrying on conversations with a child as Gray and Ayres (23) have inferred, but is often a matter of the examiner's attitude, which may vary according to the child.

† Their data have not, however, been treated in such a way as to make possible comparisons between remeasures on the *same* individual, since the measures on all infants have been grouped together.

post-natal growth, and Boas (10) for rates of growth in height and weight. Baldwin (5) (p. 107) has done this for a large number of seriatim measures on a few children. Godin (19) has shown diagrammatically the changing proportions from the foetus to the adult; the foetus has a proportionately large head and short extremities, especially legs, and the relative proportions of these change strikingly with growth.

Most of the above-mentioned studies have been concerned with the rates of growth and changes in proportion to the various body parts. A closely related field is that of the consistency in growth rates within the individual. Baldwin (5) has found, for example, that children who are tall for their age continue to be tall for their age as they grow older, while short children remain relatively short. Boas (10), on the other hand, has found recently that persons of the same adult height reach that height by different growth rates, and with different ages of most rapid growth. Hence, "acceleration or retardation within the same social group has apparently no influence upon adult stature." Merrell (31), studying rabbits, also found no relationship between rapid early growth and adult size. Studies of consistency must, necessarily, be made from seriatim measures, and must concern themselves with the independent growth curves of individuals.

Another interesting field for investigation is that of body build. Only a few studies of build have been made on infants. Bakwin and Bakwin (4) claim to have found a twofold division of body types in infants. Lederer (29) has made an elaborate classification of four types, giving them names which imply characteristic accompanying behaviour patterns or mental traits. There is considerable disagreement among investigators of body build in adults, as to whether there are distinct types, and, if so, what the types are. We may expect to find the same disagreement in respect to infants, but if we can choose a criterion (or criteria) for measuring individual differences in infants' build we may then determine whether those differences remain constant, and whether they are related to other mental or physical characteristics.

The height and weight of infants under one year have often been measured; and norms based on a large number of measures have been presented. But additional anthropometric measures on such young subjects are few, though there have been several recent studies of infants in which a large number of measures were made. Probably the most complete series of measures has been reported by the Iowa Child Welfare Research Station (26). Recent studies by Freeman (18a, 18b) and Freeman and Platt (18c) give means, S.D.'s and indices for a large number of measures made at frequent age levels on children from birth to 14 years. Other studies are those of Bakwin and Bakwin (1, 2, 3), of Boyd at Minnesota, and of Thompson at the Yale Psycho-Clinic. The results of these last two studies have not yet been published. Other studies, not quite so recent, are those of Fleischner (13), Montague and Hollingworth (32), and Taylor (36).

II. THE PRESENT STUDY.

A. SAMPLE

The present study was undertaken with a group of sixty-one infants who were measured at monthly intervals from one to twelve months, and at fifteen, eighteen, twenty-four, thirty and thirty-six months of age. They were measured on, or within a few days of, their birth date. During the first year the observations were made, if not on the birth date, within four days of that date. Longer deviations from the birth date were allowed later, if necessary (in the third year as much as two weeks), but these wide deviations were rare. This method of seriatim measures made possible a comparison of growth tendencies at different stages of development, based on measures of a sample which was identical, or nearly so, throughout the series. There were occasional absences, and a number of the children moved out of town before the end of the three-year period. Forty-nine of them were still enrolled in the study at thirty-six months.

The method of obtaining the sample caused some selection, making for a somewhat homogeneous group, although originally an attempt was made to get babies from families representative of a cross-section of the community. The subjects observed form a large proportion of Berkeley babies born in two hospitals during a six-month period. Of these babies, we included only those whose parents were white, English-speaking, and willing to co-operate by monthly visits to the Institute of Child Welfare for a series of observations. These observations included, in addition to the anthropometric measurements, tests of mental, motor and physiological development.

A comparison of this Growth Study group with a larger group of Berkeley families in which infants were born at about the same time*, show to what extent they are representative of the Berkeley population. This comparison shows that the families of this smaller group average higher in annual income, education, and occupational rating than the unselected population of families into which babies were born during this time in Berkeley. The mean annual income of the Growth Study is $2844, as compared with the Berkeley Survey mean of $2544. The mean educational level of the Growth Study parents is, for the fathers 13·8 years and for the mothers 13·1 years, while the Berkeley Survey fathers average 12·2 years and the mothers average 11·6 years of schooling. Thirty per cent. of the fathers of the Growth Study infants were classified in professional or executive occupations, as compared with 22 per cent. of the Berkeley Survey group. An equal proportion of both groups were classed in "white collar" occupations, while the smaller group had comparatively fewer in the skilled, semi-skilled and unskilled labour categories.

From these comparisons, it may be assumed that this group of infants comes, for the most part, from homes where the education and the income combine to insure hygienic living conditions and sufficient nourishing food; and hence, well-nourished, healthy babies. This selection may account, in part, for the fact that

* *The Berkeley Survey*, a study made by Frances M. Welch and Agnes Covall.

the mean measures of this group of Berkeley babies are larger, after the sixth month, than those of other published infant norms (41).

The group is probably also, because of similar racial stocks and freedom from environmental handicaps, more homogeneous than would be an unselected sample. For this reason many of the correlational values between measures will be more significant than the same values would be for a more heterogeneous group of infants.

B. The Measures: Instruments and Technique

The measures which we undertook to make were selected on the basis of two criteria: the accuracy with which they could be made on infants, and their probable value in indicating growth trends in size and bodily proportions. The measures taken were length (reclining during the first eighteen months, afterwards standing); stem length (reclining during the first eighteen months, afterwards sitting); widths of shoulders, chest and hips; chest depth; circumferences of head, chest and abdomen; head diameters, transverse and antero-posterior; and weight. This selection of measures and the methods of taking them are based, for the most part, on those used at the Iowa Child Welfare Research Station. As the technique has been described in a recent publication (26), we will indicate here only our deviations from it. In the present study we have not considered the abdominal circumference or the head diameters.

The instruments used during the first eighteen months varied from those used at Iowa in several respects. Chest depth was measured with the spreading callipers; and the shoulder, chest and hip widths with the small sliding callipers. This change was made because the large sliding callipers were found to be awkward with the young infants. From eighteen months on, the large callipers were used in the chest, hip and shoulder diameters. Circumferences at all ages were measured with a linen tape graduated in millimetres, and having a spring attachment at one end by which the pressure could be kept constant*. The tapes were replaced at intervals to prevent inaccurate measures due to their shrinkage with use. Standing length and sitting stem length were measured on a stadiometer made by the Continental Scale Works in Chicago. Since this stadiometer was calibrated in inches, the measures were read in inches and quarter-inches, and converted to centimetres for treatment of the data. Infants under about twenty pounds were weighed on a light portable spring scale whose accuracy was checked at frequent intervals; the heavier babies were weighed on a balance scale. Both scales were calibrated in pounds and ounces. These have been converted to kilograms for treatment of the data. All measures were made with all clothing removed. At about nine months, it was found to be impossible to measure most children while lying down, without vehement protest. For this reason, the head, chest, and shoulder measures were made with the child sitting, until he was able to stand with support, when all of the body diameters and circumferences were taken standing. After eighteen months, the lengths also were taken standing.

* These are made by the Narragansett Machine Co., Providence, R.I.

Further differences in procedure on individual measures are as follows: For length* of the stem in the erect posture the child sat in the stadiometer, with its spine in contact with the stadiometer, and knees flexed by placing its feet up on the platform in the position described by Dreyer (17). The experimenter then held the head in place and measured in the same way as for standing length. *Chest depth*: when the spreading callipers were used, the fixed branch was placed on the sternum at the nipple line, and the movable branch at the centre back, just below the scapulae.

In order to obtain accurate measures, every effort was made to keep the child quiet and contented. An active restless child would not maintain the optimum position long enough for a measure to be read, and the chest measures of a violently crying child varied far more widely than those of one breathing quietly. The difficulty of obtaining accurate measures increased as the infants grew older and more active until, at eighteen or twenty-four months, they could be made interested in the procedure and persuaded to stand still and submit willingly to being measured. The whole situation was made more difficult by the fact that mental, physical and motor tests and the taking of photographs were a part of the procedure during the period of observation, so that the child was often excited or tired before any anthropometric measures were taken. The experimenter measured the child with as little handling as possible, and always with a gentle friendly approach. But from about seven to eighteen months, the situation was complicated by the child's emotional reaction—at times violent—to the strange situation and to being handled by strange persons. In the difficult cases the mother usually co-operated by holding the child in position. These variable factors probably resulted in some inaccuracies in measures. For this reason, measures taken under unfavourable conditions, when they deviated widely from other measures of the same child, were not used in the treatment of the data.

Every effort was made to overcome unreliability of the measures, by carefully controlling the position of the child, and by securing its acquiescence whenever possible. Some of the measures, especially weight and height, could be made fairly accurately, even though the children were resistant, while others were greatly affected by variations in attitude toward the instruments and the examiner. An example of the latter is shoulder width. The child who shrinks from the callipers and draws its shoulders together may decrease its shoulder width by several centimetres. On the other hand, the unreliability of a measure may be a function of the dimension itself. The amount of fat or the firmness of the muscles over the points measured, in hip width, for example, may vary enough in the same child, from one month to the next, to change his relative position in the group. Other measures, such as chest depth, can be changed by assuming different body postures which change the relationships of the component parts involved in the measure. Since children so young cannot be expected to co-operate, or even to understand

* The term "length" has been substituted for "height," in this paper, to apply to both reclining length and standing height; and the symbol L. is used instead of H. in the indices and tables, i.e. $W./L.^3$.

the positions desired by the experimenter, we may expect to find such measures relatively unreliable. The most reliable measures should be those taken from bony points without intervening joints, and with little flesh covering. Some of these may be read accurately, even on a protesting child.

Most of the measures taken in this study were made by one person (Bayley), but in a few instances they were made by the other experimenter in the Growth Study, Dr L. V. Wolff. After determining a given measure, the experimenter read it aloud to a recorder who entered it on the record form.

C. Ratings of Body Build from Photographs.

In order to evaluate the validity of the different body build indices, a criterion was set up on the basis of photographs taken under standard conditions. The use of photographs made it possible to compare all of the children of a given age at one time.

(a) *Method of taking Photographs.*

The infants were photographed, with all clothing removed, at ages 1, 3, 5, 7, 9, 12, 15, 18, 24, 30 and 36 months. The pictures were taken with a Graflex camera, indoors, with standard lighting conditions, lens at $F4{\cdot}5$, and a ·1 second exposure. The infants at first lay or sat on a canvas table with the camera placed at a standard distance. When the children were able to stand alone, they were placed on "foot-prints" painted on the floor, with the camera at a new standard distance. Three exposures were always made. In the early months two exposures of the child in the dorsal and one in the prone position were taken. When a child was able to sit alone, one exposure in the sitting position was substituted for one of the dorsals. When he was able to stand alone, one front, one rear, and one side view, all standing, were taken. However, it was not always possible to persuade a child to stay where he was put for a picture, so that many of the resulting photographs are not exactly as desired. Usually at least one of the three was sufficiently in position to make possible a fair estimate of body build, while for the 30 and 36-month pictures, better co-operation of the children resulted in very satisfactory pictures.

(b) *Method of Rating.*

For the purpose of rating body build from the photographs, the pictures were sorted into groups according to age, with each child's pictures for one age fastened to a single large card. The ratings were made by comparing each child with the others of the same age. To do this the rater sorted the cards into seven stacks, if that many divisions could be made, putting pictures of similar body build together. Each age group was rated separately without reference to other ages, and each characteristic considered was rated independently. Standing pictures were not compared with sitting, nor sitting with reclining pictures.

A rating scale of seven different body proportions was prepared, with a seven-point rating for each. Since only one of these proportions—general chubbiness, or distribution on a lateral-linear scale—could be rated adequately from the

reclining and sitting pictures, it is the only one considered in this paper. Chubbiness was rated for all months at which pictures were taken, independently by two judges, one (*B*) who had measured the infants, and one (*D*) who was not familiar with the children and so was not influenced, in his ratings of the photographs, by any preformed opinions of a child's build. The pictures at 15, 18 and 24 months were rated by the writers and by Dr L. V. Wolff. The scores used for comparison with the objective measures are the combined ratings of the two (or, where available, three) judges.

III. RESULTS.

A. THE RELIABILITY OF THE MEASUREMENTS.

The best measure of the reliability of data such as these would be that afforded by the coefficients of correlation from repeated measurements taken during one examination period. This procedure was not followed in the present instance, for several reasons. In the first place, the amount of time allotted for the examination of a given child, at any one examination period, was too brief to permit the repetition of the several anthropometric measurements and still allow for the other observations which were included in the plan of testing. Again, the attempt to repeat the anthropometric measurements would have led (particularly at certain ages) to emotional disturbances in some of the infant subjects which would have rendered the second series difficult to obtain and often unreliable when they could be obtained.

In the absence of reliability coefficients derived from repeated measurements during a given examination period, the reliability of the measurements reported upon herein may be estimated in the following ways: (1) by a study of the coefficients of correlation obtained when one month's measurements are paired with those taken at the next succeeding examination period; (2) by the remeasurement of a small group of infants after an interval of one week; and (3), less directly, by a consideration of the consistency of the trends of central tendency for the group.

(1) Low coefficients of correlation when successive monthly measurements are correlated with each other could be due either to errors in the taking or in the recording of the measurements, or to pronounced shifts, during the interval between the two examinations, in the relative position of the individuals composing the sample. High correlations, on the other hand, can mean only that the measurements were made with a relatively high degree of precision and that there is a definite tendency toward stability in the rank orders of the subjects from month to month. Table I gives the self-correlations obtained in that way. Weight, total body length, and head circumference are seen to be highly stable measures. Stem length, hip width, and chest circumference rank next. Shoulder width, chest width, and chest depth are relatively unreliable, and this fact needs to be taken into consideration when the respective merits of various proposed indices of body build are under consideration. It may be noted here that our results, based upon seriatim measurements made on living infants, are in agreement with the conclusion of

TABLE I.

Self-Correlations between Consecutive Measurings.

	Months																Mean r
	1×2	2×3	3×4	4×5	5×6	6×7	7×8	8×9	9×10	10×11	11×12	12×15	15×18	18×21	21×[illegible]	[illegible]	
W.	·909	·945	·931	·947	·961	·943	·954	·961	·961	·941	·965	·957	·931	·931	[illegible]	[illegible]	[illegible]
L.	·918	·920	·898	·889	·942	·950	·939	·930	·914	·928	·932	·918	·921	·916	·917	·974	[illegible]
H.C.	·916	·934	·937	·920	·951	·942	·941	·876	·916	·911	·922	·921	·919	·912	·901	[illegible]	·924
S.L.	·772	·847	·821	·751	·879	·836	·860	·875	·887	·904	·883	·863	·871	·784	[illegible]	[illegible]	[illegible]
H.W.	·857	·810	—	—	·701	—	—	·779	—	—	·839	—	·745	·812	·851	[illegible]	·825
C.C.	·829	·821	—	—	·765	—	—	·881	—	—	·837	—	·862	·823	·839	·884	[illegible]
C.W.	·709	·963	—	—	·609	—	—	·808	—	—	·637	—	·545	[illegible]	·765	·871	[illegible]
S.W.	·766	·928	—	—	·690	—	—	·706	—	—	·829	—	·567	·511	·737	·738	[illegible]
C.D.	·670	·710	—	—	·501	—	—	·691	—	—	·580	—	·394	·677	·731	·884	[illegible]
W./L.[3]	·718	·741	·808	·857	·890	·726	·871	·874	·896	·832	·882	·874	·830	·732	·740	·911	·846

Scammon and Calkins (35), based upon repeated measurements made on human foetuses, that total body length (crown-heel length) is a more highly reliable measure than stem length (crown-rump length). Boyd (13) has also found, on remeasuring Nursery School children, that standing height is by far the most reliable of a series of length measures.

(2) As an additional check on the reliability of the measures, fifteen infants in a San Francisco baby home, ranging in age from nine days to seven months, were measured, and then remeasured by Bayley after an interval of one week. Unfortunately it was impracticable to measure more cases in this way, and since the number of cases is so small, the results must be considered as only tentative, and are interesting, mainly, because they corroborate the results of comparisons between measures made one month apart. Table II gives the means and S.D.'s, and the differences between the first and second measures. The variation is much greater for some measures than for others. The greatest mean difference is in length, the smallest in hip width. Since length is also the longest measure taken, its greater variation does not necessarily invalidate it. When we convert the mean differences into percentage of the mean of each measure (column 9) we find that the head measures are relatively most consistent, while stem length, total length, hip width and chest circumference all have mean differences of less than 2 per cent. of their mean size. Chest width shows the greatest percentage variation (3·6 per cent.), with chest depth and shoulder width also relatively unreliable.

(3) The curves of the means for all measures are very smooth when one considers that they are composed of between 50 and 60 cases at each age. This is due, in large part, to the constant sample. Gross errors, however, either in the taking of the measurements or in their recording, would presumably have led to a greater fluctuation in the means. Such fluctuation would be especially evident in the means for the sexes separately. Table IV and Figs. 2 and 4 show that these sex differences are small but consistent. If they were due to chance errors

TABLE II.

*Repeated Measures of Infants in San Francisco Baby Home**.

(Centimetres.)

1	2	3	4	5	6	7	8	9
	No.	M_1	σ_1	M_2	σ_2	M Diff.	σ Diff.	$\frac{M \text{ Diff.}}{M_1}$
Length	13	53·05	3·0338	54·08	2·0690	·75	·4005	·0139
Stem Length	13	35·56	2·0209	35·42	2·3970	·46	·4568	·0129
Shoulder Width	14	15·43	1·8941	15·36	2·1379	·44	·2660	·0285
Chest Width	15	11·37	1·3768	11·23	1·1396	·41	·3025	·0361
Hip Width	15	11·03	1·5098	11·07	1·6774	·18	·2067	·0163
Chest Depth	15	10·04	·9105	10·06	·8648	·29	·2584	·0280
Head Circumference	15	38·22	3·4906	38·15	3·4098	·21	·1752	·0055
Chest Circumference	15	35·96	3·6519	36·00	3·0633	·05	·5530	·0181
Transverse Head	13	9·07	·6467	9·08	·6182	·05	·0480	·0050
Antero-posterior Head	14	13·04	1·0645	13·04	1·0557	·04	·0534	·0031

* The ages of these children are as follows: 2 weeks, 2 cases; 1 month, 2 cases; 2 months, 4 cases; 3 months, 6 cases; 4 months, 1 case; and 7 months, 1 case.

in measurement, the direction of the difference would fluctuate, with the females sometimes exceeding the males, and *vice versa*.

As for the reliabilities of the judges' ratings of photographs, the correlations between the ratings of judges B and D range from ·71 to ·81, with a mean of ·76 (Table III). The correlations for months 15, 18, and 24 average ·85 between B and W, and ·78 between W and D. The combined ratings of all judges were correlated with themselves at consecutive ages, and these r's (Table III), when the necessarily wide age intervals between pictures are considered, are satisfactory. They range from ·65 to ·93 with a mean of ·81. A coefficient of correlation is, of course, always to be interpreted in the light of the known facts with respect to such factors as (1) the size of the sample, (2) the degree of heterogeneity present in the sample, and (3) the extent to which the sample is selected and the kind of selection involved. For example, a selection of children from the poorer social strata might include a much larger proportion of disease-affected and under-nourished children than would be found in a sample from the more prosperous classes of a population. The latter group could very well be subjected to more standardised regimes and diets, and as a result of homogeneous environment have a smaller variability in measures. It seems probable that the present sample is, in the respects of diet and regime, more homogeneous than the population of the whole community. When we consider this, and the relatively small number of cases, these correlations, based on a seven-point rating scale, show a fair relationship between the ratings of the different judges; and when the judges' ratings are combined into a single score, we have what we may consider a reliable criterion of body build against which to compare the objective indices.

TABLE III.

Correlations between Judges' Ratings of Chubbiness.

	Months											Mean r
	1	3	7	9	10	12	15	18	24	30	36	
$B \times D$	·77	·73	·75	·79	·79	·75	·81	·79	·71	·74	·73	·76
$B \times W$	—	—	—	—	—	—	·88	·88	·78	—	—	·85
$W \times D$	—	—	—	—	—	—	·77	·79	·78	—	—	·78

Self-Correlations of combined Judges' Ratings.

Months								Mean
1 & 3	3 & 5	5 & 7	7 & 9	9 & 12	12 & 15	15 & 18	18 & 24	
·70	·79	·65	·83	·93	·85	·86	·85	·81

B. Growth trends of the Individual Bodily Dimensions.

1. *Central Tendency.*

Table IV and Figs. 1—3 present the facts obtained in this seriatim study concerning the manner and relative rapidity of growth of the body characters with which we deal in this paper. The curves of the means are all of the same general type, with total body length showing far more rapid acceleration than the other body dimensions. Stem length, chest circumference and head circumference are similar in size and shape of curve throughout, though the latter shows smaller increments. Of the transverse measures, the shoulders are largest and grow the most. Next in size come hip width, chest width, and finally chest depth. Weight cannot be compared directly with these measures, but its curve is more nearly like that of height than any of the others.

Figs. 4—7 present a comparison of this sample with other samples previously reported upon (41). The data for the Growth Study, as already noted, are seriatim; all the other samples are cross-sectional. As was to be expected in the light of Woodbury's data, which showed Californian infants excelling those from other sections of the country in height and in weight, our sample, especially of the boys, is seen to average heavier and taller from the sixth month on than the samples from Iowa, New York City, and the United States at large. The Growth Study sample also averages somewhat taller and heavier than does the Californian sample above mentioned, except during the first six or eight months. During the first half-year the growth curves of all four groups are almost identical.

TABLE IV.

Table of Means, S.D.'s, P.E.'s of Means and Coefficients of Variation of nine measures made repeatedly on the same Infants from 1 through 36 months.

(Centimetres and kilograms.)

Month		Length					Stem Length					Head Circumference				
		No.	M	P.E.$_M$	σ	V	No.	M	P.E.$_M$	σ	V	No.	M	P.E.$_M$	σ	V
1	Boys	24	55·45	·317	2·31	4·16	24	36·00	·168	1·15	3·18	24	38·50	·164	1·19	3·09
	Girls	26	53·42	·319	2·41	4·52	26	34·00	·224	1·60	4·85	26	37·13	·161	1·22	3·28
	Both	50	54·39	·213	2·23	4·11	50	35·42	·167	1·65	4·65	50	37·82	·129	1·35	3·57
2	Boys	31	58·86	·222	1·84	3·12	31	38·64	·267	2·21	5·71	31	40·11	·137	1·13	2·82
	Girls	27	56·08	·280	2·16	3·79	27	37·07	·185	1·43	3·86	27	38·82	·154	1·19	3·06
	Both	58	57·98	·201	2·27	3·91	58	37·91	·180	2·03	5·36	58	39·51	·117	1·32	3·33
3	Boys	31	61·73	·204	1·68	2·72	31	40·32	·143	1·18	2·92	31	41·54	·145	1·20	2·89
	Girls	30	59·72	·276	2·24	3·75	29	38·08	·187	1·46	3·83	29	40·20	·133	1·06	2·63
	Both	61	60·74	·193	2·24	3·69	60	39·07	·128	1·47	3·70	60	40·89	·118	1·35	3·30
4	Boys	30	64·11	·227	1·85	2·88	30	41·82	·130	1·06	2·52	30	42·74	·126	1·03	2·40
	Girls	29	62·37	·224	1·79	2·88	29	40·85	·215	1·72	4·20	29	41·40	·142	1·13	2·73
	Both	59	63·25	·184	2·09	3·31	59	41·34	·135	1·54	3·73	59	42·08	·114	1·29	3·07
5	Boys	30	66·53	·253	2·06	3·09	30	43·36	·183	1·49	3·43	30	43·93	·145	1·18	2·69
	Girls	27	64·57	·307	2·37	3·67	27	41·81	·245	1·89	4·52	27	42·45	·157	1·21	2·85
	Both	57	65·60	·218	2·44	3·72	57	42·62	·172	1·92	4·52	57	43·23	·124	1·39	3·21
6	Boys	30	68·48	·266	2·16	3·16	30	44·60	·180	1·51	3·38	30	44·93	·144	1·17	2·61
	Girls	28	66·38	·272	2·13	3·21	28	43·18	·187	1·47	3·40	28	43·28	·148	1·16	2·67
	Both	58	67·46	·210	2·48	3·67	58	43·91	·145	1·64	3·73	58	44·14	·117	1·32	3·00
7	Boys	26	69·85	·309	2·34	3·35	26	45·36	·210	1·59	2·97	27	45·65	·172	1·33	2·91
	Girls	25	68·04	·357	2·65	3·89	24	44·10	·281	2·04	4·62	25	44·33	·178	1·32	2·98
	Both	51	68·99	·254	2·69	3·90	50	44·80	·181	1·90	4·23	52	45·02	·132	1·41	3·13
8	Boys	25	71·82	·333	2·47	3·43	25	46·52	·252	1·87	4·01	25	46·46	·175	1·29	2·78
	Girls	28	69·68	·355	2·78	3·99	28	45·11	·236	1·85	4·10	27	44·98	·157	1·21	2·69
	Both	53	70·69	·263	2·84	4·01	53	45·78	·179	1·93	4·21	52	45·69	·138	1·48	3·24
9	Boys	27	72·89	·297	2·28	3·13	27	47·30	·223	1·71	3·62	27	47·04	·150	1·15	2·45
	Girls	29	70·34	·314	2·51	3·56	29	45·43	·200	1·59	3·51	29	45·35	·162	1·29	2·85
	Both	56	71·57	·241	2·71	3·79	56	46·36	·173	1·92	4·14	56	46·16	·141	1·56	3·39
10	Boys	28	74·28	·287	2·25	3·03	28	47·00	·218	1·71	3·56	26	47·65	·168	1·27	2·67
	Girls	27	71·67	·327	2·52	3·58	27	46·18	·230	1·78	3·84	22	45·84	·190	1·32	2·88
	Both	55	73·00	·246	2·70	3·70	55	47·06	·172	1·89	4·01	48	46·82	·154	1·58	3·38
11	Boys	27	75·57	·333	2·57	3·40	27	48·84	·213	1·64	3·36	25	47·93	·153	1·14	2·37
	Girls	23	73·23	·388	2·76	3·77	23	46·94	·272	1·94	4·12	18	46·53	·242	1·52	3·27
	Both	50	74·49	·281	2·95	3·96	50	47·97	·188	1·98	4·11	43	47·34	·159	1·54	3·26
12	Boys	28	76·89	·323	2·53	3·30	28	49·16	·242	1·90	3·86	28	48·50	·165	1·30	2·68
	Girls	28	74·49	·335	2·63	3·53	28	47·66	·240	1·88	3·95	27	46·75	·176	1·36	2·90
	Both	56	75·69	·257	2·85	3·76	56	48·41	·184	2·04	4·21	55	47·64	·146	1·60	3·37
15	Boys	27	80·55	·370	2·85	3·53	27	51·23	·238	1·83	3·58	27	49·31	·179	1·38	2·80
	Girls	25	77·66	·372	2·76	3·55	25	49·00	·242	1·80	3·67	25	47·52	·148	1·10	2·32
	Both	52	79·16	·299	3·16	3·99	52	50·16	·197	2·11	4·20	52	48·45	·143	1·53	3·16
18	Boys	23	83·18	·470	3·34	4·02	22	52·09	·255	1·77	3·40	22	49·82	·187	1·30	2·62
	Girls	24	80·89	·393	2·85	3·53	24	49·90	·248	1·80	3·61	23	47·98	·181	1·28	2·68
	Both	47	82·01	·329	3·31	4·04	46	50·95	·205	2·06	4·05	45	48·88	·159	1·58	3·23
24	Boys	24	88·44	·452	3·28	3·71	24	53·60	·257	1·87	3·49	24	50·73	107	1·43	2·83
	Girls	19	86·08	·313	2·02	2·35	19	51·18	·185	1·20	2·34	18	49·08	184	1·15	2·36
	Both	43	87·40	·308	2·99	3·42	43	52·48	·196	1·91	3·63	42	50·00	·105	1·59	3·18
30	Boys	25	93·05	·559	4·14	4·45	25	54·10	·250	1·92	3·54	25	51·02	·170	1·33	2·60
	Girls	21	90·71	·468	3·18	3·51	21	52·60	·205	2·00	3·80	21	49·26	·180	1·22	2·48
	Both	46	91·98	·391	3·93	4·28	46	53·49	·211	2·12	3·96	46	50·22	·149	1·49	2·97
36	Boys	25	97·38	·588	4·36	4·48	25	55·08	·310	2·34	4·21	23	51·39	·187	1·33	2·59
	Girls	24	95·02	·457	3·32	3·49	24	53·40	·239	1·74	3·26	24	49·72	·143	1·04	2·08
	Both	49	96·22	·396	4·11	4·27	49	54·59	·220	2·38	4·36	47	50·49	·149	1·49	2·94

TABLE IV.—(*continued*).

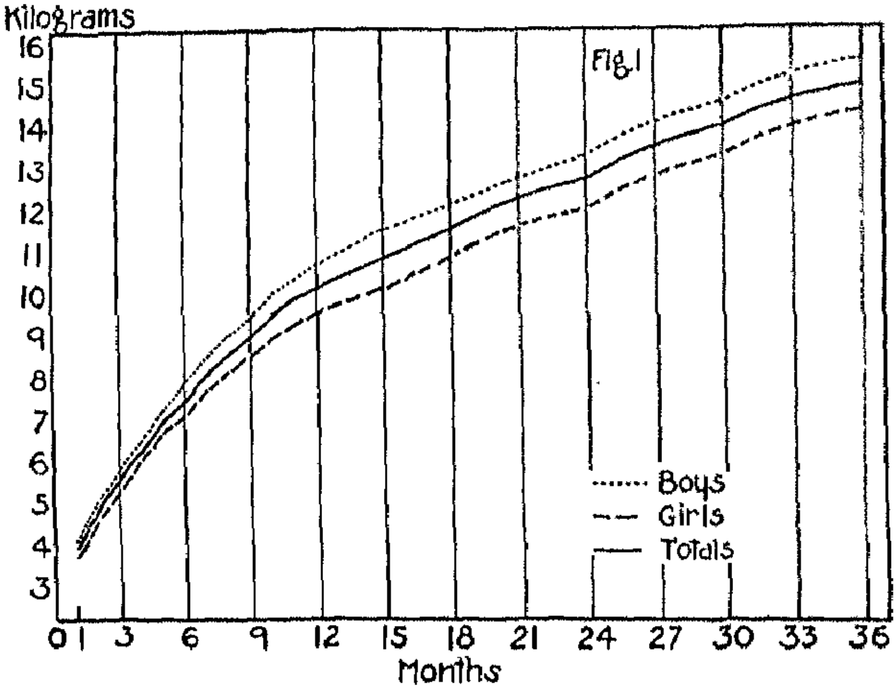
(Centimetres and kilograms.)

Month		Shoulder Width					Chest Width					Chest Depth				
		No.	M	P.E.$_M$	σ	V	No.	M	P.E.$_M$	σ	V	No.	M	P.E.$_M$	σ	V
1	Boys	23	14·86	·091	·65	4·38	22	10·46	·111	·77	7·38	24	9·89	·083	·60	6·07
	Girls	26	14·10	·082	·62	4·38	25	10·15	·101	·75	7·37	26	9·70	·081	·61	6·33
	Both	49	14·46	·068	·71	4·88	47	10·30	·073	·74	7·21	50	9·79	·059	·62	6·38
2	Boys	31	15·94	·092	·76	4·75	31	11·08	·127	1·05	9·47	31	10·35	·080	·66	6·4
	Girls	27	14·81	·143	1·10	7·45	25	10·88	·091	·67	6·16	27	10·21	·080	·69	6·77
	Both	58	15·41	·099	1·12	7·26	56	10·99	·082	·91	8·28	58	10·28	·063	·71	6·95
3	Boys	30	16·75	·072	·58	3·47	30	11·97	·139	1·13	9·44	30	10·57	·082	·67	6·34
	Girls	28	16·12	·082	·64	4·00	28	11·31	·091	·71	6·29	27	10·18	·088	·67	6·62
	Both	58	16·44	·067	·76	4·60	58	11·65	·090	1·02	8·72	57	10·38	·066	·74	7·10
4	Boys	30	17·46	·091	·74	4·25	30	11·89	·118	·96	8·08	30	10·68	·074	·60	5·59
	Girls	29	17·10	·111	·89	5·19	29	12·12	·108	·86	7·10	29	10·60	·101	·81	7·64
	Both	59	17·28	·076	·87	5·02	59	12·01	·077	·87	7·28	59	10·64	·063	·72	6·75
5	Boys	30	18·25	·134	1·09	5·98	30	12·76	·121	·98	7·68	30	11·01	·059	·48	4·32
	Girls	26	17·83	·151	1·15	6·42	26	12·36	·076	·58	4·66	26	10·72	·070	·53	4·93
	Both	56	18·05	·106	1·17	6·49	56	12·57	·078	·87	6·93	56	10·87	·052	·58	5·29
6	Boys	30	18·98	·117	·95	4·99	30	13·26	·133	1·08	8·15	29	11·11	·088	·70	6·29
	Girls	28	18·43	·148	1·16	6·32	28	12·88	·110	·93	7·26	28	10·87	·077	·61	5·59
	Both	58	18·72	·093	1·05	5·58	58	13·08	·084	·94	7·21	57	10·99	·061	·68	6·22
7	Boys	27	19·46	·128	·99	5·08	27	13·07	·102	·79	5·70	26	11·32	·085	·64	5·66
	Girls	24	19·01	·149	1·08	5·08	24	13·45	·118	·86	6·30	25	10·66	·095	·71	6·43
	Both	51	19·25	·098	1·04	5·41	51	13·57	·076	·80	5·90	51	11·15	·061	·64	5·77
8	Boys	25	19·91	·165	1·23	6·15	25	14·29	·138	1·02	7·15	25	11·35	·056	·41	3·64
	Girls	26	19·43	·129	·98	5·02	26	13·81	·115	·87	6·31	26	11·37	·113	·86	7·52
	Both	51	19·66	·111	1·18	5·98	51	14·03	·092	·98	6·97	51	11·36	·064	·68	5·97
9	Boys	27	20·33	·142	1·09	5·37	27	14·35	·125	·96	6·69	26	11·57	·082	·62	5·34
	Girls	29	19·64	·144	1·15	5·84	29	14·06	·115	·92	6·53	29	11·45	·092	·74	6·42
	Both	56	19·97	·108	1·19	5·98	56	14·20	·086	·95	6·68	55	11·51	·060	·66	5·71
10	Boys	25	20·74	·151	1·12	5·39	25	14·76	·126	·94	6·33	26	11·98	·086	·65	5·41
	Girls	22	19·93	·181	1·26	6·32	22	14·18	·149	1·04	7·33	22	11·67	·086	·62	5·34
	Both	47	20·36	·124	1·26	6·18	47	14·49	·100	1·02	7·01	48	11·84	·043	·44	3·74
11	Boys	24	21·00	·149	1·08	5·14	24	14·86	·121	·88	5·92	26	11·90	·082	·62	5·16
	Girls	19	20·25	·199	1·29	6·37	19	14·73	·127	·82	5·57	19	11·60	·100	·65	5·59
	Both	43	20·67	·126	1·22	6·01	43	14·80	·090	·88	5·93	45	11·81	·060	·60	5·08
12	Boys	25	21·09	·141	1·04	4·94	28	15·10	·120	·94	6·24	28	12·17	·088	·69	5·67
	Girls	26	20·47	·121	·91	4·45	27	14·90	·093	·72	4·83	28	11·88	·084	·60	4·20
	Both	51	20·77	·105	1·11	5·35	55	15·00	·079	·86	5·75	56	12·02	·060	·67	5·54
15	Boys	27	21·48	·172	1·33	6·18	27	15·08	·120	·93	5·70	26	12·20	·086	·65	5·28
	Girls	25	20·71	·124	·92	4·42	25	15·28	·080	·66	4·31	25	12·11	·083	·62	5·09
	Both	52	21·11	·113	1·21	5·73	52	15·04	·085	·91	5·82	51	12·20	·062	·66	5·38
18	Boys	22	22·18	·164	1·07	4·81	22	16·06	·123	·86	5·34	22	12·48	·124	·86	6·90
	Girls	22	21·40	·124	·86	4·03	22	15·53	·110	·81	5·19	23	12·35	·093	·66	5·32
	Both	44	21·79	·107	1·05	4·80	44	15·79	·093	·92	5·81	45	12·42	·072	·71	5·73
24	Boys	23	22·59	·163	1·15	5·11	22	16·45	·144	1·00	6·10	23	12·77	·115	·82	6·41
	Girls	18	21·99	·165	1·04	4·73	19	15·92	·105	·68	4·24	19	12·17	·114	·74	6·05
	Both	41	22·33	·117	1·11	4·97	41	16·20	·099	·95	5·84	42	12·50	·086	·83	6·62
30	Boys	25	22·51	·218	1·62	7·18	24	16·90	·115	·83	4·90	25	12·68	·148	1·10	8·68
	Girls	21	22·20	·130	·88	3·97	21	16·26	·088	·60	3·69	21	12·21	·128	·87	7·12
	Both	46	22·37	·132	1·33	5·94	45	16·63	·085	·85	5·08	46	12·40	·107	1·07	8·59
36	Boys	25	23·54	·132	·98	4·16	23	17·20	·122	·87	5·05	23	13·00	·127	·90	6·93
	Girls	24	23·08	·123	·89	3·87	24	16·55	·090	·69	4·19	24	12·42	·083	·61	4·87
	Both	49	23·32	·089	·92	3·94	47	16·87	·082	·83	4·93	47	12·70	·084	·85	6·68

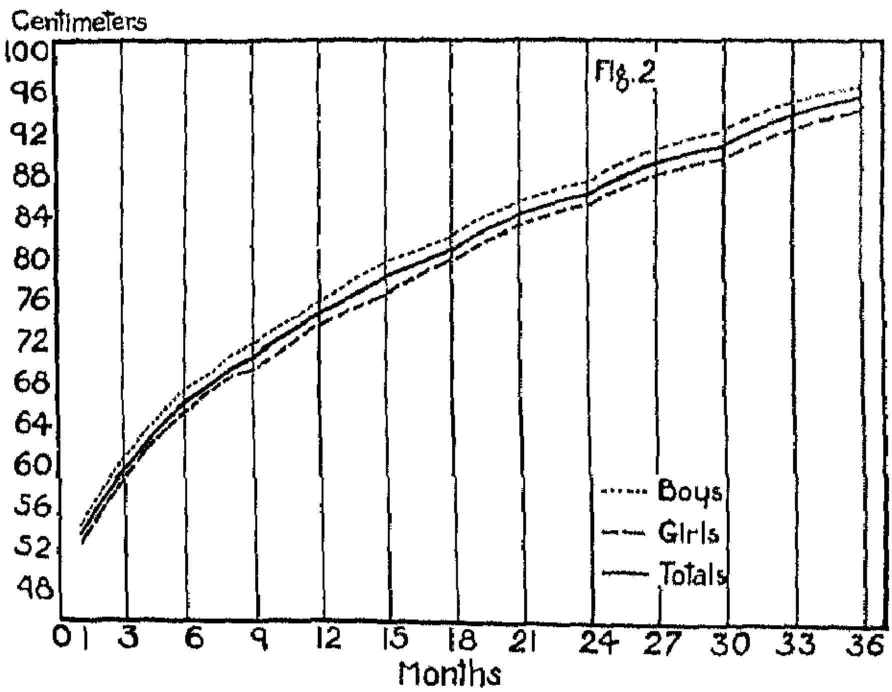
TABLE IV—(*continued*).

(Centimetres and kilograms.)

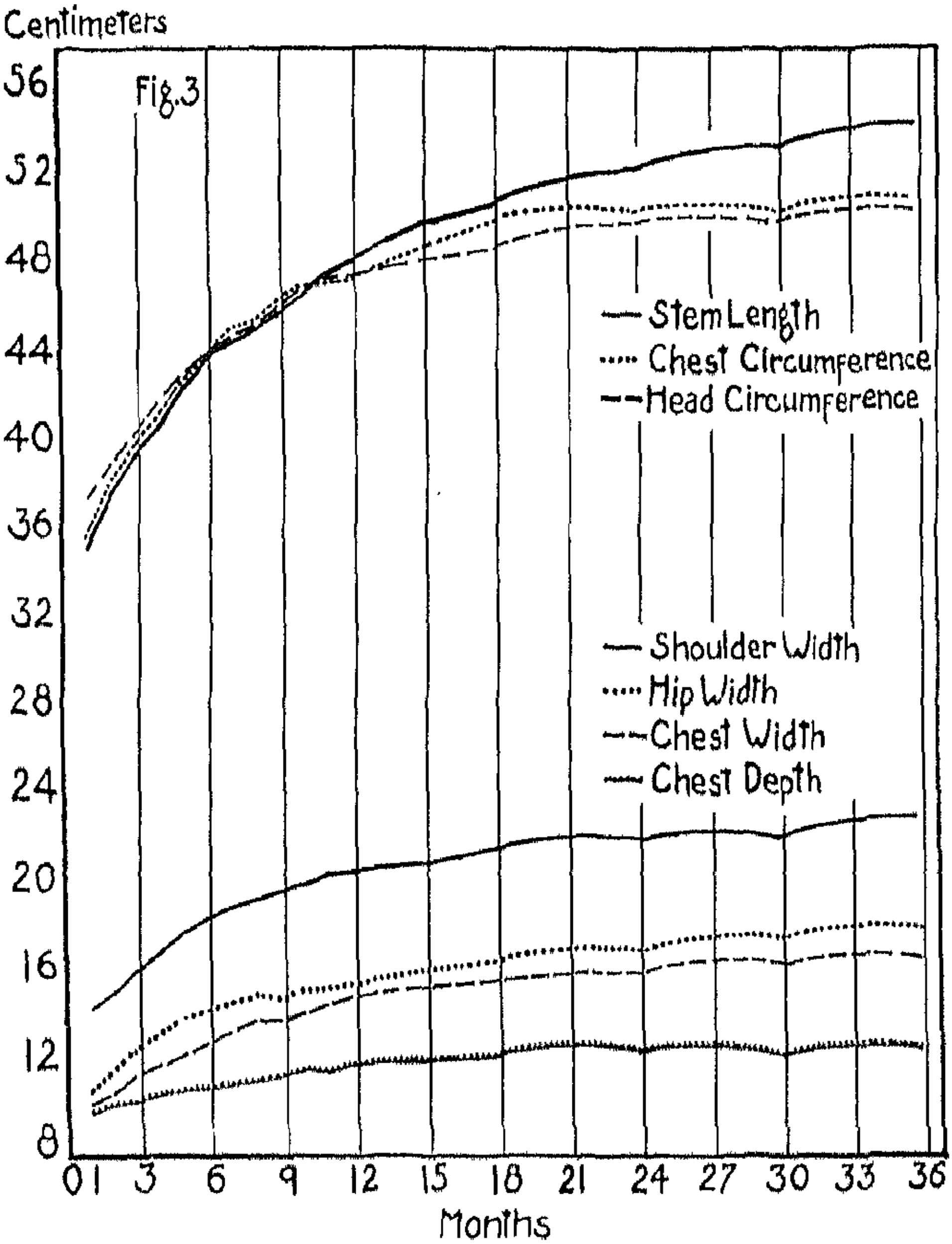
Month		Chest Circumference					Hip Width					Weight				
		No.	M	$P.E._M$	σ	V	No.	M	$P.E._M$	σ	V	No.	M	$P.E._M$	σ	V
1	Boys	24	36·49	·231	1·68	4·61	23	11·03	·111	·79	7·19	25	4·37	·065	·44	4·61
	Girls	26	35·43	·210	1·59	4·48	26	10·62	·110	·90	8·52	26	3·98	·080	·53	6·13
	Both	50	35·91	·163	1·70	4·74	49	10·82	·080	·83	7·67	51	4·17	·055	·62	5·67
2	Boys	31	38·05	·231	1·91	4·90	31	12·06	·114	·94	7·80	31	5·28	·06	·50	4·29
	Girls	27	37·87	·198	1·53	4·04	27	11·91	·124	·96	8·05	27	4·83	·08	·63	5·06
	Both	58	38·45	·159	1·80	4·67	58	11·99	·084	·95	7·95	58	5·08	·05	·59	5·31
3	Boys	30	40·44	·227	1·85	4·56	30	12·94	·137	1·12	8·62	31	6·05	·08	·66	4·01
	Girls	29	39·68	·229	1·83	4·60	28	12·84	·128	1·01	7·85	30	5·51	·08	·68	5·58
	Both	59	40·07	·161	1·84	4·58	58	12·90	·090	1·02	7·87	61	5·78	·06	·71	5·62
4	Boys	30	41·89	·262	2·13	5·08	30	13·58	·116	·94	6·93	30	6·73	·08	·66	4·47
	Girls	29	41·10	·192	1·53	3·73	29	13·48	·144	1·15	8·54	29	6·24	·09	·72	5·33
	Both	59	41·50	·169	1·92	4·63	59	13·53	·093	1·06	7·81	59	6·49	·07	·74	5·24
5	Boys	29	43·44	·238	1·90	4·38	29	14·18	·125	1·00	7·02	31	7·40	·10	·77	4·71
	Girls	27	42·67	·261	2·01	4·71	26	14·07	·154	1·17	8·28	27	6·92	·11	·84	5·54
	Both	56	43·07	·176	1·98	4·60	55	14·13	·097	1·07	7·56	58	7·22	·08	·85	5·10
6	Boys	30	44·46	·205	1·67	3·76	29	14·84	·137	1·09	7·39	31	8·12	·09	·76	4·23
	Girls	28	43·81	·321	2·52	5·75	28	14·29	·131	1·03	7·18	28	7·38	·115	·88	5·48
	Both	58	44·13	·189	2·14	4·85	57	14·55	·100	1·12	7·71	59	7·76	·08	·91	5·31
7	Boys	27	46·08	·282	2·17	4·71	27	15·15	·127	·98	6·47	27	8·70	·09	·72	3·77
	Girls	25	44·69	·289	2·12	4·75	24	14·65	·108	1·22	8·32	26	7·99	·13	·95	5·42
	Both	52	45·41	·170	1·82	4·00	51	14·92	·103	1·09	7·32	53	8·38	·08	·93	4·97
8	Boys	25	46·74	·258	1·91	4·09	25	15·48	·124	·92	5·93	25	9·30	·115	·83	4·08
	Girls	26	45·64	·263	2·01	4·41	25	15·21	·159	1·18	7·76	28	8·43	·13	·99	5·36
	Both	51	46·18	·192	2·03	4·40	50	15·34	·105	1·10	7·18	53	8·85	·09	1·02	5·25
9	Boys	27	47·34	·226	1·74	3·67	26	15·48	·103	·78	5·04	27	9·75	·10	·76	3·58
	Girls	29	46·23	·274	2·19	4·73	29	14·89	·107	1·33	8·96	29	8·77	·08	·94	3·35
	Both	56	46·77	·181	2·00	4·28	55	15·17	·101	1·14	7·51	56	9·23	·10	1·05	5·18
10	Boys	25	47·86	·250	1·92	4·02	25	15·78	·120	·89	5·65	28	10·28	·115	·88	3·89
	Girls	23	46·83	·306	2·18	4·66	22	15·08	·192	1·33	8·83	28	9·17	·14	1·04	5·19
	Both	48	47·36	·213	2·10	4·61	47	15·45	·117	1·10	7·08	56	9·72	·10	1·12	5·25
11	Boys	24	48·31	·275	2·00	4·14	24	15·98	·120	·87	5·45	27	10·63	·13	·99	4·29
	Girls	19	46·71	·311	2·02	4·31	19	15·16	·180	1·16	7·66	24	9·53	·15	1·07	5·12
	Both	43	47·60	·225	2·10	4·60	43	15·62	·110	1·07	6·86	51	10·11	·11	1·17	5·29
12	Boys	27	48·66	·205	2·27	4·66	26	16·24	·128	·97	5·95	28	10·96	·14	1·01	4·34
	Girls	26	46·68	·221	1·67	3·58	26	15·44	·130	1·05	6·81	28	9·80	·15	1·13	5·25
	Both	53	47·69	·205	2·21	4·64	52	15·84	·102	1·09	6·86	56	10·38	·114	1·22	5·36
15	Boys	27	49·85	·308	2·37	4·76	27	16·79	·130	1·05	6·25	27	11·77	·15	1·13	4·35
	Girls	25	48·06	·280	2·08	4·33	25	15·92	·147	1·00	6·83	27	10·39	·16	1·10	5·25
	Both	52	48·99	·225	2·40	4·90	52	16·37	·109	1·16	7·11	54	11·08	·125	1·34	5·34
18	Boys	22	50·60	·358	2·49	4·92	22	17·37	·138	·96	5·52	26	12·38	·17	1·24	4·57
	Girls	22	48·80	·269	1·87	3·84	22	16·18	·143	·99	6·12	25	11·14	·17	1·24	5·25
	Both	44	49·70	·242	2·38	4·79	44	16·78	·112	1·11	6·59	51	11·77	·14	1·34	5·37
24	Boys	23	51·31	·204	2·09	4·06	22	17·45	·152	1·05	6·04	25	13·53	·22	1·91	5·44
	Girls	18	49·41	·333	2·10	4·24	18	16·85	·155	·97	5·77	22	12·22	·17	1·16	4·32
	Both	41	50·48	·237	2·25	4·45	40	17·19	·104	·98	5·68	47	12·92	·15	1·56	5·49
30	Boys	25	51·32	·188	1·40	2·72	25	17·87	·102	·76	4·25	25	14·75	·23	1·65	5·14
	Girls	21	49·53	·250	1·70	3·43	21	17·35	·146	·99	5·72	21	13·55	·19	1·35	4·54
	Both	46	50·50	·233	2·34	4·64	46	17·63	·093	·94	5·31	46	14·21	·16	1·63	5·27
36	Boys	25	52·09	·310	2·30	4·42	25	18·30	·110	·89	4·83	25	15·78	·24	1·80	5·22
	Girls	24	50·15	·288	2·00	4·17	24	17·86	·130	·94	5·28	24	14·52	·18	1·35	4·24
	Both	49	51·14	·232	2·40	4·70	49	18·11	·098	1·00	5·50	49	15·17	·16	1·72	5·17



Curves of Growth in Weight.



Curves of Growth in Length.



Curves of Growth in Seven Body Dimensions (Sexes Combined).

TABLE V.

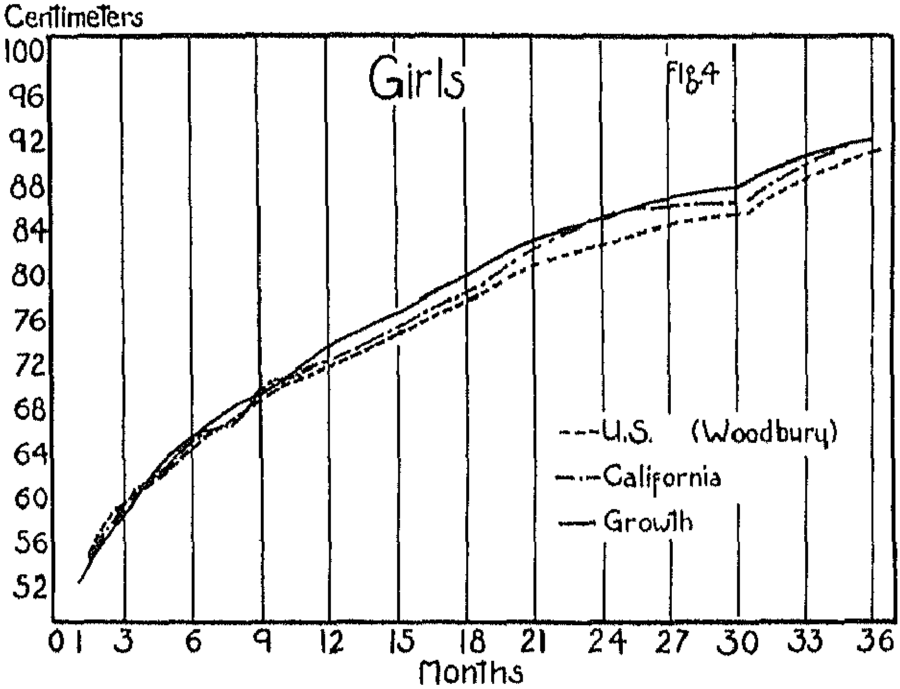
Birth Measurements.

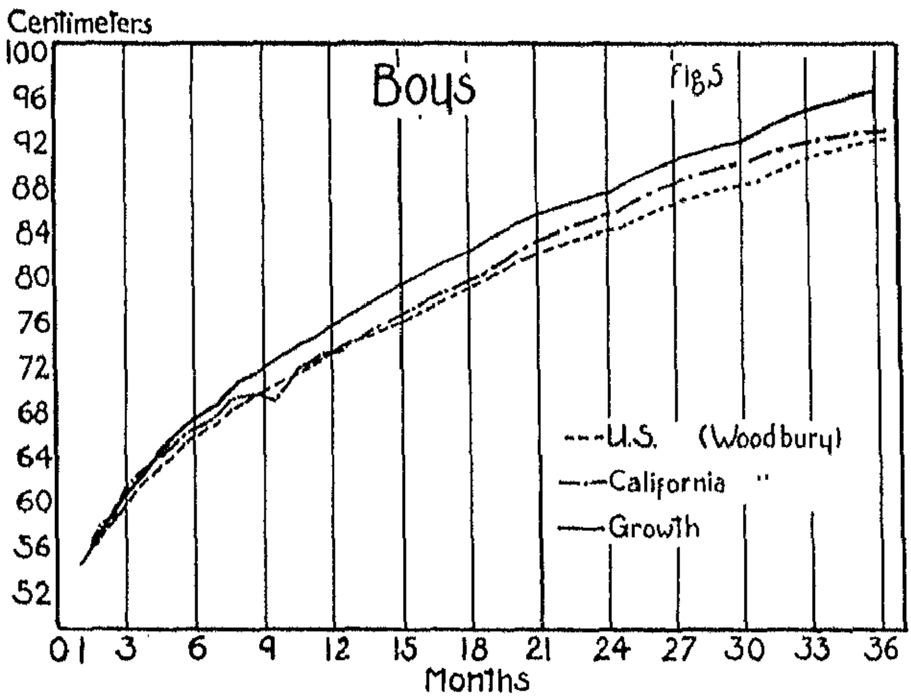
Measure	Sex	No.	Mean	Median	S.D.	Range
Head Circumference	Boys	28	35·52	35·40	1·4499	33·3 to 39·0
	Girls	25	34·52	34·50	1·3778	31·3 to 37·0
	Both	53	35·05	35·00	1·8891	31·3 to 39·0
Chest Circumference	Boys	28	33·50	33·45	1·3919	31·0 to 36·3
	Girls	24	33·51	33·75	2·1703	29·5 to 38·5
	Both	52	33·50	33·65	1·8365	29·5 to 38·5
Transverse Head Diameter	Boys	28	9·53	9·50	·6559	8·6 to 10·4
	Girls	25	9·23	9·20	·5700	8·2 to 10·4
	Both	53	9·39	9·40	·4884	8·2 to 10·4
Anterior-Posterior Head Diameter	Boys	28	12·28	12·30	·5283	11·3 to 13·4
	Girls	25	11·90	12·00	·6242	10·4 to 13·3
	Both	53	12·10	12·10	·6131	10·4 to 13·4
Weight (lb.)	Boys	31	8·17	8·19	·8971	6·75 to 9·63
	Girls	30	7·37	7·19	1·2828	5·38 to 10·50
	Both	61	7·78	7·88	1·1021	5·38 to 10·50
Weight (kg.)	Boys	31	3·70	3·71	·32	3·06 to 4·36
	Girls	30	3·35	3·27	·58	2·44 to 4·79
	Both	61	3·52	3·58	·50	2·44 to 4·79

A comparison with the measures of the Iowa Child Welfare Research Station (26) is especially interesting, since the same measurements were made, and at the same ages, in the two studies. Also, the number of cases for each age is approximately the same for both groups. The Iowa data contain many remeasures of the same children, but since the samples were not limited to children on whom seriatim measures were made, the samples for the different ages are essentially cross-sectional. The growth curves for all measures are closely similar for the two groups, except that for reasons already considered, the Growth Study curves are comparatively very smooth.

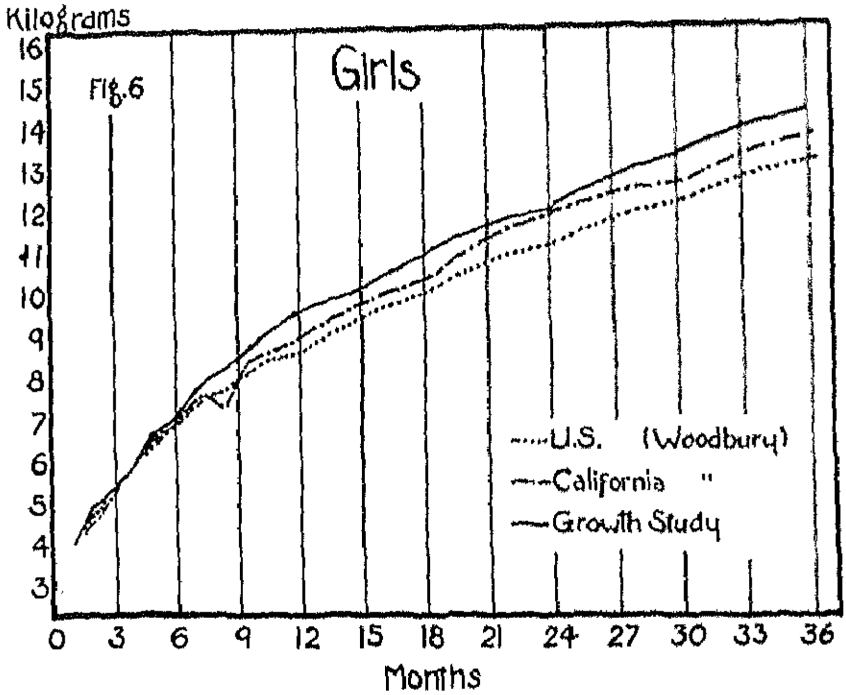
There is very little difference in the size of these two groups, except that the California babies usually measure slightly larger after six months. The greatest difference is in weight.

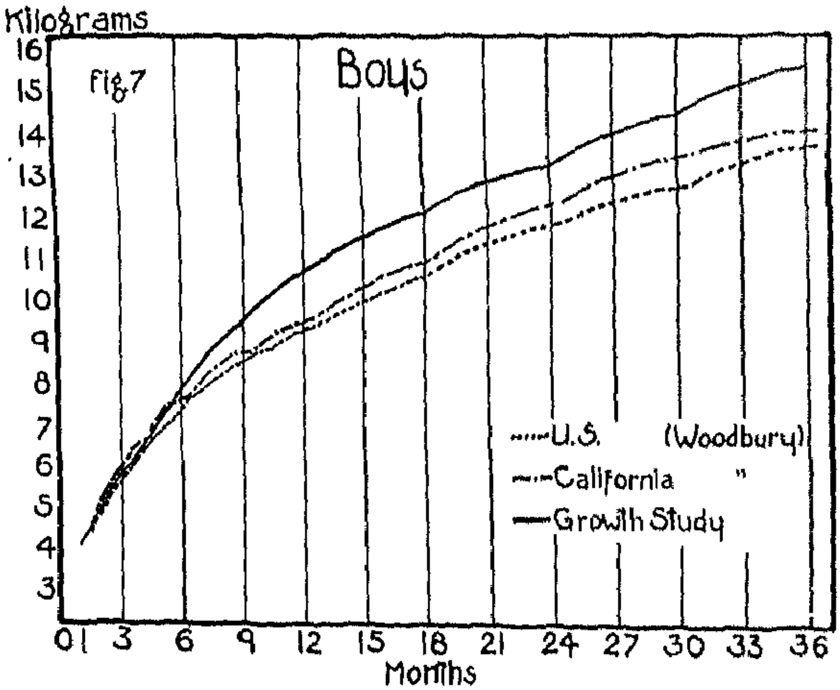
The mean weight and height of approximately 200 Berkeley infants (a representative sample) for months 12, 21 and 24 are almost identical with those of the smaller group of Berkeley infants we have been considering. The mean weights





Figs. 4 and 5. Comparative Curves of Growth in Length.





Figs. 6 and 7. Comparative Curves of Growth in Weight.

of the larger group of infants are 10·52, 12·30 and 13·11 kg. for 12, 21 and 24 months, respectively. For these same ages their mean lengths are 75·18, 84·84 and 87·88 cm. Whatever selective factors are operating to make the Growth Study infants taller and heavier than infants in other parts of the country, are also effective for the population of Berkeley as a whole.

When the growth trends of the various dimensions are made more directly comparable with each other by converting each measure into the percentage of itself at one month (Figs. 8—11), the measure which far outstrips all others is weight. This, of course, is to be expected, as it is a volumetric measure.

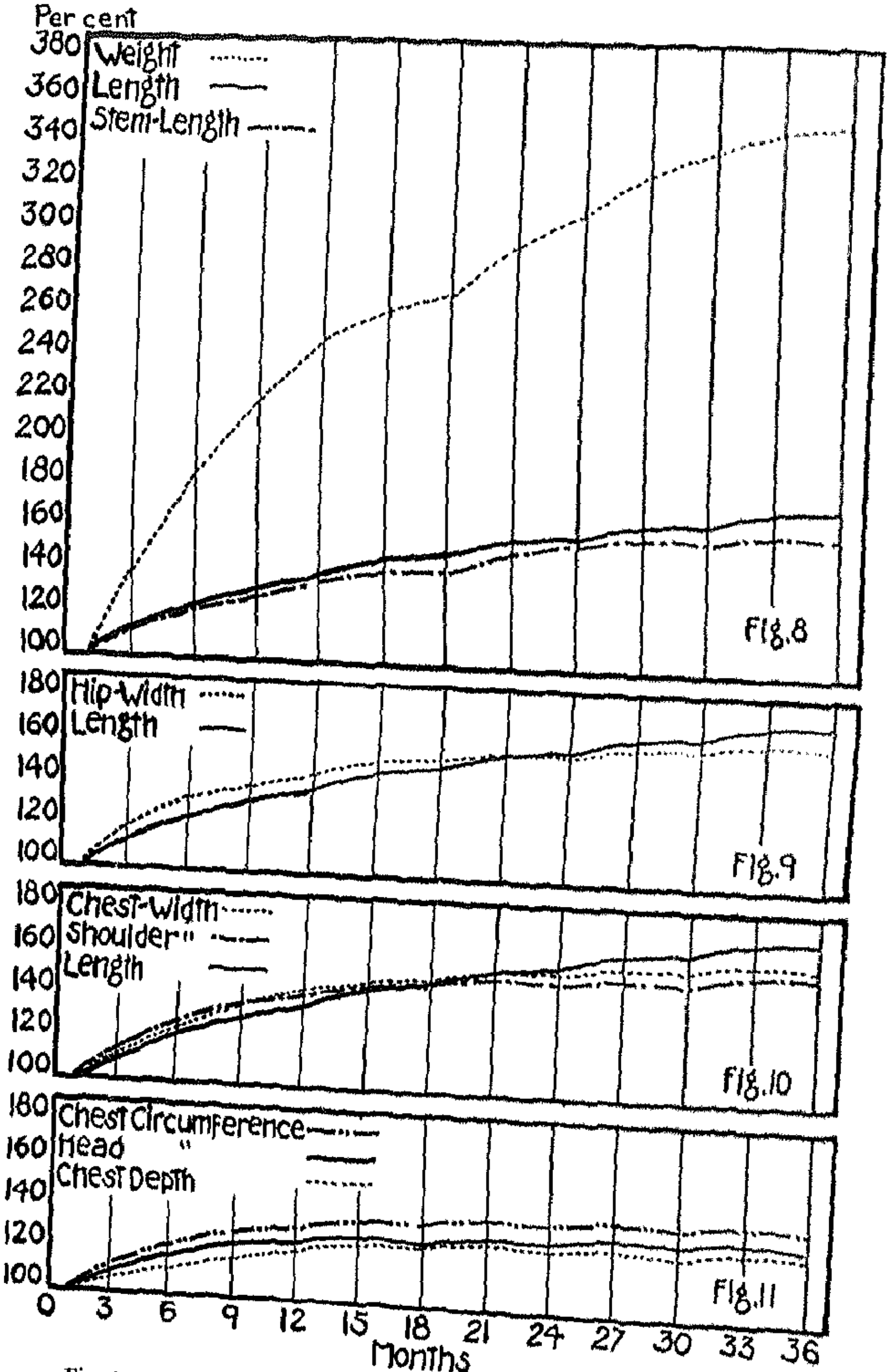
An inspection of the proportional growth curves of the linear measures reveals very interesting differences in the rates of increment of the different dimensions. Throughout the first 18 months (when growth in weight is most rapid) the measures which show the greatest increase are width measures—hips, shoulders, and chest—and it is not until the 24th-month records that we find the increments for height greater than those for the three widths. As for the height measures, the first year shows practically equal proportionate growth for the stem length and for the total length. It is only after the 12th month that the legs begin to grow rapidly and cause a relatively greater increase in the total height curve. The dimension which shows the smallest percentage increment throughout the entire 36 months, is chest depth. Head circumference shows the second smallest increment.

In general, though not consistently throughout, the cephalo-caudal direction of growth is followed in the percentage growth curves. The head size is relatively more mature at birth than are other measures and its postnatal growth is small; in the early months growth in shoulder width is relatively more rapid than chest width, and this position is reversed later; stem and leg length grow at about the same rate at first, but after the first 18 months, leg length is the most rapidly increasing dimension measured.

The changing body proportions, then, are for the first year due mainly to relatively greater growth in transverse widths, and relatively small growth in head size; while for the second and third years, the decrease in growth rate of all dimensions except total height results in changed body proportions due to rapidly increasing leg length. We find, especially during the first year, not only changes in the relative length of body parts, but also in the relative widths; and indices of body build should be formulated with this fact in mind.

C. VARIABILITY.

When we consider the variation from the mean curves of growth, the coefficients of variation (Table IV, pp. 37—39) show no age changes for any of the measures except two. Chest width and hip width decrease slightly but consistently in variation, during the three-year period.



Figs. 8—11. Percentages of First Month's Measures achieved at Later Ages.

There are no significant sex differences in variation. Although the coefficients of the girls are slightly larger in ninety-three comparisons, while the boys are larger in only fifty-seven comparisons, none of the differences is significant. The only measure in which the boys V's are greater for most ages is chest width, and since this one measure shows a contrary tendency to all the others in this respect, it may possibly be indicative of a real sex difference. There is also a slight suggestion that the girls are possibly becoming more homogeneous in length and in chest depth, toward the end of this three-year period.

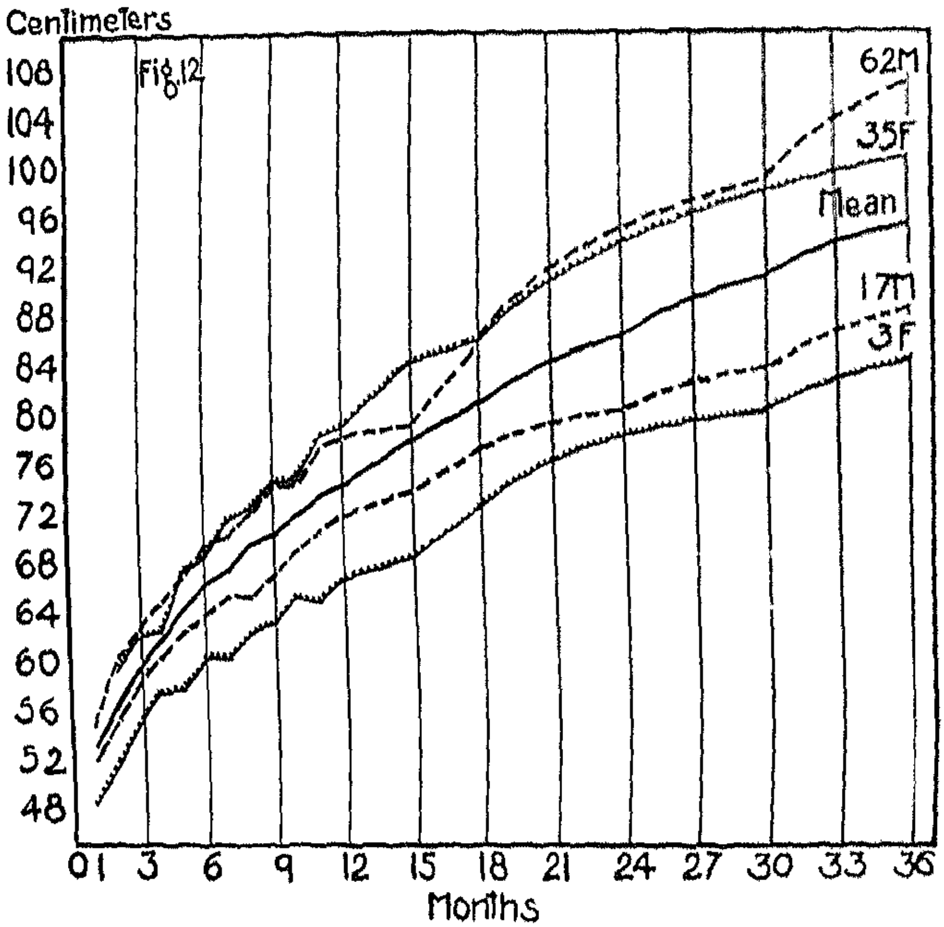
Montague and Hollingworth (32) found no tendency for sex differences in variability for seven measurements made on 2000 newborn infants. The measurements of the Iowa Child Welfare Research Station (26, 27) from one month through six years show no consistent sex differences in variation, with the possible exceptions of weight and hip width in the later ages. Both these studies do show, however, consistent tendencies for the boys to be larger. It seems unlikely, then, that any slight differences in the coefficient of variation found in the present group represent sex differences. The group is, however, typical of children, in general, so far as sex differences in size are concerned.

There are only slight tendencies for the standard deviations to increase with age, with the exception of weight, the standard deviation of which increases from ·54 kg. at month 1 to 1·72 kg. at month 36.

D. CONSISTENCY OF GROWTH.

When we select, from the 36-month measures, the tallest boy and girl and the shortest boy and girl, and plot their individual curves of growth in length (Fig. 12), we find that the tall children have been consistently tall from the first month, and the short children consistently short. The extent to which this is true for the entire group in length and in other dimensions may be shown by correlation coefficients between different ages for the same dimension. Tables VI—XV give the correlations between the measures of these children at different ages. Head circumference, length and stem length are most consistent: the correlations between the 1-month and 36-month measures are, respectively, ·72, ·68 and ·59. Weight comes fourth, with correlations which are considerably lower, the r for the same interval being only ·40. There is, however, a marked increase in the consistency of this measure after the first year. The width measures of hip and shoulder diameters and chest circumference, equally low in consistency, all show the same phenomenon as does weight—a decidedly greater consistency after the first year. Chest width and depth, relatively unreliable measures, show low consistency and only small increase in consistency with growth.

The width measures become more consistent after their period of relatively rapid growth has stopped, and weight, which is largely influenced by the width measures, exhibits the same trend.



Individual Growth Curves of Tall and Short Children compared to the Mean.

Self-Correlations showing Consistency of Growth.

TABLE VI.

In Length

Month	3	6	12	18	24	30	36
1	·87	·77	—	—	·68	—	·68
3	—	·84	—	—	·71	—	·67
6	—	—	·91	—	·83	—	·70
12	—	—	—	·91	·93	—	·88
18	—	—	—	—	·92	—	·89
24	—	—	—	—	—	·95	·92
30	—	—	—	—	—	—	·97

TABLE VII.

In Weight.

Month	3	6	12	18	24	30	36
1	·82	·54	—	—	·48	—	·40
3	—	·80	—	—	·52	—	·48
6	—	—	·72	—	·60	—	·61
12	—	—	—	·92	·91	—	·80
18	—	—	—	—	·93	—	·88
24	—	—	—	—	—	·94	·95
30	—	—	—	—	—	—	·96

TABLE VIII.

In Head Circumference.

Month	6	12	24	36
1	·70	·71	·74	·72
6	—	·80	·80	·81
12	—	—	·91	·93
24	—	—	—	·92

TABLE IX.

In Stem Length.

Month	6	12	24	36
1	·55	·53	·45	·50
6	—	·70	·71	·68
12	—	—	·86	·74
24	—	—	—	·75

TABLE X.

In Chest Circumference.

Month	6	12	24	36
1	·25	·20	·26	·29
6	—	·44	·41	·43
12	—	—	·73	·69
24	—	—	—	·82

TABLE XI.

In Chest Width.

Month	6	12	24	36
1	·14	·19	−·01	·23
6	—	·53	·37	·20
12	—	—	·54	·51
24	—	—	—	·71

TABLE XII.

In Chest Depth.

Month	6	12	24	36
1	·57	·40	·25	·12
6	—	·53	·51	·42
12	—	—	·45	·50
24	—	—	—	·63

TABLE XIII.

In Shoulder Width.

Month	6	12	24	36
1	·10	·33	·13	·21
6	—	·46	·33	·35
12	—	—	·68	·65
24	—	—	—	·71

TABLE XIV.

In Hip Width.

Month	6	12	24	36
1	·45	·37	·32	·23
6	—	·48	·60	·45
12	—	—	·80	·83
24	—	—	—	·83

TABLE XV.

Self-Correlations showing Consistency according to Sex.

Measure	Month								
	6 & 12			12 & 24			24 & 36		
	Boys	Girls	Total	Boys	Girls	Total	Boys	Girls	Total
Length	·89	·86	·91	·94	·88	·93	·93	·87	·92
Weight	·56	·78	·72	·89	·97	·91	·95	·92	·95
Hip Width	·33	·57	·48	·86	·70	·80	·84	·84	·83
$W./H.^3$	·53	·51	·44	·46	·68	·74	·70	·77	·73

The above correlations have all been computed for the total group, regardless of sex. The division into sex groups seemed unnecessary, since there were no significant differences in the growth tendencies of the two sexes. As a further check, however, consistency correlations were computed on the sexes separately for three measures, weight, length, and hip width, and the $W./L.^2$ index, for three age intervals (Table XV). These correlations give no evidence of sex differences in consistency.

E. Interrelationships among the Measures.

Inter-Correlations between Individual Dimensions.

The correlations between the measures have been computed for representative ages. Table XVI gives these correlations, ranked from highest to lowest, in the order of the mean of the r's between each pair. There is a tendency for the higher inter-correlations to be between weight and one of the more reliable width measures; and between the two length measures (which rank second and fourth in reliability). The three combinations which correlate most highly are: hip width and weight, with a mean r of ·84; stem length and total length, with a mean r of ·82; and chest circumference and weight, with a mean r of ·80. We find, also, fairly high relationships between stem length and weight, and total length and weight (·74). It is to be expected that weight should correlate highly with the various diameters and circumferences, since weight is a composite which includes them all, and since they are all positively correlated with each other.

TABLE XVI.

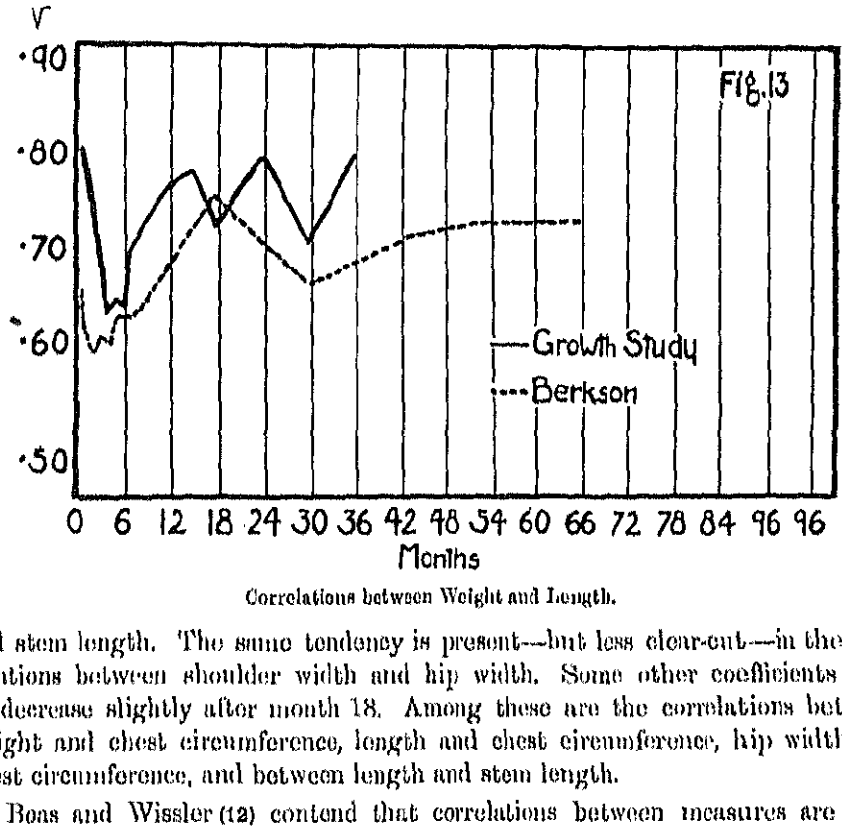
Inter-Correlations of Measures.

	Month									Mean
	1	3	6	9	12	15	18	24	30	
H.W. × W.	·898	·807	·810	·701	·821	·852	·875	·812	·854	·836
L. × S.L.	·742	·784	·810	·844	·879	·882	·797	·855	·791	·820
C.C. × W.	·740	·849	·791	·844	·792	·851	·857	·698	·770	·795
S.L. × W.	·720	·786	·685	·736	·782	·804	·797	·739	·732	·743
L. × W.*	·818	·708	·651	·734	·777	·795	·731	·808	·813	·730
S.W. × W.	·671	·748	·718	·718	·679	·666	·653	·707	·667	·694
C.C. × H.W.	·704	·711	·625	·706	·641	·732	·817	·652	·634	·691
S.W. × C.C.	·479	·697	·712	·691	·628	·700	·613	·684	·617	·641
S.W. × H.W.	·624	·586	·467	·613	·469	·635	·697	·611	·647	·594
L. × S.W.	·587	·670	·642	·562	·536	·570	·571	·658	·496	·588
L. × H.W.	·598	·489	·336	·491	·649	·655	·636	·666	·732	·584
H.W. × S.L.	·644	·579	·393	·402	·635	·716	·668	·584	·491	·580
C.C. × S.L.	·567	·627	·493	·697	·582	·678	·632	·562	·537	·579
L. × C.C.	·498	·546	·538	·648	·595	·634	·529	·669	·550	·572
S.W. × S.L.	·453	·587	·496	·496	·644	·628	·491	·513	·533	·527

* Additional correlations between length and weight have been computed. They are for months 2, 4, 5, 7, 8, 11 and 30 and are, in order, ·762, ·644, ·658, ·707, ·727, ·768, and ·719. These have been included in computing the mean of the length-weight correlations.

The r's of ·69 between chest circumference and hip width, and of ·64 between chest circumference and shoulder width, indicate a fair amount of symmetry in the transverse trunk measures, but by no means perfect symmetry. The length-width correlations are the lowest ones and fall in the last six places in the rank order table. These r's range from ·53 to ·59. One should expect, then, to find in this group tall slender infants and short chubby ones, as well as the more frequently occurring large, medium and small sizes of symmetrical proportions. In general, there is a clear tendency for all measures to be positively related. Large children are for the most part large in all their measures, and small children small in all their measures.

As for age trends in these inter-correlations, there is a tendency for some of the coefficients to decrease at and near six months. This is true of the correlations between weight and length, hip width and length, and between hip width



Correlations between Weight and Length.

and stem length. The same tendency is present—but less clear-cut—in the correlations between shoulder width and hip width. Some other coefficients tend to decrease slightly after month 18. Among these are the correlations between weight and chest circumference, length and chest circumference, hip width and chest circumference, and between length and stem length.

Boas and Wissler (12) contend that correlations between measures are high during periods of rapid growth, and conclude from this that when rapid growth occurs in one part of the body, it occurs also in other parts. Berkson (9) has computed the inter-correlations between height, weight and chest circumference for a large number of cases, and concludes that the alternating points of high and low correlations which he finds correspond with critical periods of growth. His data for the first six years are taken from Woodbury (41), and this part of his curve of length-weight correlations is given in Fig. 13 with the correlations obtained

for the present group. The Growth Study curve for length-weight correlations shows high r's at months 1 and 15, with a consistent drop from 1 to 4, and a rise from 7 to 15, followed by variations showing no directional trend between 15 and 36 months. These r's differ from the Woodbury data in being high at month 1, with a directional trend of decreasing r's from months 1 to 4*, and no significant trend toward decrement after month 15.

A number of other investigators have computed r's between body measures, and their findings may be compared with ours. Taylor's (36) correlations between weight and length of 240 newborn infants are ·79 for boys and ·81 for girls—very close to our one-month correlation of ·82.

The correlations between weight and length given by Gray and Ayres for year groups, starting at one year, fail, as ours do, to show a decreased r at three years. Instead there is a rise from one to three followed by a drop at five years.

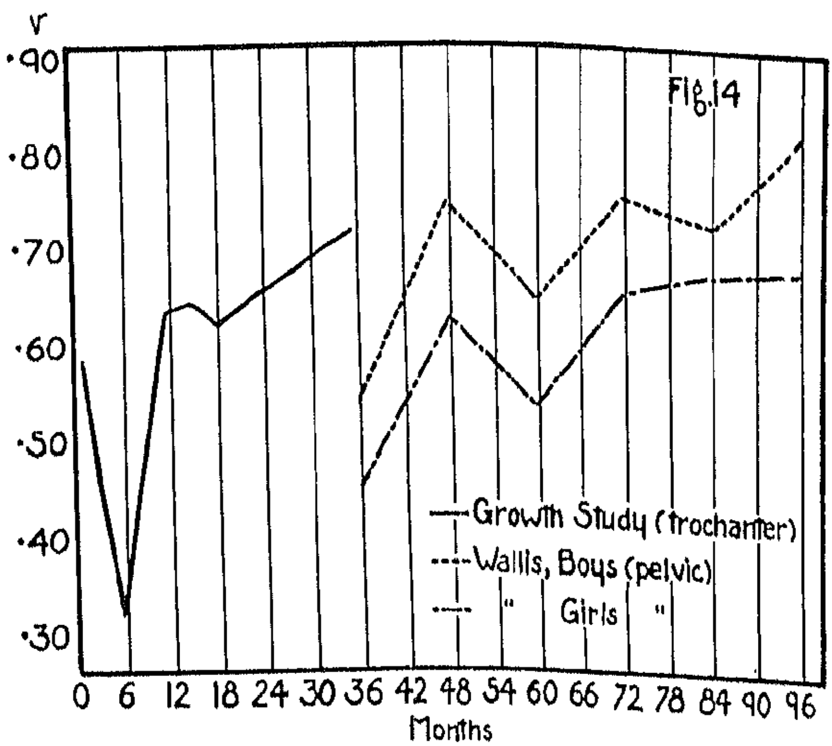
Baldwin (5) computed inter-correlations for about 120 children at each year from seven to seventeen years. He found that the correlations between different body measures were higher for the boys than for the girls, and that they were higher for both sexes in early adolescence, and lowest at 17 years. His length-weight correlations show trends very similar to those of Berkson. As has been already mentioned, Boas and Wissler (12) also find increased correlations during rapid pre-adolescent growth. Wallis (39) gives correlations between pelvic width and length of a group of New York boys from three to eight years. These are plotted in Fig. 14 with the length-hip width r's for the present group. Although the length-pelvic width r's of Wallis are smaller at three years than our length-trochanter width r's, they show the same directional trend of increasing correlation with age that is indicated in our sample after month 6.

The Woodbury measures, used by Berkson, though for large samples, were collected from many sources with the possibility of great differences in technique and reliability of the measures. In addition, it was necessary to compute the correlations from class groups in Woodbury's tables instead of from the original data. Since the measures of the Growth Study have been made by a single observer with a consistent method, and since the same children have been measured at the successive age levels, and since none of the other correlations in the studies cited show decreased r's at three years, this curve may well give the truer approximation of the changes in the body relations which occur during growth in the individual through the first three years.

So far, then, as the evidence for this short age period is concerned, any age differences in the inter-correlations of the body measures seem to be related to differences in the relative growth rates of the parts measured, and perhaps to individual differences in tendency to increased chubbiness during the first nine or

* There is some evidence for a similar trend in the Woodbury data, but in smoothing his curve Berkson does not accept this.

eleven months. Length-width correlations tend to drop from months 1 to 6 when growth in widths is more rapid and more variable among individuals, and then to rise again as the disparity in growth rates of the body parts involved diminishes. After eighteen months, the growth in chest circumference is very slow, and the correlations between it and the still rapidly growing measures—weight, length, hip width and stem length—tend to drop. For this sample, during the first three years, growth changes in correlation between the various body parts show more relation to the amount of disparity between rates of growth in these parts, than to successive periods of rapid and slow growth of the body as a whole. The situation may be



Correlations between Length and Hip Width.

very different at later ages when body proportions are changing less radically. But it is also possible that the high adolescent correlations between body measures are due to the *wider variation* of size at this time when some children start rapid growth at a younger age than do others. Boas (10) has found that, although there is greater variability in the size of adolescents than in older or younger children, "the maximum of variability of stature during adolescence disappears in groups having the same moments of maximum rate of growth." If the correlations were computed between measures on children of the same age of maximum rate of growth, rather than grouping together all children of the same age, the high adolescent r's would probably also disappear.

F. Body-build.

The recognised desirability of a quantitative measure of body-build which should reliably and adequately represent the empirically observed variations in human structure has been productive of much debate, and of a considerable amount of research as well, from the time of Quetelet up to the present. The evidence recently brought forward by Kretschmer, pointing to a close relationship between bodily proportions and temperamental predispositions, lends added significance to the the search for an objective index of build. Among the indices which have been proposed are the ratios of weight to height, weight to the square of height (Quetelet, Davenport), weight to the two and one-half power of height (Quetelet), weight to the cube of height (Rohrer and Bardeen), chest circumference to height (Davenport), hip width to height (Pryor and Lucas, Gray), stem length to height (Gray, Hejinian and Hatt), and many others. Objections have been raised to the use of an index which includes weight as one of the factors (Wertheimer and Hesketh (40)) on the ground that weight fluctuations may be of serious extent, even over a relatively small span of time, as a result of a variety of exogenous influences. This objection, however, while it may be of moment with subjects such as those used by Wertheimer and Hesketh (adult psychotics), may nevertheless be relatively inconsequential with subjects who are free from any serious ailment, either physical or behaviouristic. The self-correlations in Table I (p. 34 above) indicate that this objection is not important in the present sample. With infants and with children, at least, indices involving weight as one factor should not be dismissed *on a priori* grounds, because weight is one measure in the recording of which the personal equation plays a relatively minor rôle. We shall see that the superior reliability of measurements of body weight, with infant and child subjects such as those used in the present investigation, gives indices which are based in part upon weight measurements a definite advantage over various other proposed indices which incorporate measurements difficult to obtain, at these early ages, with a sufficiently high degree of accuracy.

Several different indices have been computed from the measures taken on the Growth Study children, and their means are given in Table XVII (p. 56) and Figs. 15—22. These curves exhibit interesting age trends in body proportions. As would be expected from the proportionate growth curves in Fig. 9, the relative hip width (Fig. 15) increases through the sixth month, but declines again, rapidly, after the seventh month. If we eliminate the leg measures by comparing hip width to stem length (Fig. 16) the initial rapid increase of hip width is still present, but after a short decrease between the seventh and ninth months, the relationship between the two measures remains constant. The ratio of stem length to total length, after a slow decrease through the ninth month, falls off very rapidly after this age. The same may be said of the ratios of chest circumference to length, and of chest circumference to stem length.

The length-weight indices vary greatly, according to the power of length used in computing the index. The simple W./L. ratio shows a growth curve of increasing

weight in relation to length, rapid at first, but slowing up in the second and third year. W./L.^{3}—a ratio which has been recommended on the ground that weight is a volumetric measure—shows a decreasing index which is rapid after the ninth month. W./L.^{2}—an index which has often been used—shows relatively increasing weight through the eleventh month, with a more gradual decrease after this age*. All three of these indices yield results on the present sample which are closely similar to age curves of the mean indices of other groups presented elsewhere. Davenport (14) gives, in tabular form, age changes in mean indices for these three† which are very similar to ours, and our W./L.^{2} means are almost identical with those for the first three years derived from Quetelet's data (34).

[* At my suggestion the authors worked out the value of I_d=W./(C.C.×H.W.×L.) as an index. It will be seen that this is an attempt to obtain weight divided by a product of three linear quantities, i.e. chest circumference, hip width and length. I did not suggest chest depth, as the authors considered it a not very reliable measure. Thus roughly we have an index of weight per volume of the body, or we have an index which seemed to me the nearest approach to what I should call "chubbiness"—a term not defined by the authors. Dr Bayley writes to me: "Chubbiness" as we used it in our ratings of photographs meant the tendency to be wide, relative to length, rather than to have excess fatty tissue, though this latter factor had a strong influence on the rating." (See below, p. 71.) The authors obtained the following results:

Index W./(C.C.×H.W.×L.)

	Age in months				
	1	6	12	18	36
Index	197	181	181	172	170

When these values are plotted it will be seen that the index decreased slowly with age, but in an almost *linear manner*. *I_d + ·0431 × (age in months) is a function which on the average is nearly constant,* = 189·46, for the first thirty-six months of life. It thus appears to be more satisfactory than several of those graphed in Figs. 15—22, the most nearly linear of which is probably W./l^3, an index which we might anticipate would have a fairly high correlation with the present index. I_d was found, however, to have *no* correlation with the judges' ratings of "chubbiness," nor with W./l^2, the index on which the writers lay stress. Now this latter index may be read as W./l^3×l, and this seems to indicate that a volumetric index of chubbiness should have no relation to length or absolute size of the infant, which I take it should be true for a real definition of "chubbiness." That this "index of chubbiness" has no relation to the judges' ratings of "chubbiness," I should, I am afraid, be inclined to account for by the fact that the judges have unconsciously paid considerable attention to size (l) in their estimate of "chubbiness," which accounts for the correlation between their ratings and the index W./l^3×l=W./l^2. Anyhow, if chubbiness equals plumpness, I think it should be practically independent of size, and that a reasonable scientific definition of it would be weight per unit of volume. Short of actually measuring the volume of the body by immersion—practically impossible with infants—we have to find rough measures of the volume of the body, or take a product of three linear dimensions which may reasonably be supposed proportional to it. I may have a different opinion from the authors of what is meant by "chubbiness," *but I think at any rate we need a scientific definition of the term. I personally am* inclined to identify it with average bodily density, and thus to free it from any *à priori* relation to size; we may afterwards show that, as is the case with the present index, density appears to decrease *physiologically with age*. For children of the same age, I should anticipate little relation of density to size. If there be, we may be running up against some other very important physiological fact. ED.]

† These tables are based on tables given by Bardeen (7), using the data of Schmid-Mounard and of Quetelet.

TABLE XVII.

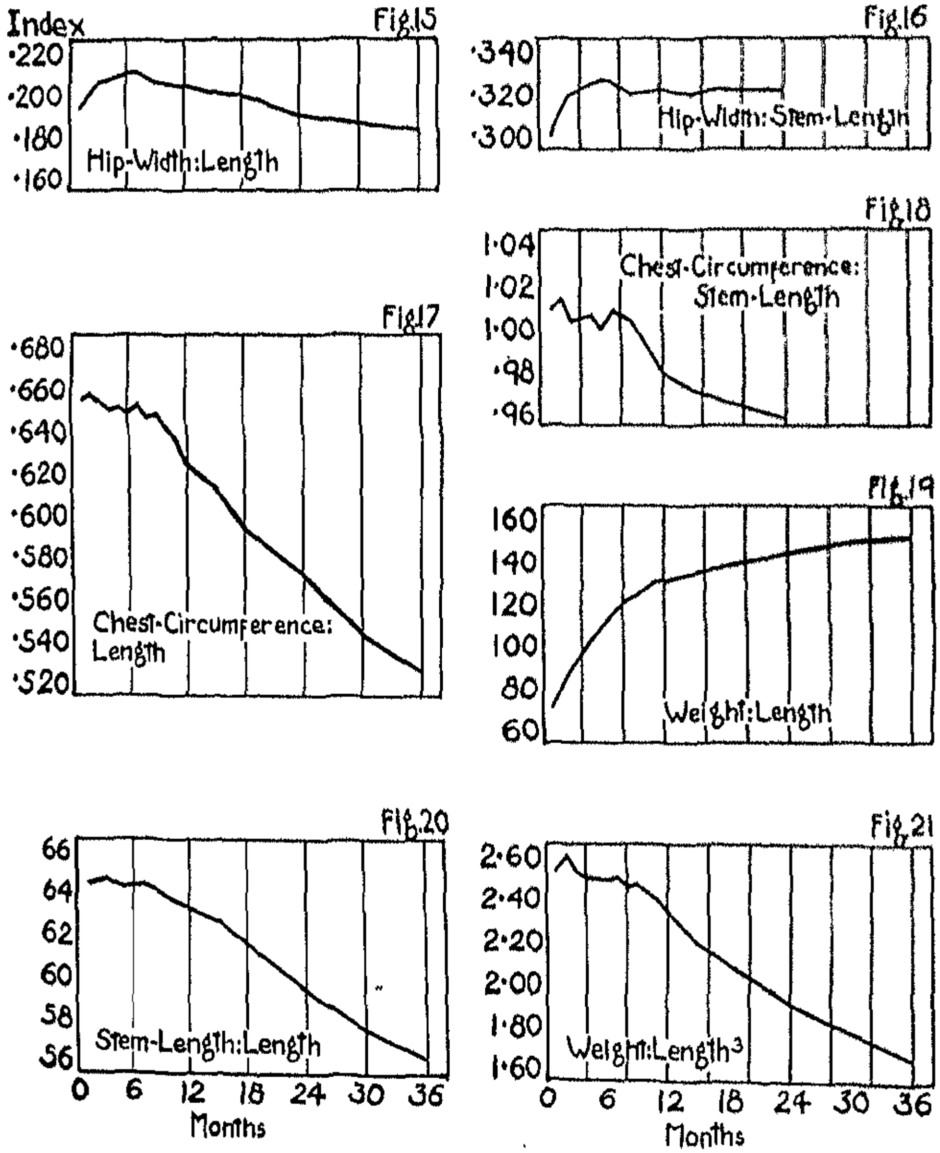
Table of Means for Boys, Girls, and Total Group of Eight Body-build Indices from 1 through 36 months.

Month		100 Weight/Length		Weight/Length²		Weight/Length³		100 Stem Length/Length		Chest Circumference/Length		Chest Circumference/Stem Length		Hip Width/Length		Hip Width/Stem Length	
		No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean	No.	Mean
1	Boys	24	79·06	24	1·42	24	2·57	24	64·94	24	·658	24	1·014	23	·199	23	·307
	Girls	26	74·26	26	1·39	26	2·60	26	65·32	26	·664	26	1·017	26	·199	26	·304
	Both	50	76·56	50	1·41	50	2·59	50	65·14	50	·661	50	1·015	49	·199	49	·306
2	Boys	29	89·14	29	1·51	29	2·55	—	—	28	·660	31	1·016	—	—	—	—
	Girls	25	86·57	24	1·56	25	2·80	—	—	23	·668	27	1·022	—	—	—	—
	Both	54	87·95	53	1·53	54	2·66	—	—	51	·663	58	1·019	—	—	—	—
3	Boys	31	97·99	31	1·59	31	2·57	31	65·32	30	·655	30	1·004	30	·210	30	·321
	Girls	30	92·01	30	1·54	30	2·60	29	65·36	28	·665	29	1·015	28	·215	27	·329
	Both	61	95·05	61	1·56	61	2·59	60	65·34	58	·660	59	1·009	58	·212	57	·325
4	Boys	28	105·06	28	1·64	28	2·56	—	—	25	·653	—	—	—	—	—	—
	Girls	25	99·75	25	1·59	25	2·55	—	—	23	·661	—	—	—	—	—	—
	Both	53	102·56	53	1·62	53	2·56	—	—	48	·657	—	—	—	—	—	—
5	Boys	30	112·60	30	1·69	30	2·55	30	65·18	28	·653	28	1·001	28	·213	28	·326
	Girls	27	106·92	27	1·65	27	2·56	27	64·74	27	·661	27	1·021	26	·217	26	·336
	Both	57	109·91	57	1·67	57	2·55	57	64·97	55	·657	55	1·011	54	·215	54	[illegible]
6	Boys	28	115·52	28	1·75	28	2·55	—	—	27	·647	30	·996	29	·217	29	·333
	Girls	24	111·53	24	1·68	24	2·54	—	—	22	·665	28	1·015	28	·215	28	·331
	Both	52	115·31	52	1·72	52	2·54	—	—	49	·655	58	1·005	57	·216	57	·332
7	Boys	26	124·45	26	1·78	26	2·56	26	64·95	26	·659	26	1·014	26	·217	26	·334
	Girls	25	117·58	25	1·73	25	2·54	24	65·08	24	·659	24	1·013	23	·215	23	[illegible]
	Both	51	121·[illegible]	51	1·76	51	2·55	50	65·01	50	·659	50	1·014	49	·216	49	·332

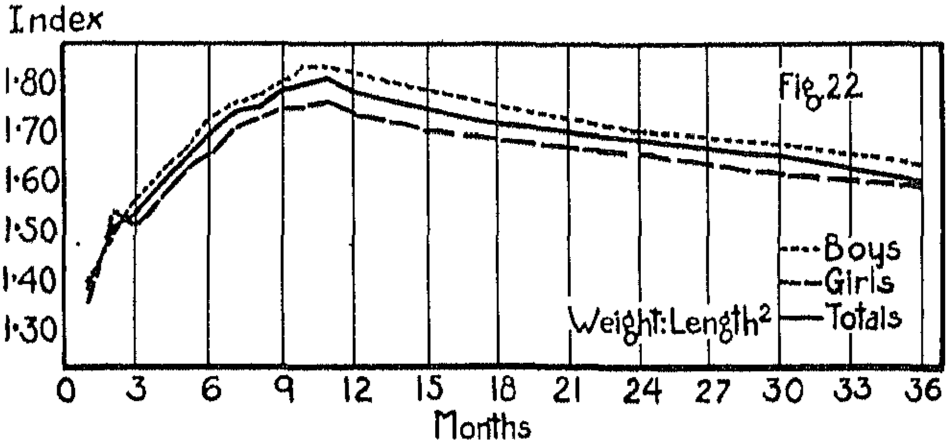
8	Boys	24	129·16	24	1·80	24	2·50	—	—	23	·649	—	—	—	—	—	—
	Girls	24	122·38	24	1·75	24	2·56	—	—	22	·657	—	—	—	—	—	—
	Both	48	125·77	48	1·77	48	2·51	—	—	45	·653	—	—	—	—	—	—
9	Boys	27	133·57	27	1·83	27	2·52	27	64·98	27	·650	27	1·000	26	·212	26	·327
	Girls	29	124·41	29	1·77	29	2·51	29	64·59	29	·657	29	1·018	29	·212	29	·328
	Both	56	128·83	56	1·80	56	2·52	56	64·78	56	·654	56	1·009	55	·212	55	·326
10	Boys	28	138·18	28	1·86	28	2·51	—	—	24	·644	—	—	—	—	—	—
	Girls	25	126·97	25	1·77	25	2·46	—	—	22	·653	—	—	—	—	—	—
	Both	53	132·89	53	1·82	53	2·49	—	—	46	·648	—	—	—	—	—	—
11	Boys	27	140·58	27	1·86	27	2·47	—	—	22	·642	—	—	—	—	—	—
	Girls	21	130·79	21	1·78	21	2·44	—	—	18	·639	—	—	—	—	—	—
	Both	48	136·30	48	1·83	48	2·45	—	—	40	·641	—	—	—	—	—	—
12	Boys	28	142·78	28	1·85	28	2·41	28	63·93	27	·633	27	·988	26	·212	26	·326
	Girls	28	131·18	28	1·76	28	2·37	28	64·00	26	·628	26	·981	26	·208	26	·328
	Both	56	136·98	56	1·81	56	2·39	56	63·97	53	·631	53	·985	52	·210	52	·327
15	Boys	27	146·12	27	1·81	27	2·25	27	63·62	27	·619	27	·973	27	·208	27	·328
	Girls	25	134·96	25	1·73	25	2·23	25	63·11	25	·619	25	·982	25	·205	25	·325
	Both	52	140·75	52	1·77	52	2·24	52	63·37	52	·619	52	·977	52	·207	52	·326
18	Boys	23	148·51	23	1·78	23	2·15	22	62·79	22	·608	21	·967	22	·200	21	·332
	Girls	24	138·11	24	1·71	24	2·13	24	61·70	23	·602	22	·975	22	·201	22	·324
	Both	47	143·20	47	1·74	47	2·14	46	62·22	45	·605	43	·971	44	·205	43	·328
24	Boys	23	153·80	23	1·73	23	1·96	24	60·51	23	·581	23	·962	22	·197	22	·325
	Girls	19	145·05	19	1·68	19	1·95	19	59·47	18	·574	18	·965	18	·196	18	·329
	Both	42	149·84	42	1·71	42	1·96	43	60·05	41	·578	41	·963	40	·196	40	·327
30	Boys	25	158·27	25	1·70	25	1·83	25	58·27	25	·552	—	—	25	·192	—	—
	Girls	21	149·13	21	1·64	21	1·81	21	58·06	21	·546	—	—	21	·191	—	—
	Both	46	154·10	46	1·67	46	1·82	46	58·18	46	·550	—	—	46	·192	—	—
36	Boys	25	161·75	25	1·66	25	1·71	25	57·21	25	·536	—	—	25	·189	—	—
	Girls	24	152·68	24	1·61	24	1·69	24	56·29	24	·528	—	—	24	·188	—	—
	Both	49	157·31	49	1·63	49	1·70	49	56·76	49	·532	—	—	49	·188	—	—

Most of these indices show rapid growth changes in body proportions, varying with the rates of growth of the parts compared. They are in accordance with our other findings, that early postnatal growth is most rapid in the transverse body dimensions, while after the first year leg length (and presumably arm length, which was not measured in this study) increases far more rapidly than any other body segment.

Any sex differences to be found in these body proportions are, again, very small, and with a sample of the size of the present study, certainly not significant. How-



Figs. 15—21. Curves of Growth in Seven Body-build Indices.



Curves of Growth in the Weight : Length² Index.

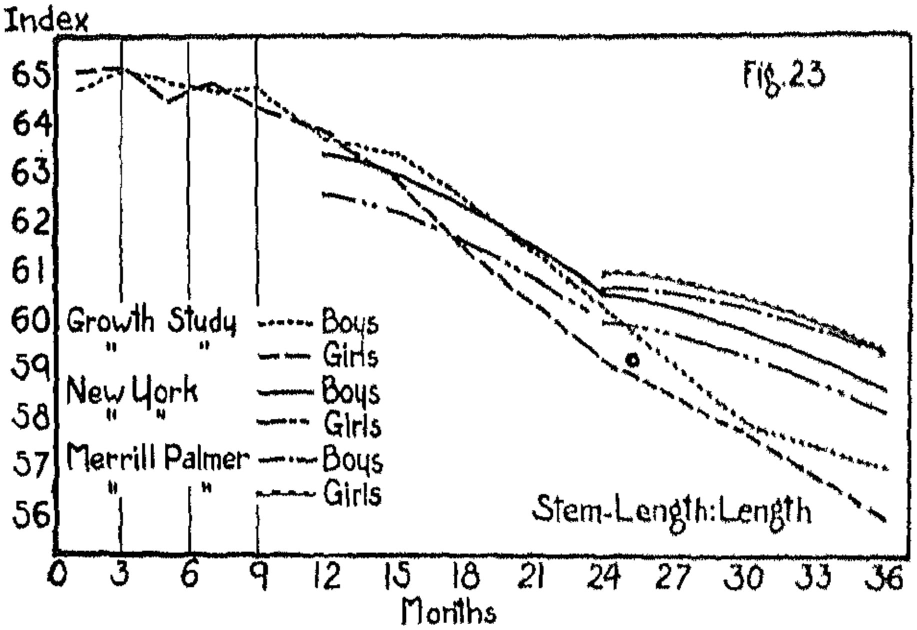
ever, so far as this group goes, during the first six months there is a tendency for the girls to have wider hips in proportion to total length and stem length (the latter is less clear); while after six months the situation is reversed, and the boys have relatively larger hips. The boys usually have a relatively longer stem length, when compared to stature, and after month 15 they have a greater chest circumference. When chest circumference is compared to stem length, however, the circumferences of the girls are larger. All of the weight-length indices show that, for the most part, the boys are heavier relative to their length.

Wallis (39) has computed recumbent stem length/length indices for New York private school children aged 1 to 8 years, and Merrill-Palmer school children, aged 2 to 5 years. Her indices, though in general their trends are similar to ours, decrease with age much less abruptly at the ages in which the Growth Study group was measured in the upright position (see Fig. 23, p. 60). The New York group shows the same sex difference as ours (higher indices for the boys) with the exception of years 7 and 8.

Davenport (15), following an extensive investigation of the statistical behaviour of the several indices cited above which are based upon weight and some power of height, concluded that either W./H.² or W./H.$^{2\frac{1}{2}}$ was to be preferred as an index of body build to either W./H. or the W./H.³ index which Bardeen (7) had argued for on logical rather than empirical grounds. The two former indices yield smaller measures of dispersion than do the two latter, with a given group of subjects at a particular age, and they also fit better the empirical picture of developing body build.

Table XVIII (p. 60) adduces new evidence in support of the W./H.² index. (Since Quetelet's data showed both W./H.² and W./H.$^{2\frac{1}{2}}$ to be good* indices, with little to choose between them, the latter index was not computed for the data from the present

[* The authors assume here that a "good" index is one that correlates highly with the judges' ratings of "chubbiness." I should anticipate the reverse of this: see my footnote, p. 55. ED.]



Comparative Curves of the Stem Length : Length Index. (The New York and Merrill-Palmer Curves are based on Recumbent Measures (89). Growth Study Measures are recumbent through month 18.)

TABLE XVIII.

Correlations of Indices of Body-build with Judges' Ratings.

Index	Months											Mean r
	1	3	5	7	9 & 10	12	15	18	24	30	36	
$W./L.^3$	·75	·71	·83	·75	·84	·70	·77	·72	·55	·84	·70	·76
$W./L.^2$	·69	·68	·79	·79	·84	·74	·70	·53	·59	·71	·68	·71
W./L.	·66	·66	·72	·59	·66	·73	·62	·74	·59	·71	·66	·67
H.W./L.	·65	·65	·75	·71	·58	·46	·65	·57	·64	·57	·63	·62
H.W./S.L.	·54	·62	·69	·63	·51	·42	·65	·58	·55			·58
$\frac{\text{Log L.} \times 1000}{\text{C.D.} \times \text{C.W.} \times \text{S.L.}}$	—	—	—	—	—	—	·49	·57				·53
C.C./L.	·38	·47	·58	·63	·61	·53	·50	·47	·10	·45	·43	·47
C.C./S.L.	·15	·32	·60	·59	·55	·48	·41	·54	·13			·41
S.L./L.	−·22	—	·28	—	—	·19	—	−·06	−·06			·03

sample.) For the first time, we believe, quantitative measures of body-build have been computed from seriatim data gathered from an approximately constant sample during the first three years of life, and have been compared with ratings of body-build based upon photographs taken on the same days on which the measurements were made. The ratings, as has been explained, were given independently by two

judges (*B* and *D*) for the eleven ages at which photographs were taken. A third judge (*W*) made ratings for months 15, 18 and 24.

For correlation with the several body-build indices the ratings of the judges were pooled. Table XVIII shows W./L.² correlating higher with the pooled photograph ratings, at almost every age, than any of the other indices which were similarly treated. The relative hip width and the relative chest girth ratios (especially the latter) appear as poor measures of body-build when correlated against the criterion furnished by the judges' ratings. An adaptation * of the Wertheimer-Hesketh index (leg length × 1000/C.D. × C.W. × S.L.) was correlated with the photograph ratings for two ages only, 15 and 18 months, and yielded coefficients of ·49 and ·57 respectively. This index affords an example of an index the validity of which suffers because of the relatively low reliability of some of the measures entering into it. Though this objection probably holds for subjects of ages similar to those used in the present investigation, it does not necessarily hold for more mature subjects. This is indicated by the high correlations reported by Wertheimer and Hesketh between their index extremes and the psychotic types with which, if Kretschmer's conclusions were soundly based, asthenic and pyknic body builds should be correlated.

The sharp drops found in the majority of the *r*'s in Table XVIII at the twenty-fourth month are probably to be attributed to a marked shrinkage in the number of cases available for measurement and photographing on that occasion, due to the illness of a number of subjects, and also to the necessity for eliminating the records of several other subjects who proved unusually refractory on their second birthday. If this one patently atypical correlation be omitted from consideration, the average correlation of the W./L.² indices with the judges' ratings, for ten ages between the limits of one month and three years, is ·78. Including the twenty-fourth month, the mean of the *r*'s is ·76.

Table I (p. 34) gives the self-correlations of the W./L.² index, for one-month intervals throughout the first year, for three-month intervals from 12 to 18 months, and for six-month intervals from 18 to 36 months. Five out of the sixteen coefficients are less than ·80, but from the seventh month through the thirty-sixth (excluding the twenty-four month correlations as atypical, for the reasons given above) these self-correlations average ·87. Including all ages, they average ·82. Considering the rapid growth changes, especially in the early months, these correlations are probably minimum measures of reliability; and remeasure-coefficients derived from measurements and remeasurements made on the same day would doubtless be found to be higher.

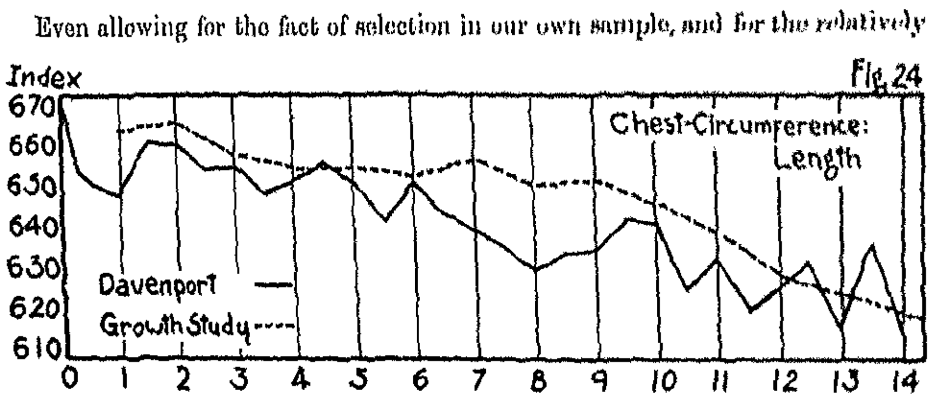
Fig. 22 (p. 59) presents a curve for the development of body-build, using W./L.² as the best quantitative measure of body-build for the ages under consideration, for the sexes separately, and for the total group. Since a low index means relative slenderness and a high index indicates relative stockiness, it is seen that these

* This adaptation was necessary, since we did not make exactly the same measures as did Wertheimer and Hesketh (40).

curves make manifest a trend toward increasing stockiness from the first to the eleventh month, at which time the curves change direction. Two judges (*B* and *W*) studied the entire series of photographs, independently, in an attempt to estimate the age at which these subjects ceased to increase in stockiness and started to grow more slender. *B* made self-comparisons of the photographs of 52 cases and estimated the peak in the trend toward increasing stockiness for each child. The mean of these was 12·3 months. *W* similarly studied the photographs of 20 cases and arrived at a corresponding estimate of 11·6 months. The close agreement between the evidence afforded by the $W./L.^3$ curves and these independent estimates from photographs, with respect to the time of occurrence of this change in trend, lends further support to claims for the validity of the $W./L.^3$ index as an indicator of body-build in infants and young children.

Before leaving this matter of the curve of development of body-build, however, we should consider some other indices, and the claims made for them. One index proposed by Davenport is that of chest girth, divided by length, or relative chest girth (14). This index presents a picture of the development of body-build different from the $W./L.^3$ curve. Davenport concludes (on the strength of the data presented graphically in his Fig. 2, p. 13) that "at six weeks the body has reached its maximum postnatal chubbiness. Another temporary loss in chubbiness occurs at about eight months, due perhaps to the cutting of the incisors." Our $W./L.^3$ curve, however, puts the time of maximum chubbiness at eleven months instead of six weeks, and there is no such sudden falling off in chubbiness at eight months as portrayed in Davenport's curve. We incline to the belief that the irregularities in Davenport's curve, including the sharp drop at eight months, are extraneous, and attributable primarily to racial heterogeneity in the cross-sectional samples, to the fewness of the cases in some of his samples (the number of cases in an age group ranges from 9 at month 12 to 37 at month 7), and to the fact that the *measurements were taken* by different examiners. This is increasingly evident when his curve is compared with our curve of relative chest girth (Fig. 24). Our curve shows an index decreasing with age from the 1st month (especially after the 9th month) which is far more regular than Davenport's in its trend, showing no radical change in plumpness at the 8th month.

Even allowing for the fact of selection in our own sample, and for the *relatively*



Davenport's Curve of Relative Chest Girth compared with the Growth Study Curve.

small number of cases, it still seems probable that the true curve of development in body-build is more nearly approached with the data from the present sample, and using the $W./L.^2$ index, than with Davenport's data, simply because the data are seriatim rather than cross-sectional, the measurements were taken by one person, and a more reliable and valid index of build is used. That our curve is typical is evidenced by the fact that the age curves of the $W./L.^2$ indices given in other studies, e.g. Quetelet (34) and Davenport (15), are all very similar, in this early portion of the curve, to our own.

Further comparisons of the indices are afforded by their inter-correlations. These inter-correlations, arranged in order of size, are presented in Table XIX.

TABLE XIX.

Inter-Correlation of Indices of Body-build.

	Months					Mean
	1	5	9	18	36	
H.W./L. × H.W./S.L.	·861	·952	·938	·824	—	·894
C.C./S.L. × C.C./L.	·729	·877	·826	·770	—	·802
H.W./L. × $W./L.^2$	·804	·753	·706	·800	·689	·750
H.W./S.L. × $W./L.^2$	·618	·653	·553	·654	—	·627
C.C./L. × $W./L.^2$	·454	·602	·624	·721	·570	·594
H.W./L. × C.C./L.	·307	·601	·549	·706	·415	·534
H.W./S.L. × C.C./S.L.	·327	·542	·623	·607	—	·525
C.C./S.L. × $W./L.^2$	·218	·503	·455	·624	—	·450
H.W./S.L. × C.C./L.	·204	·538	·482	·462	—	·422
H.W./L. × C.C./S.L.	·157	·468	·503	·498	—	·407
$W./L.^2$ × S.L./L.*	·314	·444	—	·272	·141	·272

* Additional correlations between $W./L.^2$ and S.L./L. at 12 and 24 months are +·826 and +·186, respectively.

In general, the indices which correlate most highly with each other are those in which there is high correlation (or identity) between the measures included in the two indices. There is also some evidence that the higher correlations are between indices composed of the more reliable measures. As between highly inter-correlated indices, those composed of the most reliable measures are of course more desirable as measures of build. The two highest r's are between measures in which the numerators are identical, and the denominators are length and stem length. Since the former is a more reliable measure, other things being equal, it should be considered the better index. The next three correlations, in order, are comparisons between width-length relations and $W./L.^2$. Since the latter is made up of far more reliable measures than the others, it is obviously a more reliable index, and probably, also, a more complete measure of the width-length relationship than are any of these other indices with which it is compared*.

[* The authors do not seem to have considered the influence of "spurious" correlation on the correlation of these indices. ED.]

Of the other comparisons, we must consider the fact that where there is little correlation between two indices which are composed of reliable measures, either may be a valuable measure of differing body-build traits. In this connection, the correlations between $W./L.^2$ and S.L./L., which average only ·27, are worthy of consideration. These two indices are composed of the three most reliable body measures we have taken (not including head measures), yet they show practically no relationship to each other. The former correlates closely with the judges' estimates of chubbiness, while the latter shows no relation to these estimates (Table XVIII, p. 60). It is in no way a measure of lateral versus linear tendency, but of independent linear proportions. It is possible that in considering the relationship of body-build to mental characteristics, both of these indices should be used. The Wertheimer-Hesketh index, which takes account of both relationships, may correlate less with our empirical criterion in part because of this fact. Yet for this very reason, it may be more discriminating (where the individual measurements are reliable) in picking out the extreme builds in which short legs and long thick bodies (pyknic) go together and those in which long legs and short thin bodies (asthenic) coincide. The nature of the correlations between the two separate indices is evidence that for these early years at least, there is no tendency for such body-builds to occur much more frequently than would be expected from a chance distribution in which these are the extreme deviates.

So far in our discussion, we have treated body-build as relative chubbiness or width-length relationship including subcutaneous tissue as well as skeletal structure. If we wish to limit our definition of build to the skeletal structure, we cannot with any certainty draw conclusions from our measures of external body dimensions, since these must always be affected to some extent by subcutaneous tissue. The width measure which is most reliable, hip width, is probably, because it is a simple linear measure, freer from the influence of subcutaneous tissue than is weight, and so may be a better index of skeletal proportions. The H.W./L. index (Fig. 15, p. 58) shows a change in proportions with age which does not coincide with our criterion of ratings of build. According to this index, there is a rapidly increasing lateral tendency through the sixth month, followed by an almost equally rapid decrease after the seventh month. But since the judges' ratings were made from photographs, they were necessarily affected by subcutaneous tissue, and need not be an adequate criterion of skeletal proportions alone. The apparent increasing chubbiness between the seventh and eleventh months might be due entirely to general growth in subcutaneous tissue. Since we have seen in Figs. 9 and 10 that hip width, chest width and shoulder width all continue to grow rapidly throughout the first year, we should expect increasing width relative to length for a longer period than six months. The absolutely greater increase of length, which is a much larger diameter, would mask a tendency to continued increase in chubbiness when the comparison is made with only one of the width diameters, and for this reason it is doubtful if the H.W./L. index gives us a true picture of changing build, either of skeletal proportions, or of proportions which include other tissues in addition to bony structure.

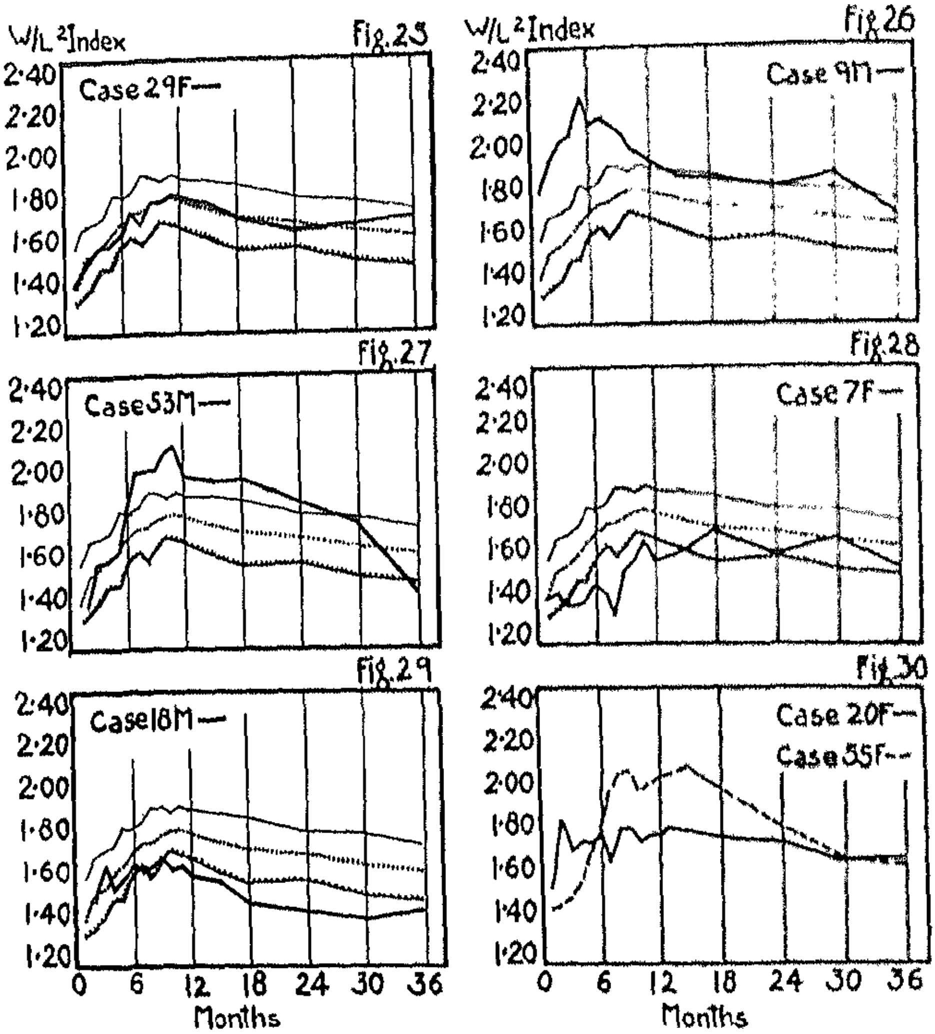
G. CONSISTENCY OF BUILD.

Of the indices of body-build which we have computed, $W./L.^2$ has been found by an empirical criterion, which is based on subcutaneous tissue as well as skeletal proportions, to be the most valid measure of growth changes in chubbiness or slenderness of build. We have also found that it shows a high degree of consistency in consecutive measurements. How consistent is it over longer periods? Granting average changes in body-build, for the total group, do the slender children remain relatively slender, and do the stocky infants continue to be the stocky three-year-olds?

(1) *Individual Curves.*

Figs. 25—29 (p. 66) show growth curves in body-build ($W./L.^2$) for selected subjects. Judge *B*, before consulting the body-build indices, named two cases whom she considered to be of definitely stocky, chubby build, two others whom she classified as definitely slender, and one other whose body-build, in her estimation, approximated the average of the group. The close correspondence between the group's average and the individual body-build curve of case 29 F (the one selected as "typical" by judge *B*) is clearly indicated in the figure. Case 9 M, one of the two subjects estimated by *B* as having a definitely stocky build, is seen to be far above the group average for the first ten months of postnatal life. He reached his peak, however, in the curve of developing stockiness, at the age of five months, as against the group average of eleven months. But although his curve from the tenth month is much closer to the group's average curve of build than was the case in the early months, case 9 M throughout is distinctly atypical in his tendency toward pronounced stockiness. Case 53 M, the other subject selected as definitely above the average in stockiness by judge *B*, presents a very different developmental picture. For the first five months this infant was fairly typical in his weight to height proportions, but between his fifth and seventh month he gained far more rapidly in weight than in length, and more rapidly than did the group as a whole, with the result that his $W./L.^2$ index changed strikingly, as indicated in the figure. The relative rôle of endogenous and exogenous factors in such sudden and unusual changes as this one is a matter which needs investigation. Since the group is a relatively homogeneous one, with the best of dietary advice equally at the command of all of the subjects' parents, the inference would seem to be indicated that such sudden shifts in metabolism are due largely to endogenous factors. When the shift is in the direction of a sudden sharp decrease in weight, rather than an increase, a history of recent illness would of course be looked for before attempting to explain the decrease as due to the influence of postnatal maturational factors. It is of interest to note that case 53 M's curve remains well above the average from the seventh month on, pointing to the establishment of a somewhat stable metabolic equilibrium in his organism at this supernormal level.

Case 7 F presents the picture of a subject who, during the first three years of postnatal life, is consistently more slender than the average. From the third month



Figs. 25—29. Individual Curves of Growth in the Weight : Length² Index compared with Curves of the Mean and Plus and Minus one Standard Deviation.

Fig. 30. Weight : Length² Curves of two Cases who differ widely in their Tendency toward Constant Weight : Length Proportions.

to the twelfth she is strikingly atypical in her tendency toward extreme slenderness. During the ten-months' interval between her eight- and eighteen-month measures she gained in weight at a relatively faster rate than in height, even though after month eleven the group as a whole was showing the contrary tendency, and this deviation from the typical developmental trend results in this subject's close approach to the normal at the eighteenth month. During the next six months, however, her curve changes direction once more and now falls away more sharply than does the average of the group, so that by the twenty-fourth month she is again well below normal. Case 18M is an individual who, except for the third

month, is also consistently more slender than the normal infant. In contrast to 7 F, however, this atypically slender baby reached his peak in the curve of increasing stockiness at the ninth month, and from that point on deviates more and more sharply from the mean.

(2) *Correlations.*

These individual curves indicate the results which we may expect when the indices are correlated over longer intervals to determine their consistency. The effect of sudden pronounced shifts—like those shown by case 53 M between the fifth and seventh months upon the rank orders of the subjects and consequently upon the correlation coefficients for the intervals of time during the course of which these shifts occurred—is obvious. When individual organisms are going through such rapid and pronounced readjustments, the correlation coefficients for the body characters involved in such readjustments will drop. A study of the consistency correlations, therefore, should be informative with respect to which developmental periods tend to be characterised by these pronounced individual divergences from the mean curve of growth, and which ones manifest relatively stable developmental characteristics on the part of the individual members of the group. Table XX gives these data.

Self-Correlations showing Consistency of Body-Build.

TABLE XX.

W./L.^{2}

Month	3	6	12	18	24	30	36
1	·68	·48	—	—	·17	—	·10
3	—	·64	—	·17	·31	—	·28
6	—	—	·44	—	·24	—	·28
12	—	—	—	·79	·74	—	·75
18	—	—	—	—	·73	—	·80
24	—	—	—	—	—	·74	·73
30	—	—	—	—	—	—	·91

It is evident at a glance that something is operating to reduce the longer-interval correlations into which the first month measurements enter. The W./L.^{2} correlations, while they are ·718 for the first and second months, and ·741 for the second and third months (see Table I), become ·68 for the first and third months (Table XX) and drop to ·48 and to ·17 for the first and sixth months and the first and twenty-fourth months, respectively. Similarly, the correlations of the sixth month indices with those for the twelfth, twenty-fourth and thirty-sixth months give the following coefficients, ·44, ·24 and ·28. But when the twelve-month indices are correlated with those for the eighteenth, twenty-fourth and thirty-sixth months we find a considerably higher degree of stability in the rank orders of the subjects at these ages, with coefficients of ·79, ·74 and ·75. Over the eighteen-month interval between the ages of 18 and 36 months, our subjects maintain their relative positions with

respect to their W./L.^{2} index to the extent indicated by a coefficient of correlation of the order of ·80, while for the thirtieth and thirty-sixth months the correlation is ·91.

The consistency correlations for the H.W./L. index (Table XXI) are very similar to those for W./L.^{2}. The r's of the former are more often larger when the first and sixth months are correlated with later ages, while the latter are higher when the twelfth and twenty-fourth months are compared with later ages. None of the differences, however, is great enough to be significant. And, in general, the H.W./L. index shows no greater consistency in build than does the W./L.^{2} index.

These correlations, and also the individual curves, show that there is little consistency in lateral-linear build during the first year—just as there is little consistency in the relative size of weight and all width measures. But after the twelfth month the correlations increase somewhat and the children appear to have established a body-build which is comparatively stable—at least over a two-year period.

Self-Correlations showing Consistency of Body-Build.

TABLE XXI.

H.W./L.

Month	6	12	24	36
1	·60	·24	·27	·13
6	—	·34	·47	·46
12	—	—	·70	·66
24	—	—	—	·72

TABLE XXII.

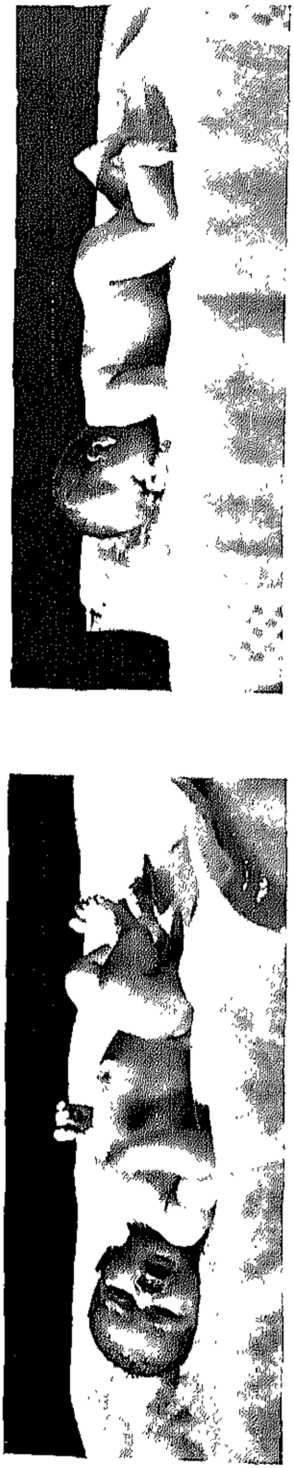
S.L./L.

Month	6	12	18	24	36
1	+·12	−·05	[illegible]	−·11	−·07
6	—	+·30		+·20	+·16
12	—	—	+·28	+·37	+·13
18	—	—			+·62
24	—	—			+·42

Consistency correlations were computed for another index composed of reliable measures, S.L./L. (Table XXII). All the coefficients are low for all ages, even though the measures used in this index are relatively consistent. It is probable that since the legs are continuing to grow rapidly through the thirty-sixth month, individual differences in growth trends would obliterate any tendency for consistent stem-stature relationships within the age limits here considered *.

Body-build types may be manifested during the first year in the growth *changes* of the W./L.^{2} index rather than in the child's relative build at any given age. As we have seen (Figs. 25—29, p. 66), when the age curves of this index are plotted for each child separately, we find wide individual difference in the trends of the curves. Some of the children (for example, cases 9 M and 18 M) show an early

* Hejinian and Hatt (25) plotted individual curves of the percentile rank of S.L./L. index for 50 children aged 24 to 60 months. *Seventy-two* per cent. of these showed consistency of build, but the others distinctly changed their percentile rank. This body proportion, then, remains unstable for many children even to five years of age.



Slender child

Stocky child

Plate I (a). 1 month

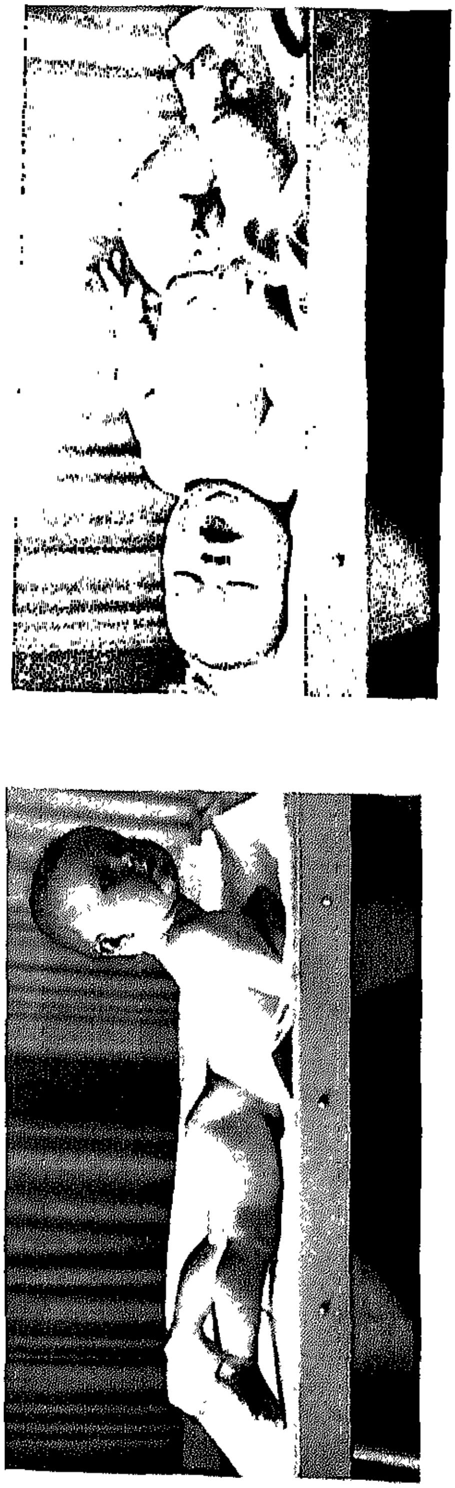
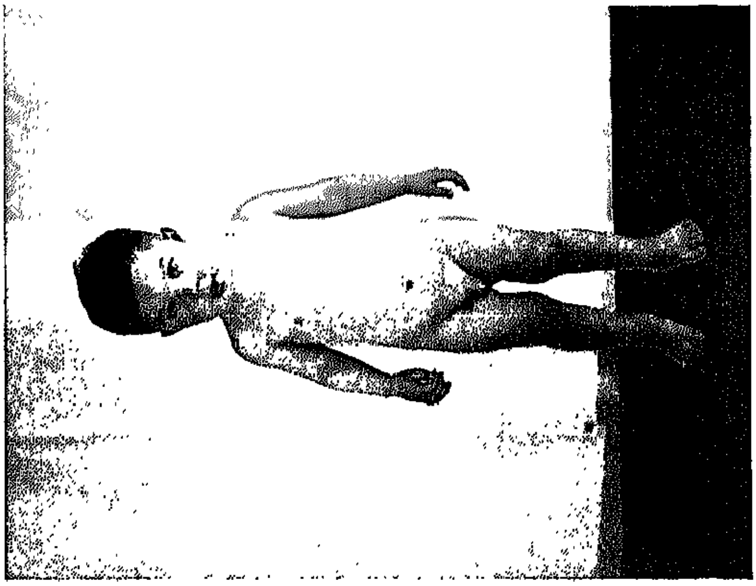


Plate I (b). 5 months

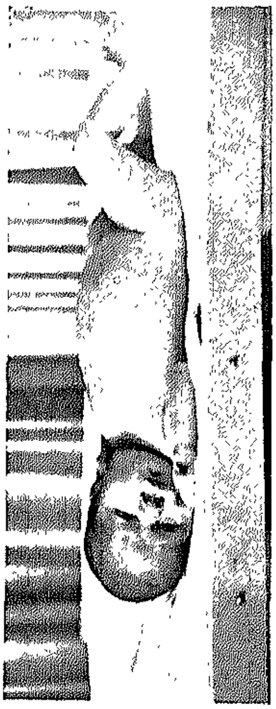
Plate I (c). 30 months



Comparisons between a slender child, *7F* (left), and a stocky child, *9M* (right), at three age levels: 1, 5 and 30 months. *9M* was extremely chubby at 5 months.

Plate II (a). 1 month

Large child



Small child

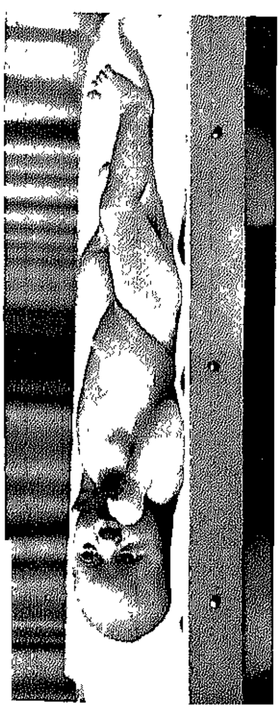
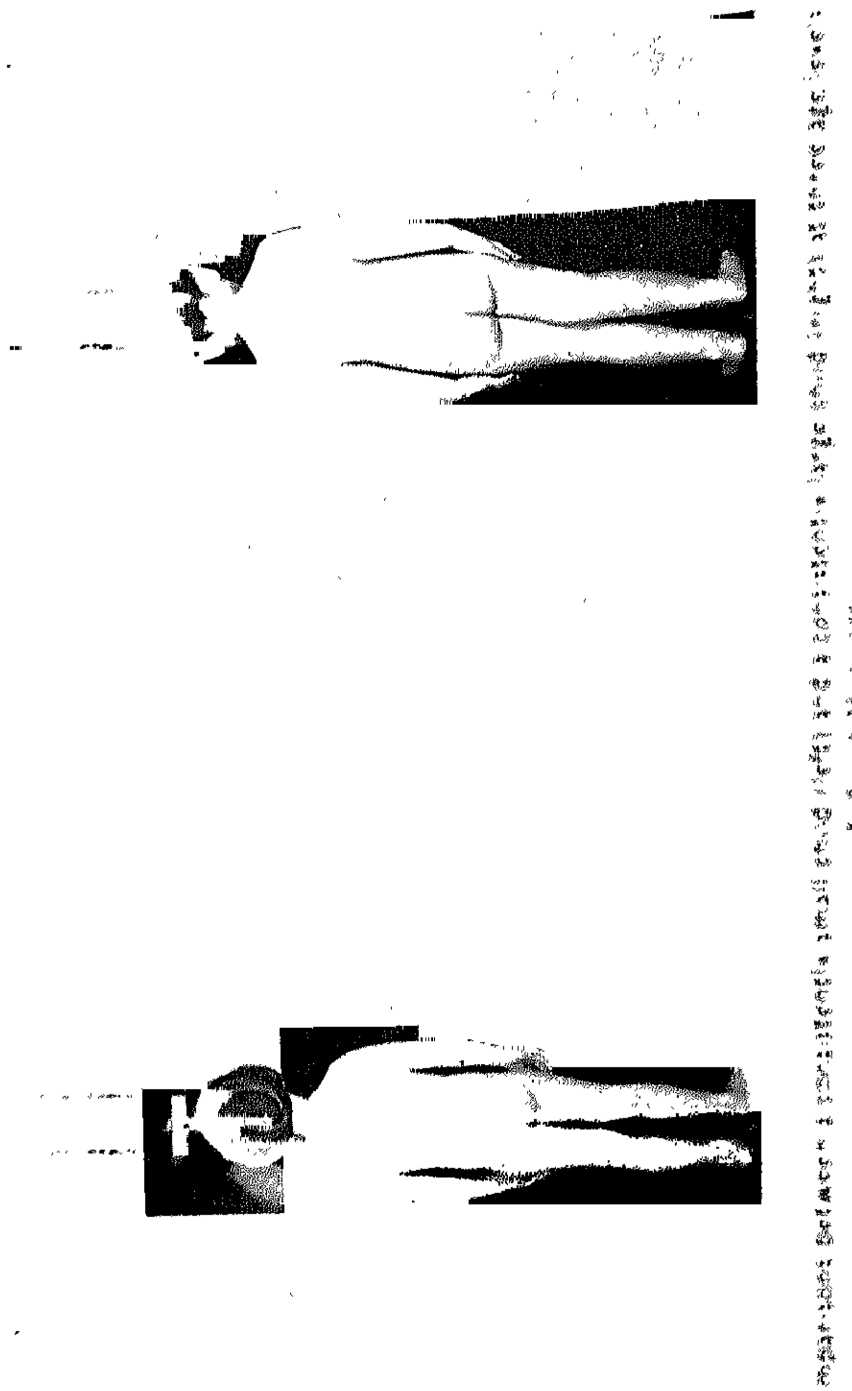


Plate II (b). 9 months

Plate II (c). 36 months



increase in chubbiness, and others a late increase (case 7 F), as compared with the majority, who follow the curve of the mean. A number of the more slender children have little or no tendency toward increasing $W./L.^3$ indices during the first year, but maintain a comparatively constant index throughout the entire three years. The $W./L.^3$ curves of two girls, cases 20 F and 55 F (Fig. 30), illustrate the wide divergence from the mean in tendency to change in body-build. Case 20 has an index which is close to the mean for the first four months. After this time it increases rapidly to the eighth month, stays high through the fifteenth month, and then decreases at a rate far more rapid than the average, until at the thirtieth and thirty-sixth months her index has returned very near to the mean. Case 55, on the other hand, has indices which are seldom far from the mean, though they are well above it during the first six months. Her index curve is almost a straight line, and, except for the rise between the first and second months, and a slow decline after the fifteenth month, shows no tendency towards a changing $W./L.^3$ index.

When the group is divided into halves, one half containing children with less than average tendency to increased $W./L.^3$ index during the first year, and the other half containing those with greater tendencies to chubbiness, the slender group, as would be expected, has a much higher consistency correlation between the sixth and twelfth months (r is ·87 for 25 cases) than is found for the total group (·44). However, their consistency correlations drop later, and are lower than for the total in the r between measures at the twelfth and twenty-fourth months (·53) and between the twenty-fourth and thirty-sixth (·57). Of the entire group of children, only one (46 M, Plate I, see p. 32) was found to be so consistently heavy that his $W./L.^3$ index did not once fall within one standard deviation of the mean for the 17 ages at which the children were measured. One other child (9 M, Plate II, see p. 32) remained above one standard deviation for all but three ages. Of the slender children, one (57 F, Plate I) remained below one standard deviation for all but two ages, while another (27 F, Plate II) remained below for all but three ages.

Infant trends in body build may be characteristic of glandular and metabolic balances in the individual and so predictive of build later in life. This we cannot know until we can compare these early trends with measures of the same individuals later. But we have found that no single index (composed of the measures we have taken) of an infant under one year can be considered as representative of a constitutional type which will continue to be one of its characteristics for any protracted length of time.

SUMMARY AND CONCLUSIONS.

1. A seriatim growth study of nine measures of body size was made on a group of from forty-six to sixty-one infants who were measured at 17 ages during their first 36 months.

2. The measures were found to be sufficiently reliable to indicate growth trends of individuals; there are indications that some measures are more reliable than

others, the highest remeasurement-coefficients being found for length, weight and stem length.

3. The group was found to be typical of other groups reported in the literature, except that after the sixth month its members averaged taller and heavier than children in other parts of the country, with the exception of selected groups such as those mentioned in the recent Iowa studies.

4. Growth in all dimensions was retarded, rapidly at first, and then more slowly through 36 months.

5. In proportion to their size at the first month, the width measures of hip, chest and shoulder increased more rapidly during the first year than any other measure except weight. After this age, the greatest proportionate growth was in length. Chest depth increased even less than did head circumference.

6. In variability there were no significant changes with growth, nor between the sexes.

7. The boys were consistently a little larger for all measures than the girls.

8. Correlations over the three-year period show that head circumference, length and stem length are the most consistent measures; that is, children with large heads at the first month tend to have relatively large heads at the thirty-sixth month and those with small heads usually continue to have small heads, while the same is true of lengths. Weight and width measures show only slight consistency before the twelfth month, but with the exception of two unreliable measures (chest width and depth) are fairly consistent after this age.

9. Hip width and weight are more closely correlated than any other pair of measures. Length and stem length are also highly correlated. Other measures of body width correlate fairly highly with weight; while the width-length comparisons show less relationship though they are still positively correlated. The smallest correlations are found between shoulder width and stem length and average ·53.

10. There is a decrease in some of the correlations between measures of widths and lengths during the first half of the first year. This is true of the *r's* between weight and length, hip width and length, hip width and stem length, and less clearly between hip width and shoulder width. These decreased *r's* appear to occur more often at ages when the *measures involved are growing at disproportionate* rates. Thus, the width-length correlations drop during the ages when most children are rapidly becoming chubbier.

11. Growth changes in body proportions are shown by curves of indices, comparing length measures with width measures and with weight. Most of these show proportionately increasing width during some portion of the first year.

12. A comparison of these indices with ratings of body build leads us to the conclusion that $W./L.^2$ is (of the indices studied) the most valid measure of relative chubbiness, or lateral-linear tendencies in build.

13. The W./L.^{2} index shows a curve of rapidly increasing chubbiness during the first eleven months, with a slow decrease in chubbiness after this age.

14. Of the three indices of build whose values were correlated for consistency over a long age interval, namely, W./L.^{2}, H.W./L. and S.L./L., none indicated consistency of build during the first year, though W./L.^{2} and H.W./L. show increasing consistency after this age.

15. We cannot predict from indices at an early age what an individual's body-build will be later. It may be possible, however, to predict later build from early trends in build. Our data give no clear evidence on this point.

16. No significant sex differences in rates of growth were found for this group, though the boys were found to be slightly but consistently heavier for their length than the girls.

NOTE. Since this memoir was set up a letter has reached the Editor from Dr Bayley, which he thinks somewhat explains their different conceptions of "chubbiness." No relation of width to length would conform to his conception of chubbiness = plumpness. Nor did he after a study of Kretschmer's work based on no extensive statistical investigation give credit to his classifications or conclusions. The following are the important paragraphs from Dr Bayley's letter:

"I have been trying out some of the indices you suggested. They should prove of value in studies of nutrition. However, in the study under consideration, I was most interested in finding an index which would differentiate types of body-build such as those described by Kretschmer (pyknic-leptosome), or Davenport (lateral-linear), and not so much interested in obtaining a measure of nutrition or health. This latter problem is one I intend to take up in connection with a study of physicians' ratings of the children's health.

"It would, of course, be of value to find an index which differentiates the children and is constant at all ages, and in which the individual children maintain the same relative position in the group. None of the indices for which I have made the comparisons has remained consistent, so that no child who is classified in body-build at one age can be expected to have the same build at another age. "Chubbiness" as we used it in our ratings of photographs meant the tendency to be wide, relative to length, rather than to have excess fatty tissue, though this latter factor had a strong influence in the rating.

"It still seems to me that the index I am looking for is one which expresses width relative to length, and all of my data indicate that such an index would show changes with age, and would not be constant. This finding in itself should be of importance in understanding the processes of growth in young children."

TABLE XXIII.

ANTHROPOMETRIC MEASURES. INFANT GROWTH STUDY.

(a) Chest Circumference Frequency Distributions.

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
40·0	1	43·0	1	44·0	2	46·5	1	47·5	2	49·5	1
39·5	2	42·5	—	43·5	—	46·0	—	47·0	2	49·0	1
39·0	—	42·0	2	43·0	1	45·5	1	46·5		48·5	
38·5	—	41·5	2	42·5	1	45·0	1	46·0	1	48·0	1
38·0	2	41·0	—	42·0	7	44·5	—	45·5	2	47·5	
37·5	1	40·5	1	41·5	4	44·0	1	45·0	2	47·0	
37·0	6	40·0	3	41·0	6	43·5	2	44·5	5	46·5	5
36·5	2	39·5	5	40·5	1	43·0	5	44·0	3	46·0	4
36·0	8	39·0	8	40·0	3	42·5	4	43·5	5	45·5	1
35·5	6	38·5	4	39·5	4	42·0	6	43·0	5	45·0	5
35·0	7	38·0	7	39·0	8	41·5	4	42·5	3	44·5	4
34·5	3	37·5	3	38·5	6	41·0	4	42·0	6	44·0	6
34·0	2	37·0	3	38·0	1	40·5	5	41·5	4	43·5	2
33·5	1	36·5	2	37·5	—	40·0	5	41·0	3	43·0	8
33·0	3	36·0	4	37·0	4	39·5	3	40·5	3	42·5	4
32·5	—	35·5	4	36·5	2	39·0	4	40·0	2	42·0	3
32·0	1	35·0	1			38·5	1	39·5	1	41·5	1
						38·0	2	39·0		41·0	1
						37·5	—	38·5	1	40·5	2
						37·0	—			40·0	
										39·5	1

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
51·0	2	50·5	—	51·0	—	51·0	4	52·0	1	52·5	1
50·5	—	50·0	3	50·5	2	50·5	1	51·5	2	52·0	
50·0	—	49·5	—	50·0	2	50·0	1	51·0		51·5	3
49·5	—	49·0	1	49·5	3	49·5	3	50·5		51·0	1
49·0	1	48·5	3	49·0	2	49·0	1	50·0	1	50·5	
48·5	1	48·0	1	48·5	2	48·5	5	49·5	5	50·0	1
48·0	2	47·5	3	48·0	2	48·0	4	49·0	2	49·5	1
47·5	4	47·0	4	47·5	5	47·5	4	48·5		49·0	3
47·0	2	46·5	8	47·0	5	47·0	5	48·0	1	48·5	2
46·5	3	46·0	2	46·5	5	46·5	5	47·5	6	48·0	2
46·0	6	45·5	6	46·0	2	46·0	3	47·0	3	47·5	3
45·5	2	45·0	1	45·5	6	45·5	4	46·5	3	47·0	3
45·0	6	44·5	3	45·0	5	45·0	1	46·0	5	46·5	2
44·5	1	44·0	4	44·5	2	44·5	2	45·5	1	46·0	5
44·0	5	43·5	4	44·0	1	44·0	3	45·0	2	45·5	3
43·5	2	43·0	2	43·5	—	43·5	—	44·5	1	45·0	3
43·0	3	42·5	—	43·0	1	43·0	—	44·0	1	44·5	1
42·5	3	42·0	1	42·5	—	42·5	2	43·5	—	44·0	
42·0	4			42·0	—			43·0	—	43·5	1
				41·5	1			42·5	—		
								42·0	1		

TABLE XXIII—(*continued*).

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
53·0	1	54·5	1	55	1	55	1	56·5	1
52·5	1	54·0	—	54	2	54	4	56·0	—
52·0	1	53·5	—	53	2	53	1	55·5	2
51·5	2	53·0	2	52	5	52	5	55·0	—
51·0	—	52·5	2	51	7	51	7	54·5	4
50·5	1	52·0	2	50	8	50	7	54·0	1
50·0	7	51·5	—	49	5	49	5	53·5	1
49·5	2	51·0	1	48	2	48	2	53·0	2
49·0	2	50·5	2	47	1	47	2	52·5	3
48·5	2	50·0	6	46	3	46	2	52·0	2
48·0	4	49·5	3	45	1	45	1	51·5	2
47·5	2	49·0	1					51·0	7
47·0	2	48·5	2					50·5	5
46·5	6	48·0	4					50·0	5
46·0	2	47·5	4					49·5	2
45·5	—	47·0	4					49·0	4
45·0	2	46·5	—					48·5	1
44·5	—	46·0	2					48·0	1
44·0	1	45·5	2					47·5	2
								47·0	3
								46·5	1

(*b*) *Hip Width Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
12·9	2	14·3	2	14·9	2	15·4	1	16·2	1	17·4	1
12·6	1	14·0	—	14·6	1	15·1	—	15·9	2	17·1	—
12·3	1	13·7	3	14·3	2	14·8	4	15·6	1	16·8	—
12·0	—	13·4	1	14·0	3	14·5	8	15·3	4	16·5	1
11·7	5	13·1	1	13·7	6	14·2	5	15·0	7	16·2	3
11·4	6	12·8	3	13·4	6	13·9	8	14·7	4	15·9	1
11·1	2	12·5	3	13·1	9	13·6	3	14·4	3	15·6	5
10·8	8	12·2	8	12·8	5	13·3	8	14·1	9	15·3	4
10·5	6	11·9	6	12·5	6	13·0	10	13·8	2	15·0	3
10·2	5	11·6	2	12·2	3	12·7	3	13·5	7	14·7	8
9·9	7	11·3	8	11·9	2	12·4	3	13·2	4	14·4	10
9·6	2	11·0	7	11·6	4	12·1	—	12·9	3	14·1	2
9·3	2	10·7	2	11·3	1	11·8	1	12·6	4	13·8	7
9·0	2	10·4	—	11·0	3	11·5	3	12·3	2	13·5	1
		10·1	2	10·7	2	11·2	—	12·0	1	13·2	4
		9·8	1	10·4	—	10·9	1	11·7	1	12·9	4
				10·1	1	10·6	—			12·6	1
						10·3	1				

TABLE XXIII—(*continued*).

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
18·0	1	18·0	1	18·3	1	17·9	1	17·8	2	18·1	2
17·5	1	17·7	—	18·0	—	17·6	—	17·5	1	17·8	1
17·0	—	17·4	1	17·7	—	17·3	2	17·2	1	17·5	1
16·5	3	17·1	3	17·4	—	17·0	4	16·9	2	17·2	2
16·0	2	16·8	2	17·1	1	16·7	1	16·6	1	16·9	—
15·5	9	16·5	1	16·8	4	16·4	4	16·3	5	16·6	6
15·0	7	16·2	7	16·5	2	16·1	2	16·0	4	16·3	4
14·5	12	15·9	1	16·2	2	15·8	4	15·7	3	16·0	3
14·0	5	15·6	3	15·9	3	15·5	5	15·4	2	15·7	5
13·5	7	15·3	6	15·6	9	15·2	5	15·1	2	15·4	3
13·0	2	15·0	5	15·3	2	14·9	5	14·8	4	15·1	1
12·5	2	14·7	7	15·0	7	14·6	4	14·5	2	14·8	1
		14·4	4	14·7	3	14·3		14·2	2	14·5	2
		14·1	—	14·4	5	14·0	4	13·9	3	14·2	2
		13·8	6	14·1	4	13·7	2	13·6	1	13·9	
				13·8	2	13·4	4			13·6	2
				13·5	1						
				13·2	1						
				12·9	—						

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
18·3	2	18·9	—	19·7	1	19·6	—	19·9	1
18·0	—	18·6	3	19·4	—	19·3	1	19·6	4
17·7	1	18·3	—	19·1	—	19·0	1	19·3	1
17·4	3	18·0	3	18·8	—	18·7	2	19·0	2
17·1	2	17·7	2	18·5	2	18·4	2	18·7	4
16·8	1	17·4	2	18·2	4	18·1	5	18·4	4
16·5	5	17·1	3	17·9	1	17·8	5	18·1	13
16·2	3	16·8	2	17·6	5	17·5	6	17·8	8
15·9	2	16·5	5	17·3	7	17·2	5	17·5	1
15·6	4	16·2	4	17·0	4	16·9	4	17·2	2
15·3	4	15·9	6	16·7	3	16·6	2	16·9	4
15·0	5	15·6	1	16·4	5	16·3	1	16·6	2
14·7	2	15·3	3	16·1	1	16·0	1	16·3	1
14·4	1	15·0	1	15·8	—	15·7	—	16·0	—
14·1	—	14·7	—	15·5	—	15·4	—	15·7	1
13·8	—	14·4	—	15·2	—	15·1	—	15·4	1
13·5	—	14·1	—	14·9	3	14·8	1		

TABLE XXIII (*continued*). (*c*) *Chest Depth Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
11·0	1	12·6	1	12·4	—	12·4	—	12·4	1	13·2	1
10·8	1	12·3	—	12·2	1	12·2	—	12·3	—	13·0	—
10·6	2	12·0	2	12·0	—	12·0	2	12·2	—	12·8	—
10·4	6	11·7	—	11·8	1	11·8	2	12·1	—	12·6	—
10·2	5	11·4	3	11·6	1	11·6	2	12·0	3	12·4	—
10·0	6	11·1	4	11·4	1	11·4	4	11·9	—	12·2	—
9·8	6	10·8	7	11·2	5	11·2	4	11·8	—	12·0	4
9·6	5	10·5	8	11·0	2	11·0	2	11·7	1	11·8	1
9·4	4	10·2	14	10·8	6	10·8	5	11·6	1	11·6	1
9·2	5	9·9	6	10·6	5	10·6	5	11·5	—	11·4	5
9·0	3	9·6	8	10·4	8	10·4	8	11·4	2	11·2	11
8·8	1	9·3	3	10·2	6	10·2	6	11·3	6	11·0	1
8·6	3	9·0	1	10·0	1	10·0	6	11·2	4	10·8	10
				9·8	6	9·8	2	11·1	1	10·6	5
				9·6	4	9·6	3	11·0	7	10·4	2
				9·4	3	9·4	1	10·9	4	10·2	3
				9·2	2	9·2	1	10·8	1	10·0	1
				9·0	1			10·7	2	9·8	—
								10·6	4	9·6	1
								10·5	2	9·4	1
								10·4	6		
								10·3	4		
								10·2	1		
								10·1	—		
								10·0	1		
								9·9	—		
								9·8	1		
								9·7	1		
								9·6	1		

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
12·6	1	13·4	1	13·8	—	13·7	1	13·2	1	13·9	—
12·5	2	13·2	1	13·6	—	13·5	—	13·0	—	13·6	1
12·4	—	13·0	1	13·4	—	13·3	—	12·8	2	13·3	1
12·3	—	12·8	—	13·2	—	13·1	—	12·6	—	13·0	—
12·2	2	12·6	—	13·0	1	12·9	3	12·4	2	12·7	3
12·1	2	12·4	—	12·8	1	12·7	1	12·2	12	12·4	4
12·0	1	12·2	1	12·6	—	12·5	2	12·1	6	12·1	12
11·9	3	12·0	2	12·4	—	12·3	3	11·9	1	11·8	10
11·8	1	11·8	4	12·2	3	12·1	5	11·7	6	11·5	8
11·7	1	11·6	5	12·0	4	11·9	10	11·5	2	11·2	5
11·6	2	11·4	6	11·8	7	11·7	3	11·3	4	10·9	2
11·5	2	11·2	9	11·6	8	11·5	6	11·1	4	10·6	—
11·4	3	11·0	6	11·4	7	11·3	7	10·9	2	10·3	1
11·3	1	10·8	2	11·2	2	11·1	2	10·7	2	10·0	
11·2	4	10·6	2	11·0	3	10·9	4				
11·1	3	10·4	2	10·8	3	10·7	—				
11·0	2	10·2	1	10·6	3	10·5	—				
10·9	1	10·0	1	10·4	1	10·3	1				
10·8	4			10·2	1						
10·7	1			10·0							
10·6	5										
10·5	2										
10·4	2										
10·3	2										
10·2	1										
10·1	1										
10·0	1										
9·9	—										
9·8	—										
9·7	1										

TABLE XXIII—(*continued*).

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
13·3	2	14·8	1	14·6	1	15·8	1	14·6	1
13·1	1	14·6	—	14·4	1	15·5	—	14·4	1
12·9	4	14·4	—	14·2	—	15·2	—	14·2	1
12·7	5	14·2	—	14·0	—	14·9	—	14·0	2
12·5	3	14·0	—	13·8	—	14·6	—	13·8	1
12·3	5	13·8	—	13·6	2	14·3	—	13·6	2
12·1	6	13·6	1	13·4	1	14·0	2	13·4	2
11·9	7	13·4	1	13·2	3	13·7	2	13·2	—
11·7	2	13·2	3	13·0	5	13·4	2	13·0	5
11·5	4	13·0	2	12·8	1	13·1	2	12·8	6
11·3	4	12·8	7	12·6	5	12·8	2	12·6	6
11·1	—	12·6	5	12·4	4	12·5	7	12·4	3
10·9	1	12·4	3	12·2	4	12·2	8	12·2	5
10·7	—	12·2	7	12·0	6	11·9	5	12·0	3
10·5	1	12·0	4	11·8	1	11·6	3	11·8	5
		11·8	3	11·6	3	11·3	3	11·6	2
		11·6	3	11·4	3	11·0	2	11·4	
		11·4	4	11·2	—	10·7	1	11·2	1
		11·2	—	11·0	1	10·4	2	11·0	1
		11·0	—	10·8	—				
		10·8	1	10·6	1				

(*d*) *Chest Width Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
11·6	4	12·8	1	14·8	—	13·7	2	15·7	1	15·8	1
11·4	2	12·5	—	14·5	1	13·5	5	15·4	—	15·5	—
11·2	2	12·2	3	14·2	1	13·3	2	15·1	—	15·2	
11·0	3	11·9	4	13·9	1	13·1	1	14·8	—	14·9	1
10·8	2	11·6	3	13·6	—	12·9	1	14·5	—	14·6	2
10·6	2	11·3	8	13·3	1	12·7	2	14·2	1	14·3	1
10·4	4	11·0	5	13·0	4	12·5	4	13·9	3	14·0	4
10·2	3	10·7	7	12·7	4	12·3	6	13·6	4	13·7	1
10·0	3	10·4	4	12·4	2	12·1	5	13·3	4	13·4	3
9·8	8	10·1	2	12·1	5	11·9	2	13·0	7	13·1	10
9·6	6	9·8	6	11·8	6	11·7	4	12·7	8	12·8	8
9·4	2	9·5	1	11·5	8	11·5	5	12·4	10	12·5	1
9·2	3	9·2	1	11·2	8	11·3	7	12·1	3	12·2	6
9·0	1			10·9	7	11·1	6	11·8	7	11·9	2
				10·6	3	10·9	4	11·5	5	11·6	1
				10·3	2	10·7	1	11·2	1	11·3	—
				10·0	3	10·5	—	10·9	—	11·0	2
				9·7	1	10·3	2			10·7	1

TABLE XXIII—(*continued*).

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
15·4	2	17·2	—	16·9	—	16·3	1	16·6	—	17·0	2
15·1	2	16·9	—	16·6	—	16·0	—	16·3	1	16·7	—
14·8	1	16·6	—	16·3	1	15·7	2	16·0	1	16·4	2
14·5	2	16·3	—	16·0	1	15·4	4	15·7	4	16·1	2
14·2	6	16·0	1	15·7	1	15·1	7	15·4	3	15·8	6
13·9	4	15·7	1	15·4	2	14·8	4	15·1	9	15·5	4
13·6	10	15·4	3	15·1	6	14·5	6	14·8	6	15·2	6
13·3	7	15·1	1	14·8	2	14·2	7	14·5	4	14·9	8
13·0	4	14·8	3	14·5	7	13·9	4	14·2	1	14·6	6
12·7	1	14·5	4	14·2	8	13·6	3	13·9	1	14·3	8
12·4	5	14·2	12	13·9	4	13·3	2	13·6	2	14·0	—
12·1	1	13·9	4	13·6	7	13·0	1	13·3	3	13·7	4
		13·6	6	13·3	3	12·7	2	13·0	—	13·4	3
		13·3	5	13·0	2	12·4	—	12·7	1	13·1	1
		13·0	2	12·7	—	12·1	—				
		12·7	2	12·4	3	11·8	—				
		12·4	—	12·1	1	11·5	—				
		12·1	2	11·8	—	11·2	1				
		11·8	2								
		11·5	—								

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
18·4	—	17·3	3	17·9	3	17·9	5	19·1	1
18·1	—	17·0	2	17·6	—	17·6	—	18·8	—
17·8	—	16·7	2	17·3	3	17·3	3	18·5	1
17·5	—	16·4	3	17·0	2	17·0	5	18·2	2
17·2	2	16·1	8	16·7	4	16·7	8	17·9	2
16·9	4	15·8	5	16·4	8	16·4	4	17·6	2
16·6	2	15·5	3	16·1	4	16·1	7	17·3	3
16·3	2	15·2	3	15·8	5	15·8	3	17·0	9
16·0	4	14·9	7	15·5	4	15·5	4	16·7	8
15·7	6	14·6	5	15·2	2	15·2	1	16·4	3
15·4	4	14·3	2	14·9	4	14·9	2	16·1	9
15·1	8			14·6	—	14·6	—	15·8	3
14·8	6			14·3	—	14·3	—	15·5	1
14·5	2			14·0	2	14·0	—	15·2	2
14·2	2					13·7	—		
13·9	—								
13·6	1								
13·3	—								

TABLE XXIII—(*continued*)

(*e*) *Head Circumference Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
41·0	1	43·0	1	44·5	1	45·0	1	46·0	1	47·0	1
40·5	1	42·5	—	44·0	—	44·5	—	45·5	1	46·5	
40·0	1	42·0	—	43·5	—	44·0	4	45·0	3	46·0	5
39·5	2	41·5	1	43·0	2	43·5	3	44·5	2	45·5	3
39·0	4	41·0	7	42·5	4	43·0	4	44·0	11	45·0	11
38·5	5	40·5	4	42·0	3	42·5	10	43·5	9	44·5	6
38·0	9	40·0	9	41·5	8	42·0	12	43·0	8	44·0	6
37·5	7	39·5	10	41·0	12	41·5	7	42·5	5	43·5	8
37·0	9	39·0	10	40·5	10	41·0	8	42·0	7	43·0	5
36·5	4	38·5	6	40·0	7	40·5	4	41·5	3	42·5	4
36·0	1	38·0	5	39·5	3	40·0	2	41·0	4	42·0	3
35·5	4	37·5	2	39·0	4	39·5	1	40·5	—	41·5	2
35·0	—	37·0	2	38·5	2	39·0	1	40·0	1	41·0	
34·5	—	36·5	1	38·0	1	38·5		39·5		40·5	
34·0	—	36·0	—	37·5	—	38·0		39·0	1	40·0	1
33·5	1	35·5	1	37·0	1	37·5	1				

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
48·5	1	48·5	1	49·5	1	50·0	1	50·0	2	51·0	1
48·0	—	48·0	1	49·0	—	49·5	1	49·5	—	50·5	1
47·5	—	47·5	3	48·5	3	49·0	1	49·0	3	50·0	1
47·0	3	47·0	6	48·0	—	48·5	2	48·5	6	49·5	2
46·5	3	46·5	6	47·5	7	48·0	6	48·0	6	49·0	3
46·0	7	46·0	4	47·0	4	47·5	7	47·5	6	48·5	7
45·5	8	45·5	10	46·5	7	47·0	5	47·0	3	48·0	5
45·0	7	45·0	7	46·0	5	46·5	4	46·5	8	47·5	9
44·5	3	44·5	7	45·5	9	46·0	6	46·0	2	47·0	5
44·0	8	44·0	1	45·0	3	45·5	5	45·5	9	46·5	3
43·5	2	43·5	1	44·5	6	45·0	3	45·0	3	46·0	5
43·0	1	43·0	3	44·0	—	44·5	4	44·5	2	45·5	—
42·5	5	42·5	—	43·5	3	44·0	1	44·0	0	45·0	1
42·0	1	42·0	—	43·0	—	43·5	—	43·5	0	43·5	—
41·5	—	41·5	1	42·5	—	43·0	—	43·0	1	43·0	—
41·0	1			42·0	—	42·5	—			42·5	1
				41·5	1	42·0	1				

TABLE XXIII—(*continued*).

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
51·5	2	51·5	4	53·0	1	53·0	1	53·5	1
51·0	2	51·0	2	52·5	1	52·5	2	53·0	1
50·5	1	50·5	2	52·0	3	52·0	2	52·5	2
50·0	4	50·0	4	51·5	3	51·5	5	52·0	1
49·5	3	49·5	2	51·0	1	51·0	3	51·5	4
49·0	9	49·0	6	50·5	2	50·5	4	51·0	9
48·5	4	48·5	10	50·0	4	50·0	6	50·5	4
48·0	8	48·0	1	49·5	3	49·5	8	50·0	8
47·5	5	47·5	8	49·0	8	49·0	1	49·5	2
47·0	5	47·0	1	48·5	4	48·5	5	49·0	4
46·5	4	46·5	3	48·0	1	48·0	2	48·5	4
46·0	1	46·0	—	47·5	4	47·5	2	48·0	—
45·5	2	45·5	1	47·0	—	47·0	—	47·5	1
45·0	—	45·0	—			46·5	—	47·0	—
44·5	—	44·5	1			46·0	1	46·5	1
44·0	1								

(*f*) *Length Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
58·0	1	62·0	1	65·0	1	67·0	2	70·5	1	72·5	1
57·5	2	61·5	2	64·5	2	66·5	2	70·0	1	72·0	1
57·0	2	61·0	2	64·0	1	66·0	3	69·5	1	71·5	1
56·5	5	60·5	4	63·5	3	65·5	4	69·0	1	71·0	2
56·0	6	60·0	4	63·0	5	65·0	6	68·5	4	70·5	3
55·5	4	59·5	2	62·5	1	64·5	1	68·0	2	70·0	2
55·0	3	59·0	4	62·0	3	64·0	6	67·5	2	69·5	3
54·5	3	58·5	4	61·5	4	63·5	3	67·0	7	69·0	2
54·0	3	58·0	9	61·0	10	63·0	7	66·5	2	68·5	3
53·5	3	57·5	5	60·5	4	62·5	4	66·0	3	68·0	4
53·0	3	57·0	6	60·0	5	62·0	4	65·5	6	67·5	7
52·5	6	56·5	—	59·5	6	61·5	3	65·0	4	67·0	7
52·0	3	56·0	6	59·0	2	61·0	6	64·5	5	66·5	2
51·5	4	55·5	2	58·5	2	60·5	2	64·0	3	66·0	4
51·0	3	55·0	—	58·0	1	60·0	3	63·5	5	65·5	3
50·5	—	54·5	3	57·5	3	59·5	1	63·0	4	65·0	4
50·0	1	54·0	2	57·0	2	59·0	1	62·5	1	64·5	1
49·5	2	53·5	2	56·5	2	58·5	1	62·0	1	64·0	4
		53·0	1	56·0	3			61·5	—	63·5	—
		52·5	1					61·0	1	63·0	1
		52·0	1					60·5	—	62·5	1
								60·0	1	62·0	—
								59·5	—	61·5	—
								59·0	—		
								58·5	1		

TABLE XXIII—(*continued*).

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
74·0	1	76·0	1	78·0	1	78·5	2	81·5	1	81·5	2
73·5	1	75·5	3	77·5	—	77·5	2	80·5	1	80·5	3
73·0	3	75·0	—	77·0	—	76·5	3	79·5	1	79·5	
72·5	1	74·5	1	76·5	1	75·5	7	78·5	2	78·5	6
72·0	1	74·0	3	76·0	—	74·5	8	77·5	8	77·5	7
71·5	2	73·5	—	75·5	4	73·5	9	76·5	5	76·5	10
71·0	2	73·0	5	75·0	1	72·5	4	75·5	4	75·5	4
70·5	5	72·5	1	74·5	—	71·5	7	74·5	8	74·5	7
70·0	4	72·0	3	74·0	1	70·5	6	73·5	5	73·5	4
69·5	4	71·5	4	73·5	3	69·5	2	72·5	6	72·5	7
69·0	2	71·0	6	73·0	5	68·5	2	71·5	5	71·5	3
68·5	2	70·5	2	72·5	6	66·5	1	70·5	3	70·5	—
68·0	4	70·0	3	72·0	4					69·5	
67·5	—	69·5	2	71·5	3			68·5	1	68·5	—
67·0	5	69·0	4	71·0	1					67·5	1
66·5	6	68·5	2	70·5	4						
66·0	2	68·0	3	70·0	2						
65·5	2	67·5	2	69·5	5						
65·0	1	67·0	4	69·0	7						
64·5	—	66·5	2	68·5	2						
64·0	—	66·0	1	68·0	2						
63·5	—	65·5	—	67·5	1						
63·0	1	65·0	—	67·0	1						
62·5	—	64·5	—	66·5	—						
62·0	—	64·0	—	66·0	—						
61·5	—	63·5	—	65·5	—						
61·0	1	63·0	1	65·0	—						
				64·5	—						
				64·0	1						

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
86·5	1	88·5	1	94·0	1	100	2	105	1
86·0	—	87·5	4	93·5	1	99	1	104	2
85·5	1	86·5	—	93·0	1	98	—	103	
85·0	—	85·5	3	92·5	—	97	2	102	1
84·5	1	84·5	6	92·0	—	96	1	101	4
84·0	—	83·5	6	91·5	2	95	2	100	1
83·5	—	82·5	5	91·0	1	94	5	99	2
83·0	3	81·5	7	90·5	—	93	3	98	3
82·5	2	80·5	4	90·0	1	92	9	97	4
82·0	—	79·5	3	89·5	3	91	2	96	5
81·5	3	78·5	5	89·0	1	90	8	95	6
81·0	5	77·5	2	88·5	7	89	3	94	10
80·5	3	76·5	—	88·0	2	88	3	93	2
80·0	3	75·5	—	87·5	1	87	1	92	2
79·5	2	74·5	1	87·0	1	86	1	91	2
79·0	3	73·5	1	86·5	3	85	1	90	1
78·5	3			86·0	5	84	1	89	2
78·0	3			85·5	—	83	—	88	—
77·5	4			85·0	3	82	—	87	—
77·0	2			84·5	3	81	—	86	—
76·5	3			84·0	1	80	1	85	1
76·0	1			83·5	2				
75·5	3			83·0	1				
75·0	3			82·5	1				
74·5	1			82·0	1				
74·0	1			81·5	—				
69·5	1			81·0	1				

TABLE XXIII—(*continued*). (*g*) *Weight Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*
11 12	1	14 8	1	16 8	1	18 8	1	20 8	1	21 8	2
11 8	1	14 0	1	16 0	—	18 0	—	20 0	—	21 0	—
11 4	—	13 8	1	15 8	2.	17 8	—	19 8	1	20 8	—
11 0	3	13 0	5	15 0	2	17 0	1	19 0	1	20 0	1
10 12	1	12 8	1	14 8	3	16 8	3	18 8	2	19 8	3
10 8	2	12 0	7	14 0	6	16 0	4	18 0	3	19 0	3
10 4	3	11 8	10	13 8	8	15 8	7	17 8	4	18 8	6
10 0	3	11 0	7	13 0	4	15 0	6	17 0	5	18 0	7
9 12	5	10 8	7	12 8	6	14 8	6	16 8	7	17 8	5
9 8	5	10 0	10	12 0	9	14 0	6	16 0	4	17 0	9
9 4	—	9 8	2	11 8	7	13 8	11	15 8	6	16 8	2
9 0	5	9 0	6	11 0	4	13 0	2	15 0	7	16 0	6
8 12	6	8 8	2	10 8	4	12 8	6	14 8	5	15 8	3
8 8	4	8 0	1	10 0	2	12 0	2	14 0	1	15 0	1
8 4	2			9 8	1	11 8	1	13 8	2	14 8	3
8 0	4			9 0	1	11 0	2	13 0	4	14 0	3
7 12	1					10 8	1	12 8	—	13 8	1
7 8	1							12 0	2	13 0	1
7 4	3							11 8	—		
7 0	1							11 0	1		
6 12	1										
6 8	1										
6 4	—										
6 0	1										

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*	Lb. Oz.	*F*
23 0	1	26 0	1	24 8	2	26 0	2	28 0	1	30 0	1
22 8	—	25 8	—	24 0	2	25 0	2	27 0	1	29 0	1
22 0	—	25 0	—	23 8	1	24 0	5	26 0	2	28 0	2
21 8	2	24 8	—	23 0	2	23 0	7	25 0	5	27 0	5
21 0	2	24 0	1	22 8	3	22 0	12	24 0	5	26 0	5
20 8	3	23 8	1	22 0	5	21 0	6	23 0	9	25 0	7
20 0	3	23 0	—	21 8	4	20 0	5	22 0	7	24 0	7
19 8	4	22 8	—	21 0	6	19 0	9	21 0	6	23 0	10
19 0	6	22 0	4	20 8	3	18 0	1	20 0	6	22 0	4
18 8	7	21 8	2	20 0	5	17 0	2	19 0	6	21 0	7
18 0	4	21 0	2	19 8	1	16 0	1	18 0	—	20 0	2
17 8	2	20 8	7	19 0	6	15 0	1	17 0	1	19 0	2
17 0	4	20 0	6	18 8	2			16 0	—	18 0	—
16 8	2	19 8	4	18 0	3			15 0	1	17 0	1
16 0	5	19 0	5	17 8	3						
15 8	2	18 8	2	17 0	4						
15 0	1	18 0	2	16 8	—						
14 8	—	17 8	7	16 0	1						
14 0	1	17 0	2	15 8	1						
13 8	1	16 8	3	15 0	—						
		16 0	2	14 8	1						
		15 8	—								
		15 0	1								
		14 8	—								
		14 0	1								

TABLE XXIII—(*continued*).

Month 15			Month 18			Month 24			Month 30			Month 36		
Lb.	Oz.	*F*	Lb.	Oz.	*F*	Lb.	Oz.	*F*	Lb.	Oz.	*F*	Lb.	Oz.	*F*
31	8	1	32	0	2	37	0	1	41	0	1	44	0	1
31	0	—	31	0	2	36	8	—	40	0		43	0	
30	8	1	30	0	3	36	0	—	39	0		42	0	
30	0	—	29	0	2	35	8	—	38	0	1	41	0	2
29	8	—	28	0	2	35	0	—	37	0	2	40	0	
29	0	2	27	0	3	34	8	3	36	0	2	39	0	2
28	8	1	26	0	10	34	0	—	35	0	3	38	0	1
28	0	—	25	0	8	33	8	—	34	0	1	37	0	3
27	8	5	24	0	2	33	0	2	33	0	2	36	0	1
27	0	3	23	0	6	32	8	1	32	0	8	35	0	4
26	8	2	22	0	6	32	0	3	31	0	7	34	0	7
26	0	1	21	0	1	31	8	—	30	0	6	33	0	6
25	8	3	20	0	—	31	0	1	29	0	2	32	0	4
25	0	4	19	0	—	30	8	2	28	0	6	31	0	5
24	8	4	18	0	1	30	0	—	27	0	4	30	0	5
24	0	6				29	8	2	26	0	1	29	0	5
23	8	1				29	0	4	25	0		28	0	1
23	0	3				28	8	5	24	0	1	27	0	
22	8	2				28	0	3	23	0		26	0	1
22	0	2				27	8	2	22	0	1	25	0	
21	8	5				27	0	1				24	0	1
21	0	2				26	8	3						
20	8	1				26	0	3						
20	0	1				25	8	2						
19	8	1				25	0	2						
19	0	—				24	8	1						
18	8	—				24	0	—						
18	0	—				23	8	1						
17	8	—				23	0	—						
17	0	—				22	8	1						
16	8	1												

TABLE XXIII—(*continued*). (*h*) *Stem Length Frequency Distributions.*

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
39·0	1	40·5	1	42·3	1	44·1	1	46·5	—	47·5	1
38·7	—	40·2	1	42·0	2	43·8	1	46·0	2	47·0	3
38·4	—	39·9	6	41·7	2	43·5	3	45·5	3	46·5	2
38·1	1	39·6	1	41·4	5	43·2	3	45·0	3	46·0	2
37·8	2	39·3	1	41·1	3	42·9	1	44·5	2	45·5	4
37·5	—	39·0	2	40·8	5	42·6	3	44·0	5	45·0	3
37·2	3	38·7	3	40·5	2	42·3	6	43·5	6	44·5	6
36·9	1	38·4	3	40·2	5	42·0	4	43·0	3	44·0	5
36·6	2	38·1	3	39·9	6	41·7	3	42·5	9	43·5	6
36·3	3	37·8	4	39·6	3	41·4	6	42·0	2	43·0	8
36·0	6	37·5	5	39·3	6	41·1	4	41·5	8	42·5	8
35·7	2	37·2	4	39·0	2	40·8	5	41·0	6	42·0	3
35·4	5	36·9	2	38·7	6	40·5	1	40·5	1	41·5	2
35·1	1	36·6	3	38·4	3	40·2	1	40·0	3	41·0	1
34·8	9	36·3	3	38·1	1	39·9	6	39·5	1	40·5	1
34·5	3	36·0	1	37·8	3	39·6	2	39·0	1	40·0	1
34·2	2	35·7	3	37·5	—	39·3	—	38·5	—	39·5	1
33·9	4	35·4	3	37·2	2	39·0	—	38·0	1	39·0	—
33·6	1	35·1	2	36·9	2	38·7	3	37·5	—		
33·3	2	34·8	—	36·6	—	38·4	—	37·0	1		
33·0	1	34·5	1	36·3	1	38·1	—				
32·7	—	34·2	—	36·0	1	37·8	1				
32·4	1	33·9	—	35·7	—	37·5	—				
32·1	—	33·6	—	35·4	1	37·2	1				
31·8	1	33·3	—								
31·5	—	33·0	1								
31·2	—										
30·9	1										
30·6	—										
30·3	—										
30·0	1										

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
48·0	3	49·5	—	50·0	1	51·5	1	52·0	1	52·5	2
47·5	4	49·0	1	49·5	4	51·0	—	51·5	1	52·0	—
47·0	2	48·5	4	49·0	3	50·5	—	51·0	—	51·5	1
46·5	1	48·0	3	48·5	1	50·0	2	50·5	2	51·0	2
46·0	1	47·5	3	48·0	4	49·5	4	50·0	2	50·5	2
45·5	5	47·0	5	47·5	6	49·0	4	49·5	3	50·0	4
45·0	3	46·5	4	47·0	3	48·5	1	49·0	6	49·5	3
44·5	8	46·0	7	46·5	6	48·0	10	48·5	5	49·0	8
44·0	6	45·5	2	46·0	6	47·5	2	48·0	4	48·5	3
43·5	5	45·0	5	45·5	3	47·0	3	47·5	4	48·0	4
43·0	5	44·5	3	45·0	5	46·5	7	47·0	5	47·5	6
42·5	2	44·0	6	44·5	5	46·0	4	46·5	2	47·0	6
42·0	1	43·5	2	44·0	3	45·5	6	46·0	4	46·5	4
41·5	1	43·0	3	43·5	1	45·0	4	45·5	4	46·0	3
41·0	—	42·5	—	43·0	—	44·5	2	45·0	3	45·5	2
40·5	—	42·0	—	42·5	—	44·0	2	44·5	—	45·0	—
40·0	—	41·5	1	42·0	1	43·5	—	44·0	—	44·5	—
39·5	—	41·0	—	41·5	—	43·0	—	43·5	—	44·0	—
39·0	—	40·5	—	41·0	—	42·5	—	43·0	—	43·5	—
38·5	—	40·0	—	40·5	—	42·0	—	42·5	—	43·0	—
38·0	1	39·5	1	40·0	1	41·5	—	42·0	—	42·5	—
						41·0	—	41·5	—	42·0	—
						40·5	1	41·0	1	41·5	—
						40·0	—			41·0	1

TABLE XXIII *(continued).*

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
55·0	1	54·5	2	57·0	1	58	1	60	1
54·5	—	54·0	2	56·5	—	57		59	1
54·0	—	53·5	1	56·0	1	56	2	58	2
53·5	1	53·0	5	55·5	3	55	6	57	5
53·0	2	52·5	2	55·0	—	54	8	56	2
52·5	3	52·0	2	54·5	1	53	10	55	8
52·0	7	51·5	5	54·0	1	52	3	54	12
51·5	2	51·0	2	53·5	5	51	3	53	7
51·0	3	50·5	4	53·0	4	50	5	52	5
50·5	4	50·0	8	52·5	3	49	1	51	5
50·0	2	49·5	3	52·0	5	48		50	
49·5	0	49·0	5	51·5	—	47		49	
49·0	4	48·5	—	51·0	5			48	1
48·5	3	48·0	1	50·5	5				
48·0	7	47·5	—	50·0	2				
47·5	—	47·0	2	49·5	1				
47·0	1	46·5	—	49·0	1				
46·5	1	46·0	—	48·5					
46·0	—	45·5	1	48·0	1				
45·5	—								
45·0	—								
44·5	—								
44·0	1								

(i) Shoulder Width Frequency Distributions.

Month 1		Month 2		Month 3		Month 4		Month 5		Month 6	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
16·5	1	17·0	3	17·7	4	19·2	2	21·4	1	21·4	1
16·0	—	16·5	4	17·4	2	18·9	1	21·1	1	21·1	
15·5	2	16·0	12	17·1	6	18·6	3	20·8		20·8	1
15·0	10	15·5	7	16·8	3	18·3	2	20·5	1	20·5	2
14·5	10	15·0	17	16·5	10	18·0	3	20·2		20·2	2
14·0	12	14·5	5	16·2	15	17·7	4	19·9	1	19·9	2
13·5	9	14·0	6	15·9	5	17·4	11	19·6	1	19·6	4
13·0	2	13·5	1	15·6	4	17·1	12	19·3	3	19·3	5
12·5	1	13·0	—	15·3	2	16·8	7	19·0	4	19·0	6
12·0	—	12·5	—	15·0	3	16·5	5	18·7	1	18·7	7
11·5	—	12·0	1	14·7	1	16·2	5	18·4	4	18·4	7
11·0	—			14·4	1	15·9	3	18·1	7	18·1	5
10·5	1					15·6	—	17·8	5	17·8	4
						15·3	—	17·5	9	17·5	4
						15·0	—	17·2	6	17·2	1
						14·7	—	16·9	8	16·9	3
						14·4	1	16·6	2	16·6	2
								16·3	2		
								16·0	—		
								15·7	—		
								15·4	1		

TABLE XXIII—*(continued)*.

Month 7		Month 8		Month 9		Month 10		Month 11		Month 12	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
22·0	1	22·3	1	22·3	1	23·5	1	22·3	3	22·9	1
21·5		22·0		22·0	2	23·0	—	22·0	2	22·6	1
21·0	2	21·7	1	21·7	2	22·5	1	21·7	4	22·3	2
20·5	2	21·4	3	21·4	2	22·0	2	21·4	4	22·0	4
20·0	6	21·1	3	21·1	6	21·5	7	21·1	1	21·7	3
19·5	10	20·8	3	20·8	—	21·0	3	20·8	3	21·4	1
19·0	9	20·5	2	20·5	5	20·5	7	20·5	1	21·1	2
18·5	6	20·2	4	20·2	2	20·0	8	20·2	5	20·8	3
18·0	8	19·9	3	19·9	4	19·5	8	19·9	3	20·5	3
17·5	2	19·6	2	19·6	3	19·0	3	19·6	2	20·2	2
17·0	1	19·3	8	19·3	7	18·5	4	19·3	2	19·9	3
16·5	1	19·0	4	19·0	2	18·0	2	19·0	3	19·6	2
		18·7	4	18·7	5	17·5	1	18·7	—	19·3	0
		18·4	3	18·4	5			18·4	—	19·0	1
		18·1	2	18·1	—			18·1	1	18·7	1
		17·8	4	17·8	—			17·8	—		
		17·5		17·5	—			17·5	—		
				17·2	—			17·2	—		
				16·9	1			16·9	1		

Month 15		Month 18		Month 24		Month 30		Month 36	
Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*	Cm.	*F*
23·5	1	25·9	—	24·5	1	25·5	1	25·4	1
23·2	1	25·6	—	24·0	4	25·0	—	25·1	—
22·9	1	25·3	—	23·5	1	24·5	2	24·8	2
22·6	—	25·0	—	23·0	4	24·0	1	24·5	6
22·3		24·7	—	22·5	10	23·5	3	24·2	3
22·0	1	24·4	—	22·0	7	23·0	4	23·9	3
21·7	2	24·1		21·5	2	22·5	8	23·6	4
21·4	6	23·8	1	21·0	5	22·0	8	23·3	3
22·1	6	23·5		20·5	2	21·5	6	23·0	9
20·8	3	23·2	—	20·0	1	21·0	1	22·7	5
20·5	4	22·9	—	19·5		20·5	1	22·4	5
20·2		22·6	3			20·0	1	22·1	3
19·9	3	22·3	9			19·5	—	21·8	3
19·6	4	22·0	2			19·0	—	21·5	2
19·3	2	21·7	3			18·5	—		
19·0	2	21·4	4			18·0	—		
		21·1	8			17·5	1		
		20·8	1						
		20·5	2						
		20·2	2						
		19·9	1						

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ENDOCRANIAL DIAMETERS AND INDICES. A NEW INSTRUMENT FOR MEASURING INTERNAL DIAMETERS OF THE SKULL.

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I. Introduction.

The close correlation between form of skull and of brain is so conspicuous a phenomenon that on superficial consideration it might seem unnecessary to make it the subject of a statistical study. When we bear in mind, however, the large bony protuberances that are characteristic for most of the prehistoric skulls, the correlation at once becomes less obvious. And everybody who has tried to make comparisons as to the shape of the brain-case between prehistoric and more recent races of man, or even between recent but remotely related races, will undoubtedly have felt the need for a statistical study of the matter. Still greater will be the need therefore in an examination of the human race's phylogenetic development, where the shape and size of the brain will always be the central point in the work of research and where also the anthropoids must be taken into consideration. The crania of these latter are, as we know, provided with large muscle ridges. In many works on the craniology of the anthropoids it is just this considerable incongruency between the shape of the brain and the outer form of the cranium that has been the technical stumbling-block. In this connection it is sufficient to mention the well-known work of Stephanie Oppenheim: *Zur Typologie des Primatencraniums* (30). It is sought to avoid the difficulties by selecting such points of measurement as will eliminate as far as possible the influence of the thickness of the cranial wall, but it cannot be said that in this way it has been found possible to secure a technique which is quite comparable with that employed in human craniology.

Schwalbe in his famous work (28) on the calotte of *Pithecanthropus erectus Dubois* mentions that the breadth-length index of this fragment cannot be directly compared with the breadth-length index for recent skulls—likewise on account of the cranium's "Aussenwerke" (outworks), as Schwalbe (after Virchow) aptly designates the Torus supraorbitalis and similar formations. Thus already in this work the main stress is laid upon the measurements and indices found on casts of the cranial cavity, partly on casts of the Pithecanthropus fragment, partly on casts of anthropoid crania examined for comparison*. In the following paper we shall deal with other more special problems connected with the greatly varying thickness of the cranial wall,

* *Note:* Later on, in 1902, Schwalbe took up the same question for renewed treatment in his "Beziehungen zwischen Innenform und Aussenform des Schädels," but unfortunately it is mainly the cranio-cerebral topography that is dealt with, i.e. a field that has only indirect interest for anthropology.

Although the anthropology of the brain has in later years taken a prominent place in the work of research—cf. Ariëns Kappers' book on the development of the nervous system, in which the chapter on "Data for an Anthropology of the Brain" occupies one-third of the number of *pages (12)—only few*

but what has already been said is sufficient to show the need for an instrument that will render possible a direct determination of the three chief diameters of the cranial cavity, without sawing through the skull and without making casts. Over a year ago I therefore addressed myself to T. Gundersen, instrument-maker, Oslo, and submitted the task to him. At that time I was not aware that H. Weinert had in 1925 already described an instrument aiming at the same purpose: "Ein neuer Messzirkel zur Ermittelung von Innenmassen" (30). This instrument, which moreover is also mentioned in the latest edition of Martin's *Lehrbuch der Anthropologie* (16), is later described in connection with a large work on the development of the Sinus frontalis (31). In the same work we are promised a more systematic comparison between the exterior and interior form of the skull—"eine eingehendere Arbeit (liegt) im Manuskript bereits vor" (S. 354), but, so far as I know, this work has not yet been published.

The measurements with the new instrument were already far advanced when I became aware of Weinert's work. Afterwards, in my study of the literature, I met with an article in *Biometrika* dealing in part with the same problems, namely, Hoadley and Pearson's "On Measurement of the Internal Diameters of the Skull in relation I. To the Prediction of its Capacity, II. To the 'Pre-eminence' of the Left Hemisphere" (9). As appears from the title of that paper two main problems are there taken up for discussion. In the first part of the work the correlation of skull capacity with the diameters of the cranial cavity is compared with the correlation of skull capacity and external measurements. In other words, the authors investigate whether it is possible by internal measurements to arrive at a more reliable formula for calculation of capacity. In the second part of the work they take up the old question of the sagittal asymmetry of the cranial cavity and consequently of the brain. The prevailing assumption that the left cerebral hemisphere, both in general and as regards certain functions (*e.g.* the speech centres), occupies a leading position as compared with the right hemisphere is, in Hoadley and Pearson's opinion, not justified by reliable morphologico-biometric observations*. And their investigations

attempts have hitherto been made to deal with the form of the brain in its entirety from an anthropological standpoint. In the literature we can find a number of biometric works on details of the different parts of the brain, but the questions which are of essential importance for anthropology are scarcely even formulated, not to speak of their being answered. And this in spite of the statement made by Klaatsch (14) more than twenty years ago: "Nicht die individuellen Verschiedenheiten der einzelnen Furchen sind es, auf die es ankommt, sondern die Kombinationen von Windungs- und Furchungssystemen, an den Hauptpartien, nicht minder aber auch auf die Gesamtform der grossen Abschnitte und des ganzen Grosshirns." Praiseworthy attempts in this direction have recently been made by the American investigators Weil (29), Pickering (21) and Connolly (7), while Hrdlička's work on the fossae cranii, published as early as 1907 (10), must be mentioned also in this connection.

[* This statement is liable to a misunderstanding on the part of the reader. Pearson and Hoadley found that the length of the right hemisphere in their material was greater than that of the left; an examination of the literature showed that the more reliable investigators found the total right hemisphere to be heavier than the total left. The writers of the paper therefore concluded for the size and weight predominance of the right. But they stated finally that:

> We do not consider that this is in any way opposed to the view that—at any rate for some functions—the left hemisphere may predominate, because we do not believe that volitional predominance is of necessity associated with such gross characters as size and weight, *loc cit.*, p. 117. ED.]

of the longitudinal development of the two halves of the cranial cavity point decisively in the opposite direction. In this investigation they deal with a very large and at the same time extremely homogeneous series (739 adult male skulls of the long 26th—30th Dynasties Egyptian Series in the Biometric Laboratory), but, as will be explained more fully later on, my measurements have nevertheless led to the opposite result.

Hoadley and Pearson make use of Weinert's instrument and, in contrast to him, they give a very full description of the technical difficulties, which are by no means inconsiderable and which, moreover, we shall revert to later. The authors, as stated, investigated only male skulls belonging to one race. By extending the investigation to both sexes and to several races it would, of course, be possible to deal with a greater number of problems. Not only should we thereby be enabled for the first time to provide a systematically prepared body of comparative material for other investigators who wish to draw parallels regarding the main form of the brain in anthropoids and in prehistoric or recent races of mankind, but we should also get an opportunity of discussing the question as to whether—and, if so, to what degree—the craniological racial differences which anthropology has demonstrated are accompanied by similar differences in the shape of the brain. Thus it is the *racial* differences that form the chief object of my investigation, and not, as in Hoadley and Pearson's work, the laws applying for the *homo sapiens* as a genus. In consequence of the constantly proceeding interaction between shape of skull and shape of brain during each individual's development it is impossible—at any rate for the present—to establish whether the point of attack for the fundamental factors of heredity is (*a*) on the brain itself, (*b*) on the bones of the skull, or (*c*) on both brain and skull. But it must be possible to ascertain the final result of this mutual interaction, provided we have sufficient material at our disposal. It is clear that the problem can here be dealt with only in broad outline, as we must limit the investigation to certain races. It is furthermore clear that when speaking of brain form and asymmetry of the brain in this paper, I mean so far as it is revealed through a study of the skull cavity. We have to disregard the influence of the meningeal membranes, but this point will certainly be of minor importance. In concluding from the transverse asymmetry of the skull cavity as to the probable transverse asymmetry of the brain, I have made one more assumption which will be mentioned in due place.

It will be understood that my task is closely connected with the racial significance of *the thickness of the cranial wall*, a matter that has received comparatively little attention in craniological literature. And as the variation in wall thickness undoubtedly leads to the *inter-racial* formulae for calculation of capacity giving relatively imperfect results, by extending the investigation to different races we also are led to examine that question.

Among other problems of immediate interest which Hoadley and Pearson do not deal with may be mentioned one connected with the asymmetry of the skull. As is known, Elliot Smith as early as 1907 (25) pointed out that the left upper

occipital fossa is generally somewhat more strongly developed than the right. A difference in this respect was found in 80 % of a rather large series of Egyptian skulls and the author, as is reasonable, considers a greater development of the left posterior lobe of the cerebrum to be the fundamental cause of this. At the same time this phenomenon is deemed to bear relation to the usually occurring greater development of the sinus transversus on the right side; Elliot Smith assumes that there exists here a simple causal relationship: "The fulness of the left occipital pole seems to be the reason for the dextral bending of the superior longitudinal sinus, the flattening of the right pole *allowing more room for the* bigger sinus on that side" (p. 577). I shall later on take up this question for further discussion. For the present I shall merely mention that M. L. Tildesley (27) from a statistical treatment of a rather large body of material has not been able to confirm Elliot Smith's finding as regards this last-mentioned correlation.

In what has been said above the plan of the present work is made clear. After *having first described the instrument and mentioned the technical difficulties*—which we hope to find considerably diminished in comparison with Weinert's procedure—we shall discuss the absolute measurements and the three main indices calculated therefrom, with special regard to their anthropological significance. In connection therewith I shall take up the question of the form of the brain in prehistoric man and in the anthropoids, where material for comparison is still unfortunately all too scanty. In Section V I shall deal with the asymmetry conditions of the cranial cavity, my task being here somewhat enlarged as compared with Hoadley and Pearson's, and I shall then finally enter upon the question of capacity calculation, where the chief interest turns upon the possibility of establishing an interracial formula which in accuracy can compare with the best intraracial formulae.

I am fully aware that the inclusion of so many problems renders it difficult to deal thoroughly with each separate question and that the non-homogeneous nature of the material constitutes in certain respects a considerable weakness. Where the results must be regarded as only provisional, that fact will appear from the text, and by constantly taking into account the error of differences (statistically determined), I hope to avoid the dangers involved in the *relative* scantiness of the material.

II. Instrument, Technique, Material, Error of Measurement.

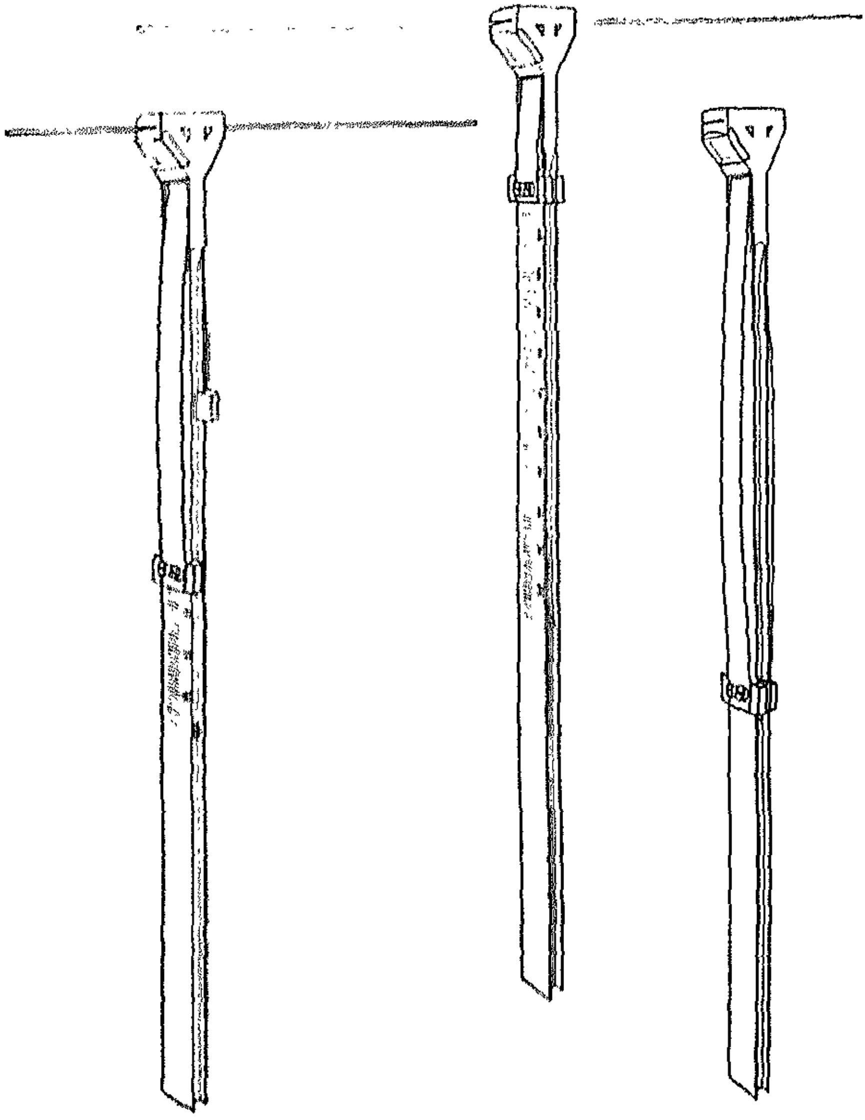
The instrument. For the purpose of comparison it is necessary first to give a very brief description of Weinert's instrument. As shown in Fig. 57 in Weinert's work, this consists of two slightly curved steel rods, the angle between which can be regulated by means of a hinge. *Along these rods run two sliders, with the* same curve as the rods themselves. After the instrument has been introduced into the cranial cavity the rods are fixed in suitable positions and the sliders are pushed forward until they touch the inside of the skull. It is noted how far the sliders

have been pushed forward, and when the instrument is then taken out it is brought back to the same position as before. The distance between the tips of the sliders is thereupon read off by aid of a millimetre-rule. As it is always a question of maximum measurements, this procedure must be repeated a few times until it is certain that the highest value has been found.

In the new instrument two flexible bands of steel are fixed to a metal bar about 12 mm. wide, which is graduated in millimetres on each side. By aid of a small slide the steel bands can be moved outwards and the distance between their tips is found as the sum of the figures read off on the scale behind the sliders. The "head" of the instrument is provided with a small roller, which ensures that the steel bands go straight outwards, and care is of course taken to preclude any lateral deviation. The bands are given a breadth of 10 mm., but their forward ends are rounded off so that the surface of contact with the bone wall is barely 2 mm. wide. The dimensions of the head are made as small as possible, namely, 12 x 24 mm., a size which renders possible an investigation of the smallest human crania, nor will the majority of anthropoids be likely to offer any difficulties in this respect. Of great importance is the fact that the instrument is provided with a mirror, so that we can inspect the interior of the skull beforehand and during the measuring can see where the tip of the band touches upon the wall. It is of course necessary to light up the cranial cavity by means of a small electric lamp, which is introduced through the foramen magnum. As the skull during measurement lies with base upwards, this presents no difficulties. The skull must be well and firmly supported on a bag filled with mustard-seed or the like. It is sufficient to provide the instrument with one mirror, as the posterior point of measurement can always be viewed directly. I here leave out of consideration the measurement of internal breadth and height, since as regards these dimensions it is of minor importance to be able to see the points of contact (Fig. 1 (a), (b), (c)).

I shall now make a comparison between Weinert's instrument and the one here described. As I personally have no experience of Weinert's instrument, and as his own particulars about the technique are scanty, I must refer to Hoadley and Pearson's above-mentioned paper in *Biometrika*. Speaking of Weinert's publication in the *Anthrop. Anzeiger* (30), they say on p. 85: "No directions for the use of the instrument are provided with it nor in the above paper, only in the latter we find a diagram illustrating what is apparently the median sagittal plane of the skull with the instrument set for measuring the length. In this diagram the maximum foraminal length is almost as great as the distance from the basion to the tip of the dorsum sellae; with such a magnitude of the foramen it is possible to place the hinge of the measuring circle (Weinert's callipers) entirely within the foramen as in the diagram. With crania with more moderate foramina we have not found this possible, and greater difficulty may then arise in taking the internal measurements; it is less easy to grope for the maximum diameter." From what has been said it will appear that the advantages of the new *instrument are the*

following: (1) there is never any difficulty in entering through the foramen magnum, (2) we can see the points of measurement*, (3) the measurements can be read off direct from the instrument, and (4) we therefore have less difficulty in



(a) (b) (c)

Fig. 1.

adjusting it to obtain the maximum value. On the other hand, it must naturally be admitted that the flexible steel bands have to be handled with a certain degree of caution. Especially, they must never be pushed forward with any great force;

[* Is this wholly in accord with the statement in the last lines of the previous paragraph? ED.]

the ends should just barely touch the walls of the skull. The fear naturally existing, that the bands will bend on meeting a solid wall, proves to be quite unfounded*. Numerous control measurements made with a view to this latter possibility have shown that the instrument's own limit of error is less than ½ mm in a range of 0 to 200 mm., and as regards our material—and for anthropological material altogether—greater dimensions are hardly likely to come into consideration. It is of course obvious that this error of the instrument itself—which may here be left out of account, since all measurements are read off to the nearest millimetre—must not be confused with the actual error of measurement in determination of the diameters of the cranial cavity, where difficulties occasioned by the peculiar nature of the material also come into consideration. As regards this point, I may refer to the last portion of the present section.

Also as regards the *technique of measurement* in general we may in all essentials refer to Hoadley and Pearson's article. Although this latter was, as stated, unknown to the author when the present material was being investigated, the mode of procedure, so far as it appears from the directions given, has been in principle the same. Of the greatest importance is Hoadley and Pearson's remark on pp. 86–87, that the *maximum longitudinal diameter* of the cavity does not lie in the median plane—of which we can easily convince ourselves by examining sectioned skulls—but about 1 cm. to the side thereof, so that the measuring points correspond to the most prominent parts of the frontal lobe or of the posterior lobe. "The reader who will examine a sectioned skull will find that the median sagittal section is ridged to a greater or less extent both anteriorly and posteriorly; we have the crest for the attachment of the falx cerebri, the crista galli of the ethmoid, and the internal occipital protuberance and the sagittal ridge associated with it" (p. 86). For this reason it is practically impossible to take an internal longitudinal measurement in the *median* sagittal plane, and apart therefrom such a measurement would be of no interest, since the purpose of the measurements of the cranial cavity is to eliminate the influence of the bone thickness. But if we are thus obliged to abandon the obtaining of a longitudinal diameter in the median plane, we must on the other hand be careful constantly to have the points of contact for measurement lying strictly sagittally; in other words, the perpendicular distances from the points of contact to the median plane represented by the crista sagittalis must be equal in front and behind, so far as it is possible to secure this by estimation with the eye. To the above directions given by Hoadley and Pearson it should furthermore be added that we must always take care that the posterior point of measurement stands free of the furrow of the *sinus sagittalis* or the latter's deflection into the sinus transversus. As the sinus transversus is generally more strongly developed on the right side than on the left, it would lead to our

* In order to strengthen the bands the author got Mr Gundersen to make a new instrument where two flexible bands are riveted together on each side of the bar. This new instrument, which has not been used as yet, looks exactly like the *former*, but the bands are very much strengthened. It might perhaps be possible to combine three bands on each side, thus securing still more strength without reducing their flexibility. [If the "*fear*" be "quite unfounded" why this need to strengthen the bands? Ed.]

getting too large a mean value for the longitudinal diameter of the right side of the cavity if we were not observant of this source of error. For the sake of completeness it should further be mentioned that—as follows moreover as a corollary from what has been said above—the measuring points for the *maximum breadth* must lie in a transverse plane and, of course, be strictly symmetrical on the two sides. But otherwise we must also here—as in every maximum measurement—take into account the individual peculiarities of the skull. It will very soon be discovered that the maximum breadth of the cavity generally lies in a plane very near to the centre of the occipital condyles, a regularity which is probably not merely fortuitous but due to mechanical factors associated with the upright manner of walking. (Cf. the anthropoids, in which the maximum breadth of the cranial cavity is constantly found considerably *in front of* this plane, provided the skull is orientated in the Frankfurt horizontal. See also Mollison's work, *Statik des menschlichen Schädels* (18). An examination of prehistoric skulls with a view to this question would not be without interest.)

It remains to be mentioned that the two height measurements, *internal basion height* and *internal opisthion height*, have been found by aid of a vertically placed rod, graduated in millimetres, which is shown in Fig. 2 (p. 96). The skull is orientated in the ear-eye plane by means of Mollison's craniophor. We are aware that the *furrow of the sinus sagittalis is included in these measurements, but at this place* the furrow is so slightly developed and probably varies so little that we may be permitted to disregard its influence. Of the two height measurements most importance is assigned to the first-mentioned, because it comes natural to compare it with the external measurement, basion-bregma height. With a view to the determination of capacity we have also calculated a third height, namely, internal *ear-bregma height*, which has been found indirectly by subtracting from the external measurement the individually determined thickness of the os parietale. This latter value was, it is true, measured at the lower posterior corner of the bone (right side and about 2 cm. above the sutura mastoidea)—through the foramen magnum by aid of ordinary callipers—and not at the bregma, but the difference is probably negligible.

Material. As to the material I need only say that it belongs to the Anatomical Institute in Oslo, the Principal of which, Professor K. E. Schreiner, has not only been so kind as to place the skulls at my disposal, but has even allowed me to make use of material assembled for another purpose. This latter concerns Norwegians and Lapps. A systematic treatment of the whole material of Lapp skulls, far more comprehensive than that dealt with here, now lies completed from Schreiner's hand (22). Although I thus, from considerations of time, have not included all the skulls of Lapps and Norwegians that stood at my disposal, yet I have not made any selection, except in so far as only adult skulls in good preservation have been employed. The two groups consisting of Australians and Maori will form the foundation for another work by myself, already commenced, on the craniology of the Polynesians.

Originally the material consisted of 328 skulls of Norwegians, Lapps, Maori and Australian aborigines. After I had finished the study, I measured 41 Eskimos to check some of the more doubtful results. The means, the σ's and v's, for all

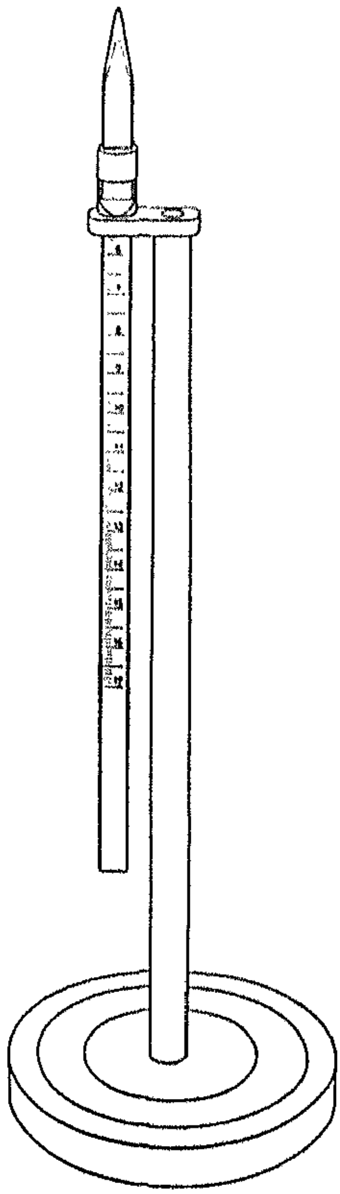


Fig. 2.

the measurements are given in Table XIII. I have also measured 10 anthropoid skulls, likewise belonging to the Anatomical Institute. These few skulls will serve to demonstrate that the same technique may be applied to this material.

It now remains to set forth a few figures illustrating *the accuracy of measurement*. What is of interest here is the error of measurement in determination of the length of the cranial cavity, where the difficulties are greatest. As described in detail by Hoadley and Pearson (p. 111), this error can be determined in the following manner: We take two sets of independent measurements and calculate $\frac{1}{2}(L_L + L_R)$ for both. If the individual observations in the first and second measurements are designated by z_1 and z_2 respectively, then $\sigma_{z_1 - z_2} \times 0{\cdot}67449$ will give "the probable error of a single measurement."

In our own material, this value was found to be 0·7406 mm. ($\sigma = 1{\cdot}098$ mm.) for 324 skulls, since 4 skulls were not available for the second measurement, whereas Hoadley and Pearson find the error to be 1·2043 mm., with use of Weinert's instrument.

We see that the error of measurement is reduced considerably (by as much as 38 %), which is not an improbably great reduction when we consider the advantages of our instrument, especially the fact that the points of measurement can be seen. With greater practice I regard it as possible to reduce the probable error to less than 0·5 mm.

III. THE ANTHROPOLOGICAL IMPORTANCE OF INTERNAL MEASUREMENTS AND INDICES. THICKNESS OF THE ROOF OF SKULL. VARIABILITY AND CORRELATION.

As mentioned in the Introduction, our first object is to investigate to what extent the craniological differences of race expressed by the three main diameters, and the indices computed on the basis thereof, are accompanied by parallel differences in the form and size of the brain. Table I gives a survey of the most important absolute measurements. Here, as in the other tables, m denotes "standard error," i.e. $\frac{\sigma}{\sqrt{n}}$. Where "probable error," i.e. $0{\cdot}67449m$, is used, this fact is specially indicated.

Norwegians and Lapps are relatively well represented, while the values for Maori and Australians can only be taken as provisional, without this fact having necessarily any great influence as to the questions here dealt with. It is seen that the difference in sex comes well into evidence, both in the external and in the internal values. The maximum length of the cranial cavity (length of the cerebrum) is in women from over 3 to about 7·5 mm. less than in men, while the breadth is from about 4·5 to about 6 mm. and the height from about 3 to 6 mm. less in women than in men. These differences are from 2 to 9 times their standard errors, only the difference in height of brain-box in Australians being statistically very doubtful (less than twice the standard error of difference). Further, it is clearly evident from the *internal* measurements that the Australians' brain is not only very narrow but also extremely short, while on the other hand it shows considerable height, not only relatively but also absolutely regarded. Before proceeding to illustrate these points by means of indices we shall see what Table I has to tell about the *bone thickness* in our 4 groups. Subject to the reservation that an

TABLE I.

External measurements	Norwegians $M \pm m$ (standard error)		Lapps $M \pm m$		Maori $M \pm m$		Australians $M \pm m$	
	♂, $n=81$	♀, $n=76$	♂, $n=63$	♀, $n=55$	♂, $n=16$	♀, $n=14$	♂, $n=13$	♀, $n=10$
1. Maximum length	187·70 ± ·65	179·28 ± ·67	179·17 ± ·86	170·96 ± ·71	186·81 ± 1·04	180·00 ± ·95	183·30 ± 1·21	173·40 ± 1·60
2. Maximum breadth	142·34 ± ·56	135·37 ± ·45	146·52 ± ·62	142·05 ± ·39	137·88 ± 1·15	132·64 ± ·97	130·85 ± 1·27	124·60 ± 1·10
3. Basion-bregma height	132·31 ± ·73	125·34 ± ·54	129·05 ± ·57	124·05 ± ·38	136·63 ± 1·15	130·21 ± 1·03	135·02 ± 1·26	130·70 ± ·65
Internal measurements								
4. Maximum length of cavity $\frac{R+L}{2}$	172·25 ± ·59	166·09 ± ·58	168·35 ± ·53	161·57 ± ·63	171·21 ± 1·06	168·04 ± 1·15	169·77 ± 1·67	170·13 ± 1·62
5. Maximum breadth of cavity ...	134·06 ± ·49	128·22 ± ·45	140·05 ± ·61	135·24 ± ·56	131·31 ± 1·12	126·57 ± 1·03	123·32 ± 1·14	117·93 ± ·96
6. Internal basion height... ...	127·67 ± ·72	121·03 ± ·52	124·86 ± ·54	119·62 ± ·55	131·44 ± ·94	125·57 ± 1·25	129·77 ± 1·42	125·30 ± 1·34

indirect determination of bone thickness can yield only approximately correct figures* and that the upper point of measurement for No. 6 does not correspond exactly to bregma (No. 3), ½ (2—5) will give the bone thickness at euryon and 3—6 the approximate thickness at bregma.

The values given below (Table II) are, however, the means of the individually computed differences (not the differences between the means)†, whereby it becomes possible to take the standard errors also into account. For comparison with the computed bregma thickness, the thickness of the os parietale (right side) at the lower posterior corner is also given, measured directly through the foramen magnum.

TABLE II.

	Norwegians		Lapps		Maori		Australians	
	♂ $n=81$	♀ $n=76$	♂ $n=63$	♀ $n=55$	♂ $n=16$	♀ $n=14$	♂ $n=13$	♀ $n=10$
Thickness at euryon	4·15 ±0·09	3·57 ±0·10	3·24 ±0·11	3·41 ±0·12	3·28 ±0·14	3·04 ±0·20	3·73 ±0·30	3·50 ±0·22
Thickness at bregma (computed)	5·25 ±0·20	4·32 ±0·18	4·16 ±0·19	3·44 ±0·23	6·19 ±0·53	4·64 ±0·47	7·15 ±0·64	4·80 ±0·77
Thickness of os parietale (direct)	5·33 ±0·17	4·38 ±0·13	4·30 ±0·15	3·98 ±0·14	6·88 ±0·35	5·70 ±0·32	7·39 ±0·56	5·70 ±0·32

We see that the difference of the directly and the indirectly determined thickness of the os parietale is *in general* small, being in 4 groups less than ¼ mm. In Lapp women, Maori men, Maori women and Australian women the difference is, however, not inconsiderable. In the two latter groups the computed bregma thickness is probably about 1 mm. too small, which may easily be explained by some irregular contours, where immediately behind the bregma the cranial roof has risen relatively much.

In the next place it is seen that the skull-cap is, as might be expected, somewhat thicker at the bregma than at the euryon and that the male skulls are in general a little thicker than the female. As regards the racial differences revealed in Table II, it is natural to compare Lapps with Norwegians and Maori with Australians. As might *a priori* be expected, the values for Lapps are lower than the values for Norwegians and those for Maori in general lower than for Australians. The difference in the former case may be regarded as statistically certain. If we,

* *Owing to the small dimensions here involved we must, in an examination aiming directly at the* thickness of the skull roof, demand a greater accuracy of measurement than 1 mm., and measurements preferably taken with an instrument specially constructed for the purpose. Such an instrument was (according to Martin) designed by Péan as early as 1896.

[† *If the two are not the same, this must be a fault of arithmetic, not of measurement.* ED.]

for example, compare the thickness at the euryon in male Lapps (3·24) and in male Norwegians (4·15), the difference is seen to be 0·91 mm., or 6·5 times the standard error of the difference (0·14). The difference between Maori and Australians, on the other hand, is statistically less well founded, but it can hardly be doubted that, in a larger body of material the difference would be well beyond the statistical error. It would be of interest to ascertain whether the Australian skull, the thickness of which at the *bregma* is very considerable, is *not* thicker at the euryon than the skulls in an average European material, as seems to appear from Table II.

Our figures accord well with what others have found: Hoadley and Pearson state the thickness at the euryon *to be* 3·52 *mm. and at the bregma** 5·46 *mm.* (male Egyptians). Todd (28), as the average of 448 sectioned "white crania" (male), found by direct measurement 3·56 mm. at the euryon and 5·88 mm. at the vertex, figures which, together with ours, confirm on the whole Broca's table of the values that must be deducted from the external breadth and external height in order to arrive approximately at the internal dimensions (see Martin, p. 726). Only we must make the reservation that the variation—as Todd points out—is so *considerable that the table can be applied only to mean values*.

We now come to *the thickness of the frontal bone at the glabella*. In order to *find this value we must from the difference between external and internal length* $\left[L - \frac{L_R + L_L}{2}\right]$ deduct the thickness of the os occipitale at the posterior point of measurement $\left[\frac{O_R + O_L}{2}\right]$. This latter dimension, which has been found directly, is likewise to be seen from the following table. (We have to disregard the fact that the anterior point of measurement for the internal length does not exactly correspond to the glabella-point from which the external length has been measured, but this circumstance will certainly have no disturbing influence on the results.)

TABLE III.

	Norwegians		Lapps		Maori		Australians	
	♂	♀	♂	♀	♂	♀	♂	♀
Thickness of os occipitale	4·21 ±0·11	3·96 ±0·13†	3·01 ±0·13	3·00 ±0·14	4·89 ±0·24	4·51 ±0·25	5·19 ±0·39	5·42 ±0·31
Thickness of os frontale at glabella	11·22 ±0·33	9·15 ±0·27	8·11 ±0·40	6·33 ±0·34	10·71 ±0·52	7·43 ±0·45	11·35 ±0·83	11·58 ±0·74

[* Hoadley and Pearson measured the bone-thickness *vertically* above the basion, which is not the bregmatic thickness. Ed.

† This is the value given in Table XIII, but if the entries in that table from which it is deduced be correct, it should be 9·23, which will modify the value for *Norwegian – Lapps* given in the following paragraph from 2·82 to 2·90. Ed.]

Taking first the occipital bone, we again note that the Lapp skulls are the thinnest and the Australian skulls the thickest, although the racial differences are not particularly pronounced. As before, the difference between Norwegians and Lapps is statistically well established, being for the male skulls 1·20 mm., or 7 times the standard error of difference, while the difference between Maori and Australians is statistically doubtful. In males it is 0·30 mm. ($m_{Dif.}$ = 0·48 mm.). Further there is seen to be a strikingly small difference between the two sexes. In Lapps and Australians the difference found actually goes in the opposite direction to what should be expected. Quite different is the situation as regards *the glabella-thickness. Here the difference between the sexes goes in the same* direction in all 4 groups, and lies between 1·78 and 3·28 mm., $m_{Dif.}$ lying between 0·43 and 1·11. The sexual difference is from 2 to 5 times the standard error of difference. And if we further compare Norwegians with Lapps and Australians with Maori, we find, to take the male skulls first, a difference between the former two of 3·11 mm., and between the latter two of 3·64 mm., that is to say, respectively, 6 and 4 times $m_{Dif.}$ (0·52 and 0·98). The female skulls give quite similar figures: Norwegians – Lapps 2·82 mm., or 6 times $m_{Dif.}$ (0·43); Australians – Maori 4·15 mm., or 5 times $m_{Dif.}$ (0·87). My figures for the glabella-thickness thus also accord very well with earlier findings: Todd's material (male "white crania") gave 11·26 mm. (cf. our Norwegians) and Hoadley and Pearson's male Egyptian skulls showed a difference between external and internal length of 15·10 mm., whereof the author's estimate ⅘ to be frontal and ⅕ occipital, that is to say, a glabella-thickness of about 9 mm. (lying between Lapp and Maori).

The marked sexual and racial difference in the glabellar region which I have here demonstrated is not surprising. In the first place, a well-developed torus supraorbitalis is one of the features to which most importance is attached in determination of sex, and as regards the Australians, Mollison (17) has emphasised the great development of the glabella particularly in contrast to the Maori. The flat glabella in Lapps is likewise a well-known phenomenon, which moreover we have seen is accompanied by generally thin skull-bones. In connection with my investigations of the thickness of the skull-wall at the euryon and bregma we can, however, from the figures here given draw certain conclusions of interest for our main subject: the correlation between form of brain and of skull.

I. Since on going over to internal measurement the length is reduced much more than the breadth and the breadth, again, more than the height, all three internal indices will probably on the average show larger values than the corresponding outer indices. The differences here concerned recur again in all 8 groups and they are so considerable that this rule may probably be deemed to apply, if not to all, at any rate to most races. The increased breadth-length index has repeatedly been observed in casts of the cranial cavity, cf. Schwalbe (23), who as early as 1899 could state, on the basis of examination of individual skulls: "Es zeigt sich, dass der Schädelinnenraum des Erwachsenen, wie bei der Mehrzahl der Affen, einen grösseren Index besitzt, als der aussen gemessene Schädel" (S. 36).

II. Even though the same tendency is present in all groups, it is not likely to make itself equally evident in all of them. In the races which show the greatest development of the glabella region the increase in the breadth-length and height-length indices will be most pronounced—in our material especially in the Australians.

By a more searching examination of my figures it would undoubtedly be possible to draw other and more detailed conclusions as to the probable "brain-indices," but we can attain the same object in a simpler manner by a direct comparison of external and internal indices, as in Table IV.

TABLE IV.

External indices	Norwegians ♂	Norwegians ♀	Lapps ♂	Lapps ♀	Maori ♂	Maori ♀	Australians ♂	Australians ♀
Breadth-length index	75·88 ±0·33	75·56 ±0·31	81·90 ±0·52	83·17 ±0·48	73·87 ±0·73	74·07 ±0·66	71·38 ±0·51	74·82 ±0·64
Height-length index	70·46 ±0·40	69·46 ±0·34	72·05 ±0·39	72·62 ±0·39	73·15 ±0·46	72·35 ±0·50	74·21 ±0·65	75·40 ±0·60
Height-breadth index	93·08 ±0·58	92·67 ±0·52	88·19 ±0·55	87·39 ±0·47	91·23 ±1·22	93·26 ±1·13	103·90 ±1·22	105·04 ±0·82
Internal indices								
Breadth-length index	77·50 ±0·37	77·24 ±0·29	83·46 ±0·50	83·77 ±0·46	77·12 ±0·71	75·38 ±0·61	75·35 ±0·62	75·27 ±0·30
Height-length index	73·80 ±0·45	73·84 ±0·39	74·45 ±0·45	74·72 ±0·42	75·94 ±0·73	74·71 ±0·52	78·67 ±0·46	80·61 ±1·21
Height-breadth index	94·82 ±0·64	94·48 ±0·54	89·40 ±0·56	89·37 ±0·51	93·28 ±1·22	93·30 ±1·14	104·48 ±1·48	107·05 ±0·71

On passing to internal measurement the breadth-length index is increased by from 0·60 to 3·97 index-units, the height-length index by from 2·10 to 5·21 and the height-breadth index from 0·05 to 2·01 index-units. These figures accord very well with what might be expected from our knowledge as to skull thickness. As regards the last-named index the increase is sometimes insignificant, and in individual cases there will of course frequently be found a deviation in the opposite direction, but for the material as a whole the increase must, in spite of the relatively large mean errors, be regarded as a fact, since the same tendency is met with in all 8 groups. Meanwhile, if we wish to examine whether the anthropological differences in shape are lessened or further increased on passing to internal measurements, it is chiefly the first two indices that will be of importance. The breadth-length index has in the highly dolichocephalic Australians increased

by 3·97 and 3·35 index-units for males and females respectively, in the Maori by respectively 3·25 and 1·29 index-units, in Norwegians (mesocephalic on the border of dolichocephaly) by 1·68 and 1·68 index-units, and in the brachycephalic Lapps by only 1·56 and 0·60 units respectively for males and females. Herein we see a distinct tendency to reduce the racial differences, once again as a result of the formation of the glabella,—the increase being also as a rule greater in males than in females.

As regards the height-length index the figures suggest a somewhat different situation. Named in order of succession from the partly hypsicephalic Australians to Norwegians on the border of chamaecephaly the following values are found for the index increase: Australians respectively 4·46 and 5·21, Maori 2·79 and 2·39, Lapps 2·40 and 2·10, Norwegians 3·43 and 3·88. The racial difference between Norwegians and Lapps is lessened on adopting internal measurement, while as between Australians and Maori it is increased. The strongly developed glabella of the Australians thus partially conceals the considerable relative development in height of brain. Had our material included a chamaecephalic type with flat glabella (if such a type exists), as well as a hypsicephalic group with strongly developed glabella, the increase in racial difference would have become still more clearly manifest.

The results of these investigations of shape of skull as contrasted with shape of brain* can thus be provisionally summed up as follows:

1. All three brain-indices have shown increase as compared with the indices for the skull. The difference is greatest in the height-length index, less in the breadth-length index and least in the height-breadth index.

2. The racial differences are constantly less clearly manifest in the breadth-length index of the brain than in that of the skull. The general validity of this rule is of course contingent on a rather strong negative (spurious) correlation between glabella-thickness and external length-breadth index. In my own material I have found the following figures for the coefficient of correlation (r) between the properties mentioned:

Norwegians, male, $-0·2960 \pm 0·101$; female, $-0·2126 \pm 0·110$.

Lapps, male, $-0·2856 \pm 0·117$; female, $-0·2621 \pm 0·126$.

Interracially this negative correlation will undoubtedly be still more pronounced.

3. On comparing a hypsicephalic race having a relatively strongly developed glabella with a chamaecephalic race with relatively flat glabella, the height-length index for the brain will show greater racial differences than the skull index.

Rules 1 and 3 need no further substantiation from our knowledge of the thickness of the skull roof at the glabella, euryon and bregma.

If we are to have any hope of penetrating more deeply into our problem of the relation between shape of skull and shape of brain, we must also take into considera-

* Let it be mentioned again—to avoid misunderstandings—that "shape of brain" is here used as synonymous with shape of skull cavity.

tion *the conditions of variation and correlation.* What is the chief interest for us here is to compare the distributions of the external with the corresponding internal dimensions (Table V).

Hoadley and Pearson find that the internal length is "considerably more variable than the external length, while in the case of the breadths and heights the difference in the coefficients is negligible, being well within the probable error." In our material there is found no conformity to rule in this respect; sometimes the internal length, sometimes the internal breadth or height is more variable than the external. The difference as a rule is slight, and for each group taken separately does not offer much to build upon (cf. the "standard error" of standard deviation), but such difference as is found generally tends in the direction that the internal measurement is more variable. If we keep to the variation coefficients and count together the + deviations for the brain, we find that 32 out of 48 point in the direction of the brain being more variable and 15 out of 48 in the opposite direction, while 1 shows no difference. On examining the absolute measurements and the indices separately the following figures are obtained: 15 +, 8 − and one 0 deviation for the absolute measurements, and 17 + and 7 − for the indices, and of these latter 7 − deviations five fall to the internal breadth-length index, while the height-length index of the brain is in all eight groups more variable than that of the skull. In all cases it can be concluded from these figures that at any rate *we must reckon with an (intraracial) variation of shape and size of the brain which is not less than that of the skull.* This result is surprising in so far as we should have expected the considerable variation in thickness of the skull to make itself strongly felt. An investigation of the correlation between internal measurement and bone thickness supplies the explanation; for Hoadley and Pearson have shown that the bone thickness is negatively correlated with the corresponding internal measurement, $r_{L_{\beta}(L-L_{\beta})}$, thus giving − 0·36448. A similar correlation is found in our material.

As appears from Table VI (p. 106) the thickness as represented by the glabella-thickness is in all four groups negatively correlated with the length of the cranial cavity. The accompanying errors are, it is true, so large that we must content ourselves with a summary consideration, but for the material as a whole hardly any doubt can prevail as to the significance of this negative correlation, especially in view of Hoadley and Pearson's investigations. We also find that the euryon-thickness in two groups is negatively correlated with the internal breadth, while in two groups the correlation is without significance.

The most natural explanation of these negative correlations is the following: either (*a*) the roof of the skull exerts an inhibitive influence on the growth of the brain, or (*b*), conversely, a relatively considerable growth of the brain will have a reducing effect on the thickness of the cranial roof, or (*c*) both factors come into play at the same time. It is not possible on the basis of the available data to penetrate more deeply into the causative relations. And neither is an investigation of the correlation between bone thickness and the three main indices of the brain particularly illuminating in this respect. As is seen, all these correlations (bone

TABLE V. *Variation of External and Internal Measurements.*

External measurements	Norwegians				Lapps				Maori				Australians			
	♂		♀		♂		♀		♂		♀		♂		♀	
	$\sigma\pm$	v	$\sigma\pm$	v	$\sigma\pm$	v	$\sigma\pm$	v	$\sigma\pm$	v	$\sigma\pm$	v	$\sigma\pm$	v	$\sigma\pm$	v
Maximum length	5·86 ± ·460	3·12	5·81 ± ·471	3·24	6·79 ± ·679	3·79	5·23 ± ·499	3·06	4·14 ± ·732	2·22	3·55 ± ·671	1·97	4·36 ± ·855	2·38	5·33 ± 1·192	3·07
Maximum breadth	5·03 ± ·390	3·53	3·94 ± ·320	2·91	5·00 ± ·445	3·41	4·39 ± ·419	3·09	4·58 ± ·810	3·32	3·64 ± ·688	2·74	4·57 ± ·896	3·49	3·47 ± ·775	2·78
Basion-bregma height	6·55 ± ·515	4·95	4·75 ± ·385	3·79	4·52 ± ·403	3·50	4·29 ± ·409	3·46	4·61 ± ·814	3·37	3·84 ± ·726	2·95	4·55 ± ·892	3·35	2·99 ± ·669	2·29
Breadth-length index	2·95 ± ·232	2·89	2·69 ± ·218	3·56	4·16 ± ·371	5·08	3·65 ± ·348	4·39	2·90 ± ·513	3·93	2·48 ± ·469	3·35	1·85 ± ·363	2·59	2·61 ± ·584	3·63
Height-length index	3·64 ± ·286	5·17	2·95 ± ·239	4·22	3·09 ± ·275	4·29	2·89 ± ·276	3·99	1·85 ± ·327	2·53	1·86 ± ·351	2·57	2·31 ± ·453	3·11	1·91 ± ·427	2·53
Height-breadth index	5·26 ± ·413	5·56	4·53 ± ·367	4·89	4·34 ± ·387	4·92	3·49 ± ·333	3·99	4·86 ± ·859	4·90	4·23 ± ·799	4·30	4·39 ± ·861	4·22	2·59 ± ·579	2·47
Internal measurements																
Maximum length $\frac{R+L}{2}$	6·24 ± ·490	3·62	5·90 ± ·479	3·55	6·56 ± ·584	3·90	4·65 ± ·443	2·88	4·25 ± ·751	2·48	4·37 ± ·826	2·60	3·85 ± ·755	2·35	5·35 ± 1·196	3·42
Maximum breadth	4·44 ± ·349	3·31	3·90 ± ·316	3·04	4·88 ± ·435	3·48	4·18 ± ·398	3·09	4·46 ± ·788	3·40	3·87 ± ·731	3·06	4·25 ± ·833	3·44	3·04 ± ·680	2·59
Internal basion height	6·48 ± ·509	5·10	4·49 ± ·364	3·71	4·64 ± ·413	3·72	4·10 ± ·391	3·39	3·74 ± ·661	2·87	3·93 ± ·743	3·13	5·12 ± 1·004	3·98	4·25 ± ·950	3·38
Internal breadth-length index	3·29 ± ·258	4·24	2·57 ± ·208	3·33	3·96 ± ·353	4·74	3·43 ± ·327	4·09	2·85 ± ·504	3·70	2·39 ± ·452	3·17	2·24 ± ·439	2·97	2·85 ± ·637	3·79
Internal height-length index	4·02 ± ·316	5·44	3·43 ± ·278	4·65	3·57 ± ·318	4·80	3·14 ± ·299	4·20	2·90 ± ·513	3·82	1·96 ± ·370	2·62	3·45 ± ·677	4·39	3·83 ± ·856	4·75
Internal height-breadth index	5·79 ± ·455	6·11	4·69 ± ·380	4·96	4·46 ± ·397	4·99	3·75 ± ·358	4·20	4·87 ± ·861	4·91	4·26 ± ·805	4·29	5·33 ± 1·045	5·10	2·24 ± ·501	2·09

TABLE VI. *Correlations.*

Correlated Pair	Norwegians ♂ $n=81$	Norwegians ♀ $n=76$	Lapps ♂ $n=64$	Lapps ♀ $n=55$
Glabella-thickness & internal length	−·3111±·098	−·2801±·105	−·2374±·119	·1067±·132
Glabella-thickness & internal breadth	−·2112±·101	−·2107±·108	·2284±·119	·0751±·134
Glabella-thickness & internal basion-height ...	−·0431±·111	−·1685±·112	+·0468±·120	−·0858±·134
Euryon-thickness & internal breadth	+·0079±·111	−·2145±·110	·1617±·123	−·0762±·134
Glabella-thickness & internal breadth-length index	+·0870±·102	+·1027±·113	+·0435±·126	+·0743±·134
Glabella-thickness & internal height-length index	+·1791±·108	+·0121±·115	+·2098±·120	+·0386±·135
Glabella-thickness & internal height-breadth index	+·0912±·110	+·0256±·115	+·1521±·123	·0304±·135
Euryon-thickness & internal breadth-length index	−·0186±·111	−·0281±·115	+·0891±·122	·0087±·135
Internal breadth-length index & capacity ...	−·0357±·111	−·2879±·105	−·1319±·123	·0038±·135
Internal height-length index & capacity	+·0210±·111	−·2227±·109	−·2238±·120	−·0110±·135
Internal height-breadth index & capacity ...	+·1393±·109	−·0131±·115	−·0216±·126	−·0751±·131
Internal length & internal breadth	+·4118±·092	+·4886±·087	+·1318±·124	+·0749±·134
Internal length & internal height	+·2303±·105	+·3583±·100	+·2135±·120	+·1299±·133
Internal breadth & internal height	+·1083±·110	+·0597±·111	+·0276±·126	+·0721±·134

thickness being represented at glabella and euryon) are very slight, only three of them exceeding $2m_r$. Neither has an examination of the correlation between brain-shape and *capacity* led to positive results. It is true that the correlation between capacity and breadth-length index is negative in all four groups, but only in one case, Norwegian females, is it of statistical value in relation to m_r. We must content ourselves with the negative conclusion that, *judging from the material here dealt with*, neither the cranial capacity nor the thickness of the roof of the skull has any essential predetermining influence on the form of the brain. Meanwhile we must bear in mind that an investigation of intra-racial correlations in a comparatively small body of material, such as mine, cannot be decisive as regards this question. I may mention in this connection that my Australian skulls were strikingly thin at the euryon, and the same was the case with the 41 Eskimo skulls (18 male, 23 female) which were examined after the main material had been dealt with (see Table XIII, no. 6 (*c*)). As we know, the Eskimo skulls are also excessively dolichocephalic. If it should prove to be the case that we are here confronted by a general law, namely, that the skulls of the dolichocephalic races are relatively thin at the euryon, then it must be permitted to draw the conclusion that a pressure from

the side wall of the skull is an essential factor in the shaping of the dolichocephalic brain. This conclusion is of all the more interest because it diverges greatly from the generally adopted answer to a much disputed question.

The last three series of coefficients for the relation between the three main diameters of the cranial cavity likewise confirm very well Headley and Pearson's results. There is a very strong positive correlation between maximum length and maximum breadth and likewise between maximum length and internal basion-height, but between breadth and height the correlation is practically zero. The positive correlation between the two first-named values suggests the presence of general factors of size (genetic or epistatic) affecting both diameters. From the fact that the correlation is by no means absolute we may conclude that there exist at the same time special shape-determining factors which act more strongly on one diameter than on the other, and such shape-determining factors create an antithesis in the relation between height and breadth of brain which is here sufficiently strong to mask the effect of the general size-factors.

IV. SHAPE OF THE BRAIN IN PREHISTORIC MAN AND IN THE ANTHROPOIDS.

It is natural to compare the above figures for the three main indices of the brain with the values previously found from casts of the cranial cavity in prehistoric specimens. As early as 1899 Schwalbe (23) compared the internal breadth-length index in the Pithecanthropus, Neanderthal-Kalotte, Spy I and Spy II with that of some isolated recent skulls. None of these prehistoric specimens showed exceptional values: Pithecanthropus 80·0 (in measurements by Dubois and Weinert (31)), Neanderthal-Kalotte 79·1 (after other measurements 78·2 – 80·0), Spy I 75·9, Spy II 81·3*. Further can be mentioned: La Chapelle-aux-Saints 78·3 (Boule and Anthony (4), p. 134), Rhodesia 79·4 (Hrdlička (11), p. 121), Gibraltar 79·0 (Hrdlička, p. 169, "close to"), Ehringsdorf 79·0 (Hrdlička, p. 239, after Weidenreich's "reconstructed 1925 skull"), La Quina 73·8 (Hrdlička, p. 291), Le Moustier 76·7 (Hrdlička, p. 301, after Weinert's reconstruction "close to").

Of the prehistoric finds here mentioned *only the La Quina has had a relatively narrow brain as compared with our recent material.* All the other figures lie above the average for our Australians and not far from the mean values for recent mesocephalic Europeans.

Schwalbe also established the fact that the brain of *the anthropoids* was relatively broad, and the figures given by him for breadth-length index have on the whole been confirmed by later investigators: Chimpanzee 85·5, Gorilla ♀ 84·5, Orang ♀ 96·0. Unfortunately Weil's (29) technique is so different from Schwalbe's and ours that we cannot make use of his measurement of "endocranial casts," and Connolly's (7) excellent paper deals with fixed brains, but it can nevertheless be regarded as established that the orang-outang is farthest removed from man as regards this index, next comes the chimpanzee, while in the gorilla the index partly coincides

* It is not quite clear from Schwalbe's work whether the two Spy figures have been based on casts of the cranial cavity. At any rate, according to Schwalbe they give "die Innenform des Schädels."

with the mean figures for recent and prehistoric man. The considerable variation found especially in the gorilla (different species?) has been pointed out by several investigators, Keith (13), Bolk (2), Harris (8). Keith (p. 78) gives the following figures for endocranial breadth-length index:

Chimpanzee ♂ 84·5, ♀ 84·1 (var. 79 – 92, n = 19).
Gorilla ♂ 78·8, ♀ 81·3 (var. 72 – 88, n = 47).

In our own material we have found the following values:

Chimpanzee: 81·9.
Gorilla: 79·5 – 82·8 – 84·2 – 82·4 – 82·2.
Orang: 85·7 – 90·2 – 87·8 – 84·1.

None of the three great anthropoids lies entirely outside of the range of variation for man!

With respect to *relative height*, it is a well-known fact that most prehistoric skulls have a height-length index which is far below that of recent human skulls and in a manner forms a bridge between homo sapiens and the great anthropoids. Here, however, we still lack reliable figures which could give us an idea of the situation as regards the brain.

For the *La Chapelle-aux-Saints* skull Boule and Anthony give an internal height-length index of

$$\left(\frac{\text{hauteur basilo-bregmatique} \times 100}{\text{longueur max.}}\right) = 68{\cdot}1.$$

Rhodesia. 70·6 (computed from Hrdlička's figures: endocranial basion-bregma-height 120, maximum length of cerebral hemisphere 170).

Gibraltar. 70·6 ("close to," computed from Hrdlička's figures for greatest length of brain 163 and basion-bregma height less thickness of frontal bone near bregma 115).

Le Moustier. 75·0 (after measurements of Weinert's reconstructed model skull, internal length 176, internal basion-height 132).

It must be deemed surprising that these values also lie within the range of variation for our recent material. As regards the Chapelle-aux-Saints skull the index is, no doubt, very low, but no fewer than 6 of our 81 male Norwegian skulls show still lower values. Neither can the internal height-breadth index be said to be exceptional in the above-mentioned skulls: La Chapelle-aux-Saints 86·9, Rhodesia 88·9, Gibraltar 89·8, Le Moustier 97·8. In this respect we find that 4 of the aforesaid 81 recent crania lie below the lowest figure, that for the La Chapelle-aux-Saints.

As regards the anthropoids Connolly's measurements of fixed brains have shown that the difference in height-length index between chimpanzee, gorilla and orang is not particularly great:

5 male chimpanzees	36·1
6 male gorillas	38·1
13 male orangs	36·5

But these figures do not of course admit of direct comparison with ours, where the technique is quite different. For our own (unfortunately scanty) anthropoid material we have found the following values for internal height-length index or $\left(\frac{\text{internal basion-height} \times 100}{\text{greatest length of skull-cavity}}\right)$, the skull being orientated in the Frankfurt plane:

Chimpanzee	69·8
Gorilla	73·2, 77·6, 71·7, 66·4, 74·6
Orang	79·0, 83·8, 81·6, 72·9

All these figures lie within the range of variation for our recent human material! We see how necessary it is also in case of the anthropoids to procure a large body of material for comparison before we can utilise the general shape of the brain for elucidation of the phylogeny. I consider it futile to proceed to speculations respecting the phylogeny on the basis of one or more indices, so long as we have not a large body of material at our disposal and so long as we have not through investigations of correlation secured a more certain basis for correct evaluation of the individual figures. So far as my figures reach, it is permissible to conclude that most prehistoric men, and especially those hitherto best investigated, had a *mesocephalic* brain with a breadth-length index which does not differ materially from that found in many recent races of man, while the height-length index was probably considerably under the mean values for recent races. This increased development in relative height of the brain will accordingly be the real problem so far as regards shape. With respect to the anthropoids, my investigations have confirmed the prevailing assumption of brachycephaly, especially for the orang-outangs, whereas we find to our surprise that the height-length index is not particularly low, provided the technique employed is kept constantly the same. Investigation of a large body of material is here highly necessary, in order to ascertain the mean figures for the different species of anthropoids.

Furthermore we can assert that the difference in general shape of brain between anthropoids and prehistoric and recent races of men is not nearly so great as craniological investigations had hitherto led us to suppose, but it is possible that future investigations will render it justifiable to attach very great importance to the difference that is found to exist. It will contribute very much to a more precise definition of the problems if we consider brain-form and skull-form to some extent separately. It must be due to a confusion between the two—or perhaps rather to deficiency in the figures—when Bolk (8) designates "the primitive races of man," including the Homo Neanderthalensis, as "dolichocephalic," while at the same time the primates are stated to be "generally brachycephalic."

If we consider the shape of the skull cavity—and otherwise we shall not be able to draw comparisons—then the primates are certainly brachycephalic as compared with man, but most of the prehistoric human skulls which have hitherto been examined must be deemed to be mesocephalic and not dolichocephalic as stated by Bolk.

V. Endocranial Asymmetry.

It might *a priori* be considered a comparatively simple matter to ascertain which hemisphere was the heavier, or which showed the greater development in length, but nevertheless Hoadley and Pearson quote a large number of papers dealing with this problem without its being possible to say that their excellent survey of the literature makes the matter very much clearer. Boyd, Broca, Topinard, Ogle, Inglessis have published data pointing to "pre-eminence of the left hemisphere." Investigations made by Thurnam, Wagner, Braune and Wilde point in the opposite direction, while Bastian and Reichard leave the question unanswered, the former after having occupied himself therewith for a number of years.

According to Braune and Wilde the greater development of the right cerebral hemisphere is associated with a preponderance of the left cerebellar hemisphere. According to Inglessis the brain is "asymmetrical, but this asymmetry is confined chiefly to the occipital part of the hemispheres, and here the left hemisphere predominates" (Hoadley and Pearson, p. 106), in strong contrast to Broca, who found the preponderance of the left side to be confined to the frontal lobe (cf. the speech centre).

In the following we shall mainly compare our results with those of Hoadley and Pearson, whose technique is the same as ours.

As average of 729 male Egyptian skulls, they find

$$\bar{L}_R = 171{\cdot}0446, \qquad \sigma = 5{\cdot}9551,$$
$$\bar{L}_L = 170{\cdot}0501, \qquad \sigma = 5{\cdot}6908,$$

that is to say, a difference of 0·9945 mm. in the length of the two hemispheres, with preponderance on the right side, a difference which, as the authors show, cannot be due either to "error of measurement" or to "random sampling." "In other words, if we may take our sample of over 700 male Egyptian skulls as representative, the right hemisphere is in the main of greater length, and therefore probably of greater capacity than the left" (p. 111). As stated in the Introduction our figures point in quite the opposite direction:

TABLE VII (*a*).

	Norwegians		Lapps		Maori		Australians	
	♂ $n=81$	♀ $n=70$	♂ $n=63$	♀ $n=55$	♂ $n=16$	♀ $n=14$	♂ $n=18$	♀ $n=10$
L_R	171·94	165·78	167·84	161·42	170·31	167·57	163·30	155·90
L_L	172·54	166·46	168·25	161·71	172·13	168·50	164·15	156·90
$L_L - L_R \pm m_{diff.}$	0·60 ±0·194	0·68 ±0·154	0·41 ±0·159	0·29 ±0·187	1·82 ±0·385	0·93 ±0·415	0·85 ±0·359	1·00 ±0·379
$L_L - L_R \pm$ probable error	0·60 ±0·131	0·68 ±0·104	0·41 ±0·107	0·29 ±0·126	1·82 ±0·260	0·93 ±0·280	0·85 ±0·242	1·00 ±0·256

In all 8 groups the left half of the cranial cavity is on the average somewhat longer than the right. The differences are from 2·3 times to 7·0 times their probable errors, using the formula

$$0{\cdot}67449\,(\sigma^2_{L_R} + \sigma^2_{L_L} - 2r_{L_R L_L}\,.\,\sigma_{L_R}\,.\,\sigma_{L_L})^{\frac{1}{2}}.$$

The differences are thus without doubt statistically reliable, not only for the material as a whole, but also for most of the separate groups. The correlations are obviously rather close:

Norwegians ♂: + 0·961, Norwegians ♀: + 0·974. Lapps ♂: + 0·983, Lapps ♀: + 0·957. Maori ♂: + 0·944, Maori ♀: + 0·937. Australians ♂: + 0·946, Australians ♀: + 0·979.

On measuring a second time the following figures were found, which we give in parentheses in order to avoid confusion:

TABLE VII (*b*).

	Norwegians		Lapps		Maori		Australians	
	♂	♀	♂	♀	♂	♀	♂	♀
L_R	(171·83)	(165·92)	(167·60)	(161·25)	(170·63)	(167·29)	(163·58)	(155·90)
L_L	(172·65)	(166·68)	(168·33)	(161·55)	(172·13)	(168·21)	(163·75)	(156·80)
$L_L - L_R$	(0·82)	(0·76)	(0·73)	(0·30)	(1·50)	(0·92)	(0·17)	(0·90)

We see that the concordance with the first set of measurements is very good. Not only do we find again in all groups the greater length of the left half of the cranial cavity, but there is also an unmistakable correlation between the lateral deviations in the first and the second set of measurements. Only in the small group of female Australians is the difference materially reduced, and here one skull was lacking at the last measurement. The conclusion we can derive from our material as compared with Hoadley and Pearson's therefore is that the asymmetry of the two hemispheres as regards length is *not, as has hitherto been supposed, a characteristic for man as a species, but varies from group to group, probably with preponderance of the left side in most cases**. This view is confirmed by certain other investigations which Hoadley and Pearson do not mention in their survey of the literature. Thus Hrdlička (10), who has measured the length of the three cerebral fossae on sectioned skulls, found the sum of the lengths to be greater for the left side: "In seven of the ten series of human adults the average of the sum of the lengths of the three cerebral fossae on the left exceeds that of the right

* It is here assumed that Hoadley and Pearson's technique corresponds exactly to ours. If we omit to light up the cranial cavity and to inspect the posterior point of measurement, this latter will sometimes fall in the furrow for the sinus transversus, which is known to be more developed on the right side. Cf. in this connection what is said later about the correlation between development of the fossa occipitalis sup., depth of the sinus transversus and length of the hemispheres. Cf. also especially Elliot Smith's article (25), which also actually concerns Egyptian material.

side by from 1 to 3 mm. There is therefore a clear, though small, excess of fossal length on the left in the mass of human adult crania" (p. 199). This excess on the left side the author finds to be due to the greater length of the left *posterior* fossa (posterior lobe), just as Inglessis found in hardened brains. (Cf. Hrdlička, p. 193: "The most striking feature of the above data is the evidence that in all the groups and series the left (posterior) fossa is the longer, which is the reverse of what was observed with the middle and especially with the anterior cavities." The author does not seem to have determined the error of measurement, nor does he add σ and σ_M to his mean figures (the article was published as early as 1907), but the material is comparatively so large (92 skulls) that it must be said to have a certain weight in this connection. A paper recently published by Connolly (7) points in the same direction. On 120 hardened brains the lengths of the two hemispheres were measured by aid of an apparatus designed by the author himself and the accuracy of the results was checked by renewed measurements, partly with change of technique. The material comprised Negroes (37 males, 9 females), Germans (36 males, 10 females), miscellaneous Whites (18 males) and Philippines (10 males). In 4 of the 6 groups (100 cases) the left hemisphere was from 1 to 3 mm. longer than the right, and only 1 group (10 cases) showed a difference of 1 mm. in the opposite direction. Of the whole material the right hemisphere was longer in 31·7 %, the left was longer in 50·0 %, and both hemispheres equally long in 18·3 % (p. 483). My 41 Eskimos are in accordance with the rule: 18 males $\overline{L}_R = 176·06$, $\overline{L}_L = 177·00$, thus giving $\overline{L}_L - \overline{L}_R = 0·94$ mm.; and 23 females $\overline{L}_R = 167·26$, $\overline{L}_L = 168·22$, thus giving $\overline{L}_L - \overline{L}_R = 0·96$ mm.

All in all, Hrdlička's, Connolly's and my own investigations, together embracing 581 cases, must be said to afford adequate proof of the greater length of the left hemisphere in certain groups of the human population. But it would of course be premature to draw from this the conclusion that the left half of the brain has in the same groups been heavier than the right. In measuring the greatest breadth of the cranial cavity it was strikingly noticeable that the distance from the median line—this latter being determined by the eye, with the longitudinal diameter of foramen magnum and the palatine suture as guiding line—was generally greater on the right than on the left side. Owing to the rough and ready determination of the median plane the figures given below make no claim to particularly great accuracy, but they are recorded here as a first attempt to determine the transverse asymmetry of the cranial cavity.

$\overline{R} - \overline{L}$, maximum distance from the "median line" of the cranial cavity:

Norwegians ♂ (81)	Norwegians ♀ (76)	Lapps ♂ (63)	Lapps ♀ (55)
3·12 mm.	1·76 mm.	2·16 mm.	1·73 mm.
Maori ♂ (16)	Maori ♀ (14)	Australians ♂ (13)	Australians ♀ (10)
1·06 mm.	1·93 mm.	2·31 mm.	2·80 mm.

If we are allowed to assume that the median plane of the skull cavity—as determined above—corresponds to the falx cerebri, or what comes to the same, that the medial surfaces of the cerebral hemispheres are nearly symmetrical to this

plane, then the above-mentioned figures go to show that the right hemisphere has been the broader in these groups.

In order to verify these observations I have measured the corresponding maximum distance from the median plane on horizontal contours through the glabella and upper edge of the orbit. As can easily be ascertained, a contour at this level will go very near to the euryon, and a possible deviation will in any case not be of importance for our problem. Such contours of 166 Lapp skulls (91 male and 75 female) were placed at my disposal by Professor Schreiner. They had been prepared for another purpose after P. and F. Sarasin's system by aid of Martin's Kubus-craniophor and diagraph. The following figures show that the same excess on the right side again appears. The measurements have been taken with an exactitude of $\frac{1}{4}$ mm.

Males	*Females*
Standard error	Standard error
$\bar{R} = 75{\cdot}17 \pm 0{\cdot}32$ mm.	$\bar{R} = 73{\cdot}10 \pm 0{\cdot}36$ mm.
$\bar{L} = 74{\cdot}20 \pm 0{\cdot}28$ mm.	$\bar{L} = 71{\cdot}33 \pm 0{\cdot}32$ mm.
$\bar{R} - \bar{L} = 0{\cdot}97 \pm 0{\cdot}238$ mm.*	$\bar{R} - \bar{L} = 1{\cdot}77 \pm 0{\cdot}290$ mm.*

The distance out to the right side is both for males and females greater than out to the left. In males the difference is $4m_{\text{dif.}}$ and in females $6m_{\text{dif.}}$. If the skulls in which the right side is the broader are designated by +, we get among the 91 males, 52 +, 29 − and 10 zero-deviations; in the 75 females, 51 +, 11 − and 13 zero-deviations. Of the total diagram-material a + deviation to the right is found in 62 %, to the left in 24 %, and no demonstrable difference in 14 %. Technical errors may of course render the above results illusory, but the probability thereof becomes less according as the material is larger and more data can be obtained pointing in the same direction. It is natural to compare my findings with the type contours which have been published in *Biometrika* since 1911. As regards the technique we may refer to Benington and Pearson's article: "Cranial Type Contours," Vol. VIII (1)†. On glabella horizontal sections the parallel No. 7 will give the most adequate expression for the greatest breadth of the skull and a comparison of the distance from the median line on each side affords very good information as to the average transverse asymmetry. Unfortunately some of the series are rather small, and if we examine n which gives the number of skulls, we see that two figures are sometimes given for one series. Obviously a single skull of the series in question has been defective with the result that the right or left side could not be measured. If the series be small, this circumstance will render it useless for our purpose. I have therefore in Table VIII omitted the series consisting of less than 10 skulls, in cases where the right or left side was not measured on one or more of the skulls.

[* If the correlations between R and L are anything like those shown on p. 111, these probable errors are curiously high. ED.

† The reader should examine this carefully as much turns on the manner in which the median lines are found in the horizontal and transverse contours. Von Bonin's technique is not that of the Biometric Laboratory. ED.]

Altogether the table embraces 42 series. These show a + deviation to the *right in 32 cases and to the* left in 10 cases, and on adding our own investigations *of the dimensions of the cranial* cavity the numbers will be 40 and 10 respectively, as we here disregard the *craniograms of our* Lapp material, since this has in part been included in the 40. Moreover, *as already mentioned, these* craniograms also proved to be in accordance with the rule. Of course, *the* distribution in some of the smaller groups will be fortuitous, while *technical errors* undoubtedly may be responsible for some of the minor deviations (Congo Batetelu, *Eskimo, Morant's and* Woo's Egyptian crania, Bantam-Batavia and Mittel Java, as well as Negro crania *from* Teita Hills), but *for the larger series with average* difference in breadth of more than 1 mm. it is very improbable *that this source of* error has played any important rôle. This assumption can easily be confirmed by comparison with the corresponding "transverse vertical contours" which are likewise shown in Table VIII. It is seen that on these latter contours + deviation *in 27 cases goes to* the right, in 14 cases to the left, while in 1 case the deviation is 0. If we confine the examination *to the groups in* which the number of skulls equals or exceeds 20 and where the excess in breadth on one side is equal to or more than 1 mm. on transverse contours, we find that in 14 out of 15 cases the excess is on the right side and only in 1 case on the left, which altogether is a good proof that the technical errors have not been so great that the rule (the *preponderating + deviation on the right side)* loses its general validity, even though we cannot venture to attach any decisive weight in this question to the individual investigations. It must therefore be said to be of minor importance that 37 Basque crania or a single group of 31 Eskimo skulls show + deviation to the left, but it is of greater interest to note that out of 9 English groups there is only 1 (Hythe female crania) which on horizontal contours diverges from the rule. Or, if we prefer also to take into account the transverse contours, we find that of 18 deviations 16 go to the right and only 2 to the left (male Anglo-Saxons, transverse contours, and female Hythe crania, horizontal contours). Of Egyptian material we have 6 series which without exception, both on transverse and on horizontal contours, are in accordance with the rule, whereas the Oriental material (Burmese, Tibetan A and B, Nepalese, Hindu) strikingly often show a + deviation to the left. I thus obtain confirmation for my earlier expressed assumption that cerebral asymmetry is probably a racially determined characteristic (associated with the quantitative proportion of left-handedness in the population?), although for hitherto investigated European material it can be regarded as a general rule that the left hemisphere is the longer and the right side of the cranial cavity is the broader*.

In this connection I must mention a paper published by Woo (34) which to some extent confirms my findings. Woo has, like Hoadley and Pearson, investigated a very large body of Egyptian material—887 male skulls from the 26th to the 30th Dynasty—with a special view to asymmetry conditions. His technique differs

* The plausible conjecture that the greater breadth of one side represented a compensation for the greater length of the other has not been confirmed by our material. The correlation $(L_L - L_R$ and $B_R - B_L)$ is practically equal to 0 and is in any case without statistical significance. $(r = -0{\cdot}060 \pm 0{\cdot}035.)$

TABLE VIII.

Author	Biometrika Vol.		Horizontal Contours								Transverse Contours					
			♂				♀				♂			♀		
			n	R	L	$R-L$	n	R	L	$R-L$	R	L	$R-L$	R	L	$R-L$
Benington, C. ...	VIII	Congo cr., *a*	50	67·3	66·7	0·6	—	—	—	—	66·6	65·4	1·2	—	—	—
		,, *b*	20	68·9	66·3	2·6	—	—	—	—	66·8	66·5	0·3	—	—	—
		,, Batetelu	41	68·7	68·8	−0·1	—	—	—	—	67·7	66·4	1·3	—	—	—
		Egyptian cr., XXVI to XXX D.	100	68·2	67·5	0·7	—	—	—	—	66·5	65·5	1·0	—	—	—
		Eskimo cr.	31	66·0	66·3	−0·3	—	—	—	—	66·5	67·1	−0·6	—	—	—
		Guanche cr.	14	70·0	68·8	1·2	—	—	—	—	67·7	66·0	1·7	—	—	—
		English cr., 17th century ...	100	69·8	69·4	0·4	—	—	—	—	68·6	68·5	0·1	—	—	—
Thomson, E. Y. ...	XI	Moriori cr.	33	69·1	68·3	0·8	21	67·1	65·8	1·3	68·6	67·2	1·4	65·7	64·4	1·3
Tildesley, M. L. ...	XIII	Burmese cr., A	44 (43)	70·7	70·2	0·5	39	68·5	66·6	1·9	69·7	70·0	−0·3	66·9	66·5	0·4
		,, B	7	69·6	67·6	2·0	17	67·3	65·7	1·6	67·4	68·3	−0·9	65·8	65·8	0·0
		,, C	8	69·0	68·0	1·0	18	64·1	62·9	1·2	67·4	67·8	−0·4	63·6	62·7	0·9
Morant, G. M. ...	XIV	Tibetan cr., A	17*	68·7	70·5	−1·8	—	—	—	—	68·4	69·8	−1·4	—	—	—
		,, B	15	68·1	70·6	−2·5	—	—	—	—	69·2	68·8	0·4	—	—	—
Morant, G. M. ...	XVI	Oriental cr., Tibetan A	35	68·8	70·2	−1·4	—	—	—	—	67·3	67·6	−0·3	—	—	—
		,, Nepalese	46	67·0	66·1	0·9	—	—	—	—	63·2	63·7	−0·5	—	—	—
		,, Hindu	10	63·3	64·2	−0·9	—	—	—	—	62·2	63·4	−1·2	—	—	—
Morant, G. M. ...	XVIII	Anglo-Saxon cr.	37	70·5	67·8	2·7	39	66·0	65·0	1·0	68·8	69·2	−0·4	66·2	66·1	0·1
Morant, G. M. ...	XVII	Egyptian cr. I D.	(31)	66·9	66·6	0·3	—	—	—	—	66·1	64·9	1·2	—	—	—
Hooke, B. G. E. ...	XVIII	17th-century Londoners	70 (69)	70·5	69·2	1·3	65 (64)	66·4	65·3	1·1	69·5	67·4	2·1	65·4	63·5	1·9
Stoessiger, B. N. ...	XIX	Badarian cr. [Egypt]	34	65·9	63·1	2·8	21	64·7	62·6	2·1	63·6	62·6	1·0	63·7	61·1	2·6
Morant, G. M. ...	XXI	Basque cr.	37 (36)	69·2	70·6	−1·4	—	—	—	—	68·9	69·7	−0·8	—	—	—
Woo, T. L.	XXII	Sedment cr. [Egypt, IX D.] ...	38	67·4	67·1	0·3	30	65·5	64·5	1·0	66·3	65·8	0·5	63·9	63·2	0·7
Bonin, G. von ...	XXIII	Dajaks	30	68·8	67·0	1·8	—	—	—	—	66·2	68·0	−1·8	—	—	—
		Bantam-Batavia	54	69·7	69·5	0·2	—	—	—	—	69·9	67·3	2·6	—	—	—
		Mittel-Java	30	70·2	69·9	0·3	—	—	—	—	72·1	68·0	4·1	—	—	—
		Tagalen	29	68·7	67·6	1·1	—	—	—	—	67·7	66·6	1·1	—	—	—
		Aëtas	30	70·5	71·8	−1·3	—	—	—	—	71·5	69·5	2·0	—	—	—
Bonin, G. von ...	XXIII	Easter Islander cr.	12	65·3	66·5	−1·2	—	—	—	—	63·4	68·1	−4·7†	—	—	—
Morant, G. M. ...	XXIII	Spitalfields cr.	116 (115)	71·7	70·2	1·5	22 (21)	67·1	65·9	1·2	70·6	70·0	0·6	66·1	65·7	0·4
Kitson, E.	XXIII	Negro cr., Teita hills	46	64·7	64·3	0·4	49	62·8	62·4	0·4	63·7	63·8	−0·1	61·4	61·6	−0·2
Stoessiger and Morant	XXIV	Hythe cr. (English)	112	73·6	72·4	1·2	87	69·3	70·0	−0·7	72·4	72·3	0·1	69·5	68·8	0·7

* These 17 Tibetan A are part of the 35 Tibetan A below.

† "The fact that the Easter Islanders show an asymmetry of the skull reverse to that seen in the Anglo-Saxons may be noticed in passing" (p. 265).

very much from mine, as only external measurements were taken and then chiefly confined to the individual bones. While paying full regard to possible sources of error, the author concludes that a preponderance on the right side is more frequent than on the left. Out of 25 measurements 16 show a + deviation to the right and, as appears from his tables, this excess is most pronounced for the os parietale and the os frontale. On the other hand it is noteworthy that the occipital arc from the lambda to the asterion "*is very significantly greater on the left*," and although the two other occipital measurements give contrary results, it therefore seems rather premature when the author on the basis of the excess on the right side in the frontal and parietal region draws the conclusion that the right hemisphere is more strongly developed (p. 332)*. To my former conclusion I shall here merely add that the question of a general preponderance of the right or of the left hemisphere can hardly be decided by cranial measurements, and in carrying out direct weighing—which will certainly be adopted very largely also in the future—the anthropological character of the material ought not to be entirely disregarded. It would be of great interest to examine the larger lobes more systematically than has hitherto been the case, and judging from my material I should expect to find *right-sided preponderance of the parietal and left-sided preponderance of the occipital lobe, here again with certain reservations for anthropological reasons.*

Relation between fossa occipitalis sup., sinus transversus dexter et sinister and length of the hemispheres.

So far as can be seen, Elliot Smith was the first to point out that the upper posterior fossa, which so clearly marks off the caudal pole of the posterior lobe of the brain, is in general deeper and more fully formed on the left side than on the right (25). It was natural to connect this finding with the main outflow of the venous blood through the right sinus transversus. As mentioned in most text-books on anatomy and as can easily be ascertained on inspecting a series of crania, the furrow is more strongly developed on the right side than on the left—Le Double (15)—while the foramen jugulare also shows distinct difference on the two sides. I must agree with Elliot Smith that it is natural to suppose—but only as a provisional hypothesis—that the greater development of the left posterior lobe is *the cause* of the main outflow of the venous blood being to the right. And the next step in the process of deduction really follows of itself: the greater development of the left posterior lobe is in some way or other connected with the predominating use of the right hand, so that by examining the fossa occipitalis superior and the sinus transversus in prehistoric man and in anthropoids we can obtain interesting information as to the phylogenetic development of "right-handedness." Elliot Smith has later reverted to this question on several occasions, for instance in an article on "Right and Left-handedness in Primitive Men" (26), where it is stated, *inter alia*, that the Pithecanthropus skull-cap and the London skull both show inverse occipital asymmetry, for which reason both of them are

[* It is in the frontal and parietal regions on the *left* side that functional centres have been authoritatively placed and are asserted to give predominance in weight and size to that side. Ed.]

thought to have belonged to left-handed individuals. Here, however, we already meet with the first difficulties, since it is found that the sinus transversus in both of these cases runs to the right. For there at once arises two questions of fundamental importance in this connection: (1) Does the deeper and more marked fossa occipitalis sup. sinistra give expression for a greater development of the left posterior lobe? (2) Can there be demonstrated any correlation between the craniological characters, the development of the fossa occipitalis sup. and the depth of the sinus transversus?* To the first question Elliot Smith originally thought he could give an affirmative answer. For he found on hardened brains that the area striata, i.e. "the area of cortex containing Gennari's stria," extended far more widely over the outer convex surface of the hemisphere on the left side, and he interpreted this finding as a sign that the left area strata was much more diffused, but later in 1907 he adds—"which subsequent investigation has shown to be erroneous" (p. 574), and he describes in this connection an attempt to measure the area striata directly on 10 brains. As the results did not confirm the original assumption, he advances the theory that it is the greater development of certain other cortical regions which causes the right area striata to fold itself largely on the medial surface of the hemisphere, and he mentions specially as a possibility Flechsig's large parietal association centre, which is stated to be much larger on the right side. This, too, is of course merely a conjecture, but it seems to be worth mentioning on the background of my own and Woo's investigations respecting cranial asymmetry. That the left region of vision is not more developed than the right has subsequently been established by Cohn and Papez (6). Measurements of the fissura calcarina on 100 hardened brains—and according to Campbell (5) the length of the fissura calcarina gives expression for the extension of the region of vision—gave 8·8 cm. for the right side and 7·9 cm. for the left. In no less than 72 % the furrow was longer on the right side, as against 10 % longer on the left. These measurements are unaccompanied by information as to the influence of error of measurement, but the authors verify their results by direct planimetric measurements, which led to practically the same result. I shall pass from this question without bringing into the discussion the difficult problem of "left-handedness." I refer to the above-mentioned paper by Cohn and Papez, and above all to Woo and Pearson's (32): "Dextrality and Sinistrality of Hand and Eye," where the latter authors' reserved attitude to the question finds expression in the accompanying motto: "Keine Erklärung ist aber doch besser als eine oder mehrere irrthümliche."

Further Elliot Smith believed he could answer affirmatively the second question. He states that in a series of Egyptian skulls (which we may certainly be permitted to assume to have been large in number) about 80 % show marked asymmetry, with left-sided preponderance of the fossa occipitalis sup. (25, p. 576), and at the same place it is said that in the cases where there was distinct divergence the main

* A third and not less important question is whether the inverse occipital asymmetry really is associated with left-handedness. This question Elliot Smith claims to be able to answer in the affirmative and he refers to a comparative investigation made by Wood-Jones in 1925 (35) respecting the length of the bones of the upper arm.

direction of the transverse sinus was to the left. No statistical *proof* of this relationship is furnished, however, and Tildesley, who *took up the same question in 1920*, arrived at the following negative result: "One has to conclude therefore that in this series—127 Burmese skulls—there is no significant association between outer occipital asymmetry and the direction in which the main occipital sinus is turned when its course is changed from longitudinal to lateral" (27, p. 257). Meanwhile Tildesley points out that this *external* examination of the squama occipitalis is not decisive for the question discussed by Elliot Smith, which thus still remains unanswered, at any rate from a statistical standpoint. It is on this point that our material is calculated to cast some light.

The characters here concerned are in their manifestation of a quantitative nature, *although it is not inconceivable that the underlying biological causative factors* are alternative (dextrality or sinistrality). Meanwhile it is practically impossible *quantitatively to measure the size of the posterior fossa or of the sinus transversus.* On the first examination of the material I have noted the depth (size) of the sinus transversus in three categories, *r*, *l* and *?*, and on the second investigation noted the development of the posterior fossa in the same manner. The cases where the one groove was deeper, but the other of greater breadth, are assigned to the ?-group. As regards the statistical treatment of the data, I have arranged my material into 9 cell-systems and to these applied "the root-mean-square contingency," the formula for which is

$$\phi^2 = \frac{1}{n} \Sigma \frac{(n_{pq} - v_{pq})^2}{v_{pq}},$$

where n_{pq} stands for observed and v_{pq} for theoretically expected frequencies. This dimension is compared with

$$\phi^2 \text{ for zero-correlation} = \frac{c-1}{n} \pm \cdot 67449 \sqrt{\frac{2c}{n}},$$

where c is the number of "cells," and further, for the sake of completeness, C_2 (coefficient of mean-square contingency) has been reckoned out from the formula

$$C_2 = \sqrt{\frac{\phi^2}{1 + \phi^2}}.$$

I have found it justifiable, on the basis of our earlier investigations of asymmetry, which revealed the same conformity to rule in all groups, to deal with the material collectively, without regard to sex or race.

We will first examine the relation between *length of hemisphere* and *size of fossa occipitalis sup.*

Greater Length of Hemisphere

Larger fossa occipitalis		Right	Equal	Left	Totals
	Right	24	14	12	50
	Equal	19	45	45	109
	Left	9	53	104	166
	Totals	52	112	161	325

From the vertical marginal totals it is seen that our material shows the same regularity as regards the fossa occipitalis sup. as Elliot Smith's Egyptian series: 51·1% are larger on the left side, against 15·4% on the right side. The group "equal" forms as much as 33·5%, which shows that our estimation has been very cautious. We see that the group "left-left" is by far the most numerous and the group "right-right" considerably more numerous than the dissimilar combinations.

$\phi^2 = 0{\cdot}1846$, as against $0{\cdot}0246 \pm 0{\cdot}0088$ for zero contingency and $C_2 = 0{\cdot}3947$ (without "class index correction"). We can conclude from these figures that the greater development of the left posterior fossa is a very important, *possibly* the only, cause of the greater length of the left hemisphere in our material.

Relation between fossa occipitalis sup. and sinus transversus:

Greater Groove

Larger fossa occipitalis	Right	Equal	Left	Totals
Right	12	15	23	50
Equal	73	24	12	109
Left	135	25	6	166
Totals	220	64	41	325

From the horizontal marginal totals it can be reckoned out that the transverse sinus in 67·7 % of the skulls goes preponderatingly to the right, as against 12·6 % to the left, which accords well with earlier observations made by Le Double (68·5 % right, 14·5% left), Elliot Smith and Tildesley (70·1% right, 19·7% left). Further we find that 158 skulls are in accordance with Elliot Smith's law—right-left and left-right combinations, while only 18 skulls form an exception.

$\phi^2 = 0{\cdot}1973$, which on comparison with the earlier mentioned value for zero contingency furnishes a good proof of the validity of the law. ($C_2 = 0{\cdot}4059$.)

The relation between the sinus transversus and length of the cranial cavity is seen from the following figures:

Greater Length of Hemisphere

Greater Groove	Right	Equal	Left	Totals
Right	12	70	140	222
Equal	11	33	20	64
Left	30	10	2	42
Totals	53	113	162	328

It is seen that the dissimilar combinations are by far the most numerous, the contingency is even higher than before:

$\phi^2 = 0.3810$, while ϕ^2 for zero contingency is about as before.

$C_2 = 0.5786$.

We thus find a high contingency between length of hemisphere and main direction of the sinus transversus, assuming that length of cranial cavity forms an adequate expression for length of hemisphere. We have thereby succeeded in furnishing statistical proof of the validity of Elliot Smith's law, and by a method which—as far as regards one of the characters—is independent of all subjective judgment. It is reasonable to *suppose*, as does Elliot Smith, that there exists a comparatively simple causality relation between the characters, but we cannot *a priori* preclude the possibility that the correlation is due to other circumstances, for example, the influence of some common factor (position of the fœtus in the uterus?). It may possibly be supposed that such (hypothetical) factors may be the cause of the correlation not being absolute. In our last table there are two cases where the left hemisphere has the greater length while at the same time the flow of blood goes mainly to the left, and in twelve cases the situation is the opposite. In these cases it may be that factors of otherwise secondary importance have played a decisive part.

Especially it must be noted that our figures tell nothing as to the relation to left-handedness. In this respect a systematic investigation of length of hemisphere in connection with the size of the sinus transversus and the length of the humerus on each side would undoubtedly be illuminating.

The results of our investigations as to the asymmetry of the skull—and, indirectly, of the brain—may be briefly summed up as follows:

1. Left hemisphere longer than right. (+ deviation to left in 49·4 %, + deviation to right in 16·2 %.)

2. Right hemisphere broader than left. (+ deviation to right in 67·1 %, *certain* + deviation to left only in 2·2 %.)

3. Left fossa occipitalis sup. more developed than right (Left fossa greater in 51·1 %, right fossa greater in 15·4 %.)

4. Right sinus transversus larger than left. (Right in 67·7 %, left in 12·6 %.)

Hereunto may be added that, according to Elliot Smith as well as to Cohn and Papez, the right area striata is greater than the left, while at the same time we may remember that Hoadley and Pearson's investigations of length of hemisphere in their Egyptian material give results different from ours. The most important problems remaining in this field accordingly are the following:

(1) Will direct measurements on fixed brains confirm our findings as to the greater breadth of the right hemisphere, and if so, what is the cause of the greater breadth (parietal lobe)?

(2) How are we to explain the far greater development of the left fossa occipitalis sup. when at the same time the area striata is less developed on the left side?

(3) How can we explain the cases in which the sinus transversus deviates from the rule?

VI. DIMENSIONS OF CRANIAL CAVITY AND PREDICTION OF CAPACITY.

It would lead too far to give a detailed account of the many attempts that have been made to set up formulae for calculation of capacity. In this respect we may refer to Martin's text-book, while a recently published paper by Neuert (19) also gives a survey of the literature on the subject. It will be found that in the main two principles have been followed. The investigators have *either* (1), as do Broca and Manouvrier, begun by taking the product of the chief cranial dimensions (L, B, H), occasionally with measurement of arcs and circumferences, and then introduced a number of corrections in order to reduce the error to the smallest possible amount, *or else* (2) they have, like Lee and Pearson, computed the correlation between the directly observed capacity values and the product of the three main diameters (eventually taking arcs and circumferences) and on the basis of this correlation established the best possible regression formula. This latter mode of procedure (No. 2) might be designated the *empirical-statistical* method and there can hardly be any doubt but that it is scientifically better founded than the former, which is of *a priori* nature*. The English-speaking, German and Scandinavian anthropologists have by degrees united in adopting the English method, and the formulae most frequently employed are the following†:

I. Lee's interracial formula, computed from 8 different races.

$$\text{Male } C = 0{\cdot}000\,370 \,.\, L \,.\, B \,.\, OH + 321{\cdot}16 \text{ cm.}^3$$

$$\text{Female } C = 0{\cdot}000\,375 \,.\, L \,.\, B \,.\, OH + 296{\cdot}40 \text{ cm.}^3$$

II. Pearson's interracial formula.

$$\text{Male } C = 0{\cdot}000\,365 \,.\, L \,.\, B \,.\, OH + 359{\cdot}34 \text{ cm.}^3$$

$$\text{Female } C = 0{\cdot}000\,375 \,.\, L \,.\, B \,.\, OH + 296{\cdot}40 \text{ cm.}^3$$

In case the ear-height is not known, other formulae come into consideration, as to which we may refer to Martin's text-book. The same applies to the *intra-racial formulae*, which are of course to be preferred in many cases. The importance of having a good interracial formula is, however, obvious. Sometimes the race is unknown or doubtful, as, for example, in dealing with prehistoric crania, while in other cases an adequate intraracial formula is lacking. Meanwhile it is maintained in several quarters that these mathematically determined interracial formulae are not

* Instead of linear measurements Neuert adopts measurement of plane surfaces (cephalograms), taken in different planes of the skull, sometimes several surfaces in the same plane. The advantage of this method is that peculiarities in the *shape* of the cranium find better expression. The method seems rather complicated and can hardly be said to be fully elaborated statistically, but judging from Neuert's paper the results are very good. For X-ray examination of living individuals it undoubtedly has a future before it.

[† As to the attribution of these formulae, see my Note following this Memoir. ED.]

particularly satisfactory, and the reason for this may be found in four factors: (1) Influence of bone thickness, (2) development of the sinus frontalis, (3) peculiarities in the shape of the skull and (4) fortuitous circumstances in connection with the composition of the material forming the basis for the formulae, both with respect to selection of races and to the numerical proportion of the individual races. In the paper which has already several times been mentioned Hoadley and Pearson(8) therefore investigate the correlation between the product of the three chief internal diameters, L_i, B_i and H_i (by H_i is understood here the internal basion-height, perpendicular on the Frankfurt horizontal) and the capacity. The latter was determined "by packing the skull tightly with mustard seed and then weighing." It was found that the correlation had increased from 0·82081 (between capacity and external measurements) to 0·89586 (between capacity and internal measurements). The possibility of obtaining a better formula was therefore obvious. With a view to rendering the method more serviceable in practice the authors tried to determine L_i, B_i and H_i from L, B and H, but this attempt failed owing to the slight correlation between thickness of bone and external measurement. The authors therefore conclude by stating that "we must either be contented with the degree of accuracy provided by the external diametral product, or if we wish to improve on that accuracy we must practise the rather difficult technique of internal measurement" (p. 93). Owing to the great technical difficulties the authors express doubt as to the utility of the new mode of procedure. As my new instrument renders the technique simpler and more time-saving, while at the same time the accuracy of measurement becomes greater, fresh interest is created in the importance of internal measurements in determination of capacity. And there can be no doubt that the correlation between the internal measurements and the capacity has also in our material been considerably increased as compared with the external measurements, but I have nevertheless refrained from this mode of procedure, as our material comprises too few races—two of the races especially (Maori and Australians) being too badly represented—to form the basis for an empirical-statistical inter-racial formula. Such a formula would, in fact, not have a long life. According as more races were investigated it would be necessary to make small changes in its different factors. I have therefore selected another mode of procedure, which no doubt is theoretically to be compared with the *a priori* method, but which I think rests upon a better foundation than all previous attempts in that direction, and which in practice will give better results than Lee's and Pearson's formulae. If we disregard the individual peculiarities of the cranial cavity and compare it to a regularly shaped space, then it is most natural to think of the shape of an egg, and the stereometrically known body that will first come into consideration is *the ellipsoid*, the simple formula for which,

$$C = \frac{L \,.\, Br \,.\, H}{6} \,.\, 3{\cdot}14159,$$

it should be possible to apply to the three main diameters of the cranial cavity. As height of the ellipsoid we have chosen the internal ear-height (external ear-bregma height minus thickness of the os parietale) instead of the internal

basion-height or internal opisthion-height, which owing to the greatly varying clivus-angle would give less satisfactory results. And instead of the maximum breadth of the cavity alone I have for the breadth of the ellipsoid used the mean of greatest breadth of cavity + maximum frontal breadth. If we inspect a series of skulls from one or more race in the norma verticalis, we soon become aware of the greatly varied tapering of the anterior pole of the skull. It is mainly this circumstance that forms the basis for Sergi's classification of skull types into Pentagoids, Ellipsoids, Rhomboids, etc.; cf. contours on pp. 688—689 of Martin's text-book. By introducing the *maximum frontal breadth* in addition to the internal measurements we therefore hope to eliminate or diminish three of the previously mentioned sources of error. The ideal procedure would of course be to use the *internal frontal breadth*, but this dimension is very difficult to obtain and, besides, the thickness of the bone at this place is of very little importance.

The formula we employ is therefore the following:

$$C = 0{\cdot}0005236 \, . \, L_i \, . \left(\frac{B_1 + B_2}{2}\right) . \, H_i,$$

where $L_i = \dfrac{L_L + L_R}{2}$, B_1 = greatest breadth of the cranial cavity, B_2 = greatest frontal breadth, H_i = internal ear-bregma-height and and $0{\cdot}0005236 = \dfrac{3{\cdot}14159}{6 \times 1000}$.

We shall first see how this formula suits our material without an additive term. The directly observed capacities in the following table have been found by means of the mustard-seed method and measuring-glass. Ranke's bronze skull has invariably been employed for checking the packing.

TABLE IX.

	Norwegians		Lapps		Maori		Australians	
	♂ $n=81$	♀ $n=76$	♂ $n=63$	♀ $n=55$	♂ $n=16$	♀ $n=14$	♂ $n=18$	♀ $n=10$
Directly observed C	1456 $\pm 14{\cdot}3$	1293 $\pm 11{\cdot}4$	1473 $\pm 13{\cdot}8$	1301 $\pm 11{\cdot}3$	1418 $\pm 20{\cdot}2$	1327 $\pm 23{\cdot}3$	1278 $\pm 21{\cdot}9$	1113 $\pm 17{\cdot}3$
Computed C... ...	1234	1093	1247	1123	1195	1109	1072	931
Observed-computed C	222	200	226	178	223	218	206	182

The best possible intraracial formulae are accordingly obtained by adding to the expression $0{\cdot}0005236 \, . \, L_i \left(\frac{B_1 + B_2}{2}\right) . \, H_i$ as additive term the values given in the lowest column. The next step in the investigation should be to apply these formulae so as to ascertain the individual divergencies. Here we lack, however, intraracial

formulae for comparison, and therefore we proceed at once to use the interracial formula. It is seen that the additive terms show remarkably little variation. This is of course a necessary condition for a good interracial formula when the other factor should at the same time be kept constant. On weighting the additive terms with their respective n-values we get the following interracial formula, which we will at once compare with Lee's and Pearson's*:

$$\text{For } ♂ \text{ skulls } C = 0{\cdot}0005236\, L_i \cdot \left(\frac{B_1 + B_2}{2}\right) \cdot H_i + 222 \text{ cm.}^3$$

$$\text{For } ♀ \text{ skulls } C = 0{\cdot}0005236\, L_i \cdot \left(\frac{B_1 + B_2}{2}\right) \cdot H_i + 192 \text{ cm.}^3$$

The following Table X gives us an opportunity of comparing the mean figures for the three methods with the direct measurements.

TABLE X. *Mean figures.*

	Mustard Seed	Lee	Pearson	Ellipsoid formula	Mustard Seed Lee	Mustard Seed Pearson	Mustard Seed Ellipsoid formula
Norwegians ♂	1450	1443	1463	1456	+13	+ 7	6
" ♀	1293	1279	1279	1285	+14	+14	+ 8
Lapps ♂	1473	1419	1444	1469	+54	+29	+ 4
" ♀	1301	1294	1294	1315	+ 7	+ 7	14
Maori ♂	1418	1428	1450	1417	−10	32	+ 1
" ♀	1327	1300	1301	1301	+27	+26	+26
Australians ♂	1278	1333	1357	1291	−55	−79	−13
" ♀	1113	1178	1178	1123	−65	−65	10

It is seen that in six groups the ellipsoid formula gives better results than Lee's and Pearson's methods and in one group (Lapps ♀) the difference is only 7 cm.³ in the opposite direction. In the group Maori ♀ the difference is insignificant. Lee's and Pearson's formulae suit very badly for our Australian skulls. Somewhat better results would here be obtained by following Wacker's instruction to add 50 cm.³ for thick-walled skulls (Martin, p. 647—really intended it only to apply to Lee's method). The results would, however, still be worse than with the ellipsoid method, and moreover Lee's and Pearson's formulae would in that way lose their inter-racial character.

Table XI shows the mean individual divergencies, without regard to plus or minus sign.

It is seen that the ellipsoid formula brings us on the average from 5 to 53 cm.³ nearer to the directly observed values.

* In the additive terms we entirely neglect the decimals, which are without practical importance. The exact figures would be 222·35 and 192·66 cm³. After the figures 222 and 192 had been employed one of the Australian skulls was rejected as being defective, which explains the slight divergency as regards v.

TABLE XI.

Average Individual Divergency, without regard to sign.

	Lee	Pearson	Ellipsoid formula
Norwegians ♂	54	55	47
" ♀	45	45	33
Lapps ♂	69	56	35
" ♀	40	40	35
Maori ♂	28	41	23
" ♀	42	42	37
Australians ♂	69	85	32
" ♀	65	65	22

Here again it is the Australian material that gives the worst results by Lee's and Pearson's methods. A deduction of 50 cm.³ renders the figures somewhat more favourable, but they still continue to be worse than by my new method, the average divergency being 45 or 53 cm.³ for males and 29 cm.³ for females as against respectively 32 and 22 cm.³ by the ellipsoid formula.

An investigation of the standard deviation of the divergencies with zero as mean has given the following results:

TABLE XII.

Standard Deviation of Divergencies from directly observed C.

Race	Mustard Seed Lee	Mustard Seed Pearson	Mustard Seed Ellipsoid
Norwegians ♂	70·1 ± 5·5	72·8 ± 5·7	58·0 ± 4·6
" ♀	55·5 ± 4·5	55·5 ± 4·5	41·0 ± 3·3
Lapps ♂	82·5 ± 7·3	69·1 ± 6·2	44·3 ± 3·9
" ♀	53·2 ± 5·1	53·2 ± 5·1	41·3 ± 3·9
Maori ♂	33·8 ± 6·0	46·2 ± 8·2	31·5 ± 5·6
" ♀	62·2 ± 11·8	62·2 ± 11·8	46·8 ± 8·8
Australians ♂	75·1 ± 14·7	94·3 ± 18·5	37·6 ± 7·4
" ♀	72·3 ± 16·2	72·3 ± 16·2	28·6 ± 6·4

Now it might naturally be objected against the comparison here made that the ellipsoid formula has been applied to a material which had served as basis for the establishment of the same formula. As the additive term is the only factor which varies, the question to be considered is the influence of that factor on the comparison. There are two circumstances which reduce the weight of the objection. Firstly, the fact that our material includes two such widely divergent races as Lapps and Australians, wherefore it is probable that the formula will give correspondingly good results also for races which as regards shape, capacity and thickness of skull lie between these extremes. And, secondly, the objection will not apply at all to the

two groups, female Lapps and female Maori, where the mean figures for the capacities computed by *the ellipsoid formula are not better than those obtained by* Lee's and Pearson's methods. *It is seen from Table XI that even in these cases the average individual divergency is reduced, although only by 5 cm.*^{3} Although the advantages of the new method can hardly be doubted, yet for control I have applied it to an *entirely new material.* For this purpose a series of Eskimo skulls, 18 males and 23 females, likewise belonging to the Anatomical Institute, was selected. On account of the peculiar wedge-shaped tapering of the parietalia this material should provide an especially good test. From the figures given below it appears that here also the ellipsoid formula gives much better results than the other two methods:

Eskimos.

Mean values		*Male*	*Female*
Directly observed C	...	1526 cm.^{3}	1335 cm.^{3}
Lee	...	1480 „	1318 „
Pearson	...	1502 „	1318 „
Ellipsoid formula	...	1522 „	1336 „
Average divergency			
Lee	...	75 cm.^{3}	48 cm.^{3}
Pearson	...	69 „	48 „
Ellipsoid formula	...	40 „	35 „

The standard deviation of the divergencies from the directly observed capacities:

For males: $\sigma_{\text{Mustard Seed: Lee}}\ 75{\cdot}5 \pm 12{\cdot}6$,

$\sigma_{\text{Mustard Seed: Pearson}}\ 77{\cdot}0 \pm 12{\cdot}8$, $\sigma_{\text{Mustard Seed: Ellipsoid}}\ 51{\cdot}3 \pm 8{\cdot}55$.

For females: $\sigma_{\text{Mustard Seed: Lee}}\ 53{\cdot}5 \pm 7{\cdot}9$, $\sigma_{\text{Mustard Seed: Ellipsoid}}\ 45{\cdot}8 \pm 6{\cdot}8$.

TABLE XIII.

	Norwegians ♂, $n=81$					Norwegians ♀, $n=76$					Lapps ♂, $n=63$				
	M	±	σ	±	v	M	±	σ	±	v	M	±	σ	±	v
External measurements															
1. Maximum length	187·70	0·65	5·86	0·46	3·12	179·28	0·67	5·81	0·47	3·24	179·17	0·86	6·79	0·60	3·79
2. Maximum breadth	142·34	0·56	5·03	0·40	3·53	135·37	0·45	3·94	0·32	2·91	146·52	0·63	5·00	0·45	3·41
3. Basion-bregma height	132·31	0·73	6·55	0·51	4·95	125·34	0·54	4·75	0·39	3·79	129·05	0·57	4·52	0·40	3·50
4. (a) Ear-bregma height	113·28	0·55	4·99	0·39	4·41	108·16	0·42	3·66	0·30	3·38	113·02	0·45	3·55	0·32	3·14
(b) $4(a)$ − thickness of parietal bone $(6(a))$ ...	107·96	0·52	4·72	0·37	4·37	103·78	0·39	3·39	0·27	3·27	108·71	0·45	3·60	0·32	3·31
5. (a) Maximum frontal breadth	118·16	0·60	5·41	0·43	4·58	113·77	0·42	3·67	0·30	3·23	120·59	0·56	4·42	0·39	3·67
(b) $(5(a)+14)/2$	126·105	0·52	4·71	0·37	3·72	120·99	0·36	3·10	0·25	2·56	130·33	0·51	4·03	0·36	3·08
6. (a) Thickness of parietal bone	5·33	0·17	1·57	0·12	[29·45]	4·38	0·13	1·17	0·09	[26·71]	4·30	0·15	1·23	0·11	[28·60]
(b) $3-15$	5·25	0·20	1·80	0·14	[34·29]	4·32	0·16	1·57	0·13	[36·34]	4·16	0·19	1·52	0·14	[36·54]
(c) $\frac{1}{2}(2-14)$	4·15	0·09	0·86	0·07	[20·72]	3·57	0·10	0·86	0·08	[24·09]	3·24	0·11	0·84	0·07	[25·92]
7. (a) Thickness of occipital bone, R	4·30	0·13	1·16	0·09	[26·98]	4·10	0·14	1·21	0·10	[29·51]	3·12	0·14	1·13	0·10	[35·90]
(b) Thickness of occipital bone, L	4·10	0·13	1·21	0·10	[29·51]	3·86	0·13	1·13	0·09	[29·27]	2·90	0·13	1·03	0·09	[35·52]
(c) $(7(a)+7(b))/2$	4·21	0·11	1·00	0·08	[23·75]	3·98	0·13	1·12	0·09	[28·28]	3·01	0·13	1·04	0·09	[34·55]
8. $1-13(c)-7(c)$	11·22	0·33	2·97	0·23	[26·47]	9·21	0·27	2·32	0·19	[25·36]	8·11	0·40	3·21	0·29	[39·58]
9. Breadth-length index $(2\times100)/1$	75·88	0·33	2·95	0·23	3·89	75·56	0·31	2·69	0·22	3·56	81·90	0·52	4·16	0·37	5·08
10. Height-length index $(3\times100)/1$	70·46	0·40	3·64	0·29	5·17	69·96	0·34	2·95	0·24	4·22	72·05	0·39	3·09	0·28	4·29
11. Height-breadth index $(3\times100)/2$	93·08	0·58	5·26	0·41	5·65	92·67	0·52	4·53	0·37	4·89	88·19	0·55	4·34	0·39	4·92
Internal measurements															
13. (a) Maximum sagittal length of skull cavity, R	171·94	0·68	6·14	0·48	3·59	165·78	0·67	5·88	0·48	3·55	167·84	0·80	6·37	0·57	3·80
(b) Maximum sagittal length of skull cavity, L	172·54	0·70	6·32	0·50	3·66	166·46	0·68	5·92	0·48	3·56	168·25	0·85	6·74	0·60	4·01
(c) $(R+L)/2$	172·24	0·69	6·24	0·49	3·62	166·12	0·68	5·90	0·48	3·55	168·05	0·83	6·56	0·58	3·90
14. Maximum breadth of skull cavity	134·05	0·49	4·44	0·35	3·31	128·22	0·45	3·90	0·32	3·04	140·05	0·61	4·88	0·43	3·48
15. Internal basion height	127·07	0·72	6·48	0·51	5·10	121·03	0·52	4·49	0·36	3·71	124·88	0·58	4·64	0·41	3·72
16. Internal opisthion height	129·28	0·67	6·02	0·47	4·66	126·09	0·58	5·03	0·41	3·99	127·60	0·73	5·76	0·51	4·51
17. Internal breadth-length index $(14\times100)/13(c)$	77·56	0·37	3·29	0·26	4·24	77·24	0·29	2·57	0·21	3·33	83·46	0·50	3·96	0·35	4·74
18. Internal height-length index $(15\times100)/13(c)$ ...	73·89	0·45	4·02	0·32	5·44	73·84	0·39	3·43	0·28	4·65	74·45	0·45	3·57	0·32	4·80
19. Internal height-breadth index $(15\times100)/14$ ...	94·82	0·64	5·79	0·45	6·11	94·48	0·54	4·69	0·38	4·96	89·40	0·56	4·46	0·40	4·99
20. Capacity (mustard seed)	1456·0	14·3	128·6	10·1	8·8	1293·0	11·4	99·8	8·1	7·7	1473·0	13·8	109·3	9·7	7·4
21. Capacity (Lee)	1443·0	10·3	92·5	7·3	6·4	1279·0	8·5	74·4	6·0	5·8	1419·0	10·0	79·2	7·1	5·6
22. Capacity (Pearson)	1463·0	10·1	90·9	7·1	6·2	1279·0	8·5	74·4	6·0	5·8	1444·0	9·8	77·5	6·9	5·4
23. Capacity (ellipsoid formula)	1456·0	12·2	110·1	8·7	7·6	1285·0	9·0	78·5	6·4	6·1	1469·0	12·4	98·6	8·8	6·7

TABLE XIII (*continued*).

	Lapps ♀, $n=55$					Maori ♂, $n=16$					Maori ♀, $n=14$				
External measurements	M	±	σ	±	v	M	±	σ	±	v	M	±	σ	±	v
1. Maximum length ...	170·96	0·71	5·23	0·50	3·06	186·81	1·04	4·14	0·73	2·22	180·00	0·95	3·55	0·67	1·97
2. Maximum breadth ...	142·05	0·59	4·39	0·42	3·09	137·88	1·15	4·58	0·81	3·32	132·64	0·97	3·64	0·69	2·74
3. Basion-bregma height ...	124·05	0·58	4·29	0·41	3·46	136·63	1·15	4·61	0·82	3·37	130·21	1·03	3·84	0·73	2·95
4. (*a*) Ear-bregma height ...	109·51	0·52	3·88	0·37	3·54	116·06	0·78	3·11	0·55	2·68	112·07	1·28	4·79	0·91	4·27
(*b*) 4 (*a*) − thickness of parietal bone (6 (*a*)) ...	105·53	0·50	3·68	0·35	3·49	109·19	0·73	2·92	0·52	2·67	106·29	1·30	4·87	0·92	4·58
5. (*a*) Maximum frontal breadth ...	116·27	0·58	4·27	0·41	3·67	112·63	0·98	3·90	0·69	3·46	109·50	0·87	3·25	0·61	2·97
(*b*) (5 (*a*) + 14)/2 ...	125·74*	0·49	3·61	0·34	2·87	121·78*	0·92	3·69	0·65	3·03	118·39*	0·82	3·08	0·58	2·60
6. (*a*) Thickness of parietal bone ...	3·98	0·14	1·07	0·10	[26·88]	6·88	0·35	1·41	0·25	[20·49]	5·79	0·32	1·20	0·23	[20·73]
(*b*) 3 − 15 ...	3·44*	0·23	1·67	0·16	[48·54]	6·19	0·53	2·10	0·37	[33·93]	4·64	0·47	1·77	0·33	[38·15]
(*c*) ½ (2 − 14) ...	3·41	0·12	0·86	0·08	[25·07]	3·28	0·14	0·56	0·10	[17·07]	3·04	0·20	0·75	0·14	[24·67]
7. (*a*) Thickness of occipital bone, R ...	3·07	0·15	1·12	0·11	[36·48]	5·06	0·29	1·14	0·20	[22·53]	4·50	0·26	0·98	0·19	[21·78]
(*b*) Thickness of occipital bone, L ...	3·02	0·15	1·09	0·10	[36·09]	4·72	0·22	0·86	0·15	[18·22]	4·57	0·28	1·05	0·20	[22·98]
(*c*) (7 (*a*) + 7 (*b*))/2 ...	3·06*	0·14	1·02	0·10	[33·33]	4·89	0·24	0·94	0·17	[19·22]	4·54	0·28	1·05	0·20	[23·13]
8. 1 − 13 (*c*) − 7 (*c*) ...	6·33*	0·34	2·51	0·24	[39·65]	10·71	0·52	2·09	0·37	[19·51]	7·43	0·45	1·70	0·32	[22·88]
9. Breadth-length index (2 × 100)/1 ...	83·17	0·49	3·65	0·35	4·39	73·87	0·73	2·90	0·51	3·93	74·07	0·66	2·48	0·47	3·35
10. Height-length index (3 × 100)/1 ...	72·62	0·39	2·89	0·28	3·99	73·15	0·46	1·85	0·33	2·53	72·35	0·50	1·86	0·35	2·57
11. Height-breadth index (3 × 100)/2 ...	87·39	0·47	3·49	0·33	3·99	99·23	1·22	4·86	0·86	4·90	98·26	1·13	4·23	0·80	4·30
Internal measurements															
13. (*a*) Maximum sagittal length of skull cavity, R	161·42	0·64	4·77	0·45	2·96	170·31	0·99	3·95	0·70	2·32	167·57	1·19	4·44	0·84	2·65
(*b*) Maximum sagittal length of skull cavity, L	161·71	0·62	4·61	0·44	2·85	172·13	1·14	4·55	0·80	2·64	168·50	1·14	4·26	0·81	2·53
(*c*) (R + L)/2 ...	161·57	0·63	4·65	0·44	2·88	171·21	1·06	4·25	0·75	2·48	168·04	1·17	4·37	0·83	2·60
14. Maximum breadth of skull cavity ...	135·24	0·56	4·15	0·40	3·06	131·31	1·12	4·46	0·79	3·40	126·57	1·03	3·87	0·73	3·03
15. Internal basion height ...	120·62	0·55	4·10	0·39	3·39	130·44	0·94	3·71	0·66	2·47	125·37	1·05	3·93	0·74	3·13
16. Internal opisthion height ...	124·29	0·61	4·55	0·43	3·66	[illegible]	1·28	5·15	0·91	4·06	[illegible]	1·18	4·42	0·84	3·41
17. Internal breadth-length index (14 × 100)/13 (*c*)	83·77	0·46	3·43	0·33	4·09	77·12	0·71	2·85	0·50	3·70	75·36	0·64	2·35	0·45	3·17
18. Internal height-length index (15 × 100)/13 (*c*) ...	74·72	0·42	3·14	0·30	4·20	75·84	0·73	[illegible]	0·51	3·82	74·71	0·52	1·96	0·37	2·62
19. Internal height-breadth index (15 × 100)/14 ...	89·37	0·51	3·75	0·36	4·20	99·28	1·22	4·87	0·86	4·91	[illegible]	1·14	4·25	0·81	[illegible]
20. Capacity (mustard seed) ...	1340·0	11·3	84·9	8·0	6·3	[illegible]	20·2	[illegible]	14·3	5·7	[illegible]	23·3	87·2	16·5	[illegible]
21. Capacity (Lee) ...	1294·9	8·7	64·4	6·1	5·0	[illegible]	17·4	69·4	12·3	4·9	[illegible]	[illegible]	[illegible]	13·4	[illegible]
22. Capacity (Pearson) ...	1294·9	8·7	64·4	6·1	5·0	[illegible]	17·0	68·0	12·0	4·7	[illegible]	[illegible]	[illegible]	13·4	[illegible]
23. Capacity (ellipsoid formula) ...	1315·0	10·4	77·0	7·3	5·9	[illegible]	17·7	70·8	12·5	5·0	[illegible]	23·3	86·1	16·3	[illegible]

[* These values are not in accordance with the statements in the first column. Replace by 125·755, 3·43, 3·045, 6·345, 121·97 and 118·035 respectively. Ed.]

	Australians ♂, $n=13$					Australians ♀, $n=10$				
External measurements	M	±	σ	±	v	M	±	σ	±	v
1. Maximum length	183·30	1·21	4·36	0·85	2·38	173·40	1·69	5·33	1·19	3·07
2. Maximum breadth	130·85	1·27	4·57	0·90	3·49	124·60	1·10	3·47	0·78	2·78
3. Basion-bregma height	135·92	1·26	4·55	0·89	3·35	130·70	0·95	2·99	0·67	2·29
4. (a) Ear-bregma height	113·85	1·26	4·55	0·89	4·00	107·80	1·04	3·28	0·73	3·04
(b) 4 (a) − thickness of parietal bone (6 (a)) ...	106·46	0·76	2·74	0·54	2·57	102·10	1·08	3·42	0·76	3·35
5. (a) Maximum frontal breadth	111·23	1·35	4·85	0·95	4·36	104·90	0·91	2·88	0·64	2·75
(b) (5 (a) + 14)/2	117·31	1·25	4·49	0·88	3·83	111·25	0·88	2·78	0·62	2·50
6. (a) Thickness of parietal bone	7·39	0·56	2·02	0·40	[27·33]	5·70	0·32	1·00	0·22	[17·54]
(b) 3 − 15	7·15	0·64	2·32	0·45	[32·45]	4·80	0·77	2·44	0·55	[50·83]
(c) ½ (2 − 14)	3·73	0·30	1·07	0·21	[28·69]	3·50	0·22	0·71	0·16	[20·29]
7. (a) Thickness of occipital bone, R	5·08	0·44	1·57	0·31	[30·91]	5·40	0·35	1·11	0·25	[20·56]
(b) Thickness of occipital bone, L	5·31	0·45	1·61	0·32	[30·32]	5·45	0·08	0·26†	0·06	[4·77]
(c) (7 (a) + 7 (b))/2	5·19	0·39	1·42	0·28	[27·36]	5·42	0·31	1·01	0·23	[18·63]
8. 1 − 13 (c) − 7 (c)	14·35	0·83	2·99	0·59	[20·84]	11·58	0·74	2·35	0·53	[20·29]
9. Breadth-length index (2 × 100)/1	71·38	0·51	1·85	0·36	2·59	71·92	0·83	2·61	0·58	3·63
10. Height-length index (3 × 100)/1	74·21	0·64	2·31	0·45	3·11	75·40	0·60	1·91	0·43	2·53
11. Height-breadth index (3 × 100)/2	103·99	1·22	4·38	0·86	4·22	105·04	0·82	2·59	0·58	2·47
Internal measurements										
13. (a) Maximum sagittal length of skull cavity, R	163·30	1·10	3·97	0·78	2·43	155·90	1·60	5·09	1·14	3·26
(b) Maximum sagittal length of skull cavity, L	164·15	1·08	3·88	0·76	2·36	156·90	1·76	5·58	1·25	3·56
(c) $(R+L)/2$	163·77*	1·07	3·85	0·76	2·35	156·40	1·69	5·35	1·20	3·42
14. Maximum breadth of skull cavity	123·38	1·18	4·25	0·83	3·44	117·60	0·96	3·04	0·68	2·59
15. Internal basion height	128·77	1·42	5·12	1·00	3·98	125·90	1·34	4·25	0·95	3·38
16. Internal opisthion height	129·08	1·16	4·18	0·82	3·24	126·30	1·15	3·64	0·81	2·88
17. Internal breadth-length index (14 × 100)/13 (c)	75·35	0·62	2·24	0·44	2·97	75·27	0·90	2·85	0·64	3·79
18. Internal height-length index (15 × 100)/13 (c) ...	78·67	0·96	3·45	0·68	4·39	80·61	1·21	3·83	0·86	4·75
19. Internal height-breadth index (15 × 100)/14 ...	104·48	1·48	5·33	1·05	5·10	107·05	0·71	2·24	0·50	2·09
20. Capacity (mustard seed)	1278·0	21·9	79·1	15·5	6·2	1113·0	17·3	54·8	12·3	4·9
21. Capacity (Lee)	1333·0	22·8	82·3	16·1	6·4	1178·0	22·3	70·5	15·8	6·3
22. Capacity (Pearson)	1307·0	22·4	80·9	15·9	6·2	1178·0	22·3	70·5	15·8	6·3
23. Capacity (ellipsoid formula)	1294·0	19·2	69·1	13·6	5·3	1123·0	20·8	65·9	14·7	5·9

* ? 163·72⁵.

[† This value and the results on either side dependent on it are remarkable, and if correct show how little dependence can be placed on series of 10. Ed.]

TABLE XIII (*continued*).

	Eskimos ♂, $n=18$					Eskimos ♀, $n=23$				
External measurements	M	±	σ	±	v	M	±	σ	±	v
1. Maximum length	190·44	1·17	4·97	0·83	2·61	179·65	0·95	4·58	0·68	2·70
2. Maximum breadth	137·56	1·06	4·49	0·75	3·26	132·43	0·96	4·60	0·68	3·47
3. Basion-bregma height	139·39	1·13	4·79	0·80	3·44	133·52	0·86	4·13	0·61	3·09
4. (*a*) Ear-bregma height	119·56	0·86	3·64	0·61	3·04	114·52	0·65	3·10	0·46	2·71
(*b*) 4 (*a*) − thickness of parietal bone (6 (*a*)) ...	114·92	0·81	3·42	0·57	2·98	110·30	0·73	3·48	0·51	3·16
5. (*a*) Maximum frontal breadth	113·94	0·83	3·54	0·59	3·11	110·04	0·91	4·36	0·64	3·96
(*b*) (5 (*a*) + 14)/2	122·36	0·84	3·57	0·59	2·92	117·93	0·93	4·44	0·65	3·76
6. (*a*) Thickness of parietal bone	4·64	0·26	1·10	0·18	[23·71]	4·22	0·24	1·14	0·17	[27·01]
(*b*) 3 − 15	5·78	0·51	2·15	0·36	[37·20]	5·65*	0·51	2·46	0·36	[43·54]
(*c*) ½ (2 − 14)	3·33*	0·30	1·29	0·21	[38·74]	3·30	0·25	1·20	0·18	[36·36]
7. (*a*) Thickness of occipital bone, *R*	5·67	0·48	2·04	0·34	[35·98]	5·15	0·29	1·39	0·20	[26·99]
(*b*) Thickness of occipital bone, *L*	5·08	0·40	1·68	0·28	[33·07]	5·20	0·28	1·32	0·19	[25·38]
(*c*) (7 (*a*) + 7 (*b*))/2	5·36	0·44	1·86	0·31	[34·57]	5·17	0·25	1·22	0·18	[23·60]
8. 1 − 13 (*c*) − 7 (*c*)	8·38*	0·60	2·56	0·43	[30·55]	6·74	0·52	2·51	0·37	[37·24]
9. Breadth-length index (2 × 100)/1	72·30	0·84	3·57	0·59	4·94	73·73	0·58	2·78	0·41	3·77
10. Height-length index (3 × 100)/1	73·23	0·66	2·79	0·46	3·81	74·36	0·54	2·59	0·38	3·48
11. Height-breadth index (3 × 100)/2	101·44	1·07	4·54	0·76	4·48	100·93	0·88	4·22	0·62	4·18
Internal measurements										
13. (*a*) Maximum sagittal length of skull cavity, *R*	176·06	1·10	4·67	0·78	2·65	167·28	0·94	4·50	0·66	2·69
(*b*) Maximum sagittal length of skull cavity, *L*	177·00	1·02	4·32	0·72	2·44	168·22	0·99	4·75	0·70	2·82
(*c*) (*R* + *L*)/2	176·53	1·08	4·59	0·76	2·60	167·74	0·94	4·50	0·66	2·68
14. Maximum breadth of skull cavity	130·78	1·13	4·81	0·80	3·68	125·83	1·07	5·14	0·76	4·08
15. Internal basion height	133·61	1·16	4·68	0·78	3·50	127·21	0·79	3·81	0·56	2·99
16. Internal opisthion height	131·38	1·23	5·20	0·87	3·96	127·43	1·07	5·11	0·75	4·01
17. Internal breadth-length index (14 × 100)/13 (*c*)	74·15	0·87	3·69	0·61	4·98	75·04	0·59	2·81	0·41	3·74
18. Internal height-length index (15 × 100)/13 (*c*) ...	73·73	0·70	2·97	0·49	3·92	76·31	0·65	3·10	0·46	4·06
19. Internal height-breadth index (15 × 100)/14 ...	102·31	1·21	5·15	0·86	5·03	101·83	1·04	4·97	0·73	4·88
20. Capacity (mustard seed)	1528·0	18·3	77·5	12·9	5·08	1335·0	19·1	91·8	13·5	6·88
21. Capacity (Lee)	1450·0	14·3	69·5	10·1	4·80	1318·0	12·2	58·4	8·6	4·43
22. Capacity (Pearson)	1572·0	14·0	59·6	9·9	3·87	1318·0	12·2	58·4	8·6	4·43
23. Capacity (ellipsoid formula)	1522·0	14·4	61·1	10·2	4·01	1336·0	17·1	81·8	12·1	6·13

[* * Read instead of these respectively 3·39, 5·535 and 5·61. Ed.]

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NOTE ON SECTION VI OF DR K. WAGNER'S MEMOIR.

BY KARL PEARSON.

1. I find it difficult to follow the reasoning of a portion of Dr Wagner's most valuable paper. But before I enter into any criticism, it is needful to remind the reader of one or two points which all of us are apt to overlook. There are two types of formulae respectively termed by me many years ago, *intraracial* and *interracial*. If we wish to predict a character like the capacity of the skull which is laborious to measure, or it may be impossible, from more easily ascertained lengths, such as external diameters, then we require to know the most probable value and the standard error, which determines the range of variation about this most probable value. The statistical method is definite—provided we make certain assumptions—i.e. we measure the dependent character desired and the independent characters in as many individuals of the same race as are available and construct a regression surface, and theory then gives us the probable value and the standard variation about that value of the character desired in terms of the independent variables. On the assumption that cranial characters approximately follow what are termed normal distributions, the resulting regression surface is planar, and the standard variation depends on the multiple correlation coefficient. The result thus obtained enables us to predict from the given set of independent variables the desired character in any number of the same race with a given likelihood of accuracy. Such a construction formula is termed an *intraracial* formula.

As long as the hypothesis of normality is sensibly true, and therefore our formula linear, the mean of any desired character in a number n of crania of the same race will be obtained with the same accuracy by inserting in the formula the means of the independent variables.

If we now take the means of the desired character in a number of races and the corresponding means of the independent characters, these means will follow normal distributions, if the individual characters do; and we construct in the same manner a planar regression surface for ascertaining the mean desired character in any new race from the means of its independent variables. Such a regression surface is termed an *interracial* formula.

The process is perfectly straightforward if laborious from both the measurement and computing sides, once the hypothesis of normal distribution of cranial characters is accepted; and, as theory demonstrates, no other planar formula with the *same independent* variables can give as good, much less better, results. The formulae, whether obtained by the theory of multiple regression or by the method of least squares, are identical.

In the above manner intraracial reconstruction formulae were worked out for five races, in 1901, by Dr Lee and myself*. We worked them out for various combinations of the Length, Breadth and Height of the skull, including the product of these diameters. These intraracial formulae gave by no means bad results and average formulae from our best material, Bavarian, Aino and Naqada, gave us general formulae for the two sexes (*loc. cit.* p. 243), namely,

$$\left.\begin{array}{l}(\male)\ C = {\cdot}000337\,LBH + 406{\cdot}01\\(\female)\ C = {\cdot}000400\,LBH + 206{\cdot}60\end{array}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

for an *individual* skull of any race where C is the capacity in cubic centimetres and the diameters are measured in millimetres.

L is the maximum (glabellar-occipital) length, B is the maximum breadth taken *on the parietals*, H the height measured from the auricular line *perpendicular to the Frankfort Plane*.

* *Philosophical Transactions R. S.* Vol. CXCVI A. pp. 225–264.

It must be remembered that the mean capacity of a number of crania can only be obtained approximately from these formulae by inserting the mean values of LBH, for the mean value of the product (LBH) is not the product of the mean values of L, B, and H.

Now let us turn to interracial formulae. Few investigators have measured enough crania of a large number of sufficiently diverse races to obtain a good interracial formula, whether it be a linear or product formula. The use of a large number of racial capacities measured by different observers may be most misleading; not only may the technique be different, but variations due to personal equation are excessive. The older French measurements cannot be very fairly pooled with the German, and the older English measurements with either*. In the course of the Royal Society memoir of 1901 Dr Lee and I reached by the method of least squares interracial formulae (*loc. cit.* p. 247) for male and female series. These are as follows:

$$\left.\begin{array}{l}(\male)\ C = \cdot 000365\,\bar{L}\times\bar{B}\times\bar{H} + 359\cdot 34 \\ (\female)\ C = \cdot 000375\,\bar{L}\times\bar{B}\times\bar{H} + 296\cdot 40\end{array}\right\} \qquad \text{(ii)},$$

where $\bar{H}$ is again the mean auricular height measured perpendicular to the Frankfort Plane.

These results were obtained from the mean capacities, and the mean diameters of 9 or 10 races; for this reason they cannot be applied to an individual skull of any special race. They are formulae for *mean* values: cf. what has been said on p. 133.

I have referred to the difficulty of getting capacity measured in the same manner by different observers and techniques, and accordingly in a footnote Dr Lee and I gave the male formula with three doubtful series omitted, i.e.

$$(\male)\ C = \cdot 000370\,\bar{L}\times\bar{B}\times\bar{H} + 321\cdot 10 \qquad \text{(iii)}.$$

To my surprise Dr Wagner breaks up Dr Lee's and my joint work, and provides two sets of formulae. He attributes to Dr Lee (ii) and to me (iii) for males and the second of (ii) for females. It has the appearance of our having provided two sets of different and partly discordant formulae! Why he should separate out a result in a footnote to a joint memoir as peculiarly my fundamental formula for racial capacities I am puzzled to say!

On the same page (*loc. cit.* p. 247) are given the interracial formulae

$$\left.\begin{array}{l}(\male)\ C = \cdot 000266\,\bar{L}\times\bar{B}\times\bar{H}' + 524\cdot 6 \\ (\female)\ C = \cdot 000156\,\bar{L}\times\bar{B}\times\bar{H}' + 812\cdot 0\end{array}\right\} \qquad \text{(iv)},$$

where $\bar{H}'$ is the mean height of the skull measured from the basion to the point of the skull *vertically above* the basion when orientated to the Frankfort horizontal plane.

Now the reader will note that neither H ($= OH$ of later notation) nor H' are measured to the *bregma*.

It would have been theoretically more reasonable to take instead of L the length L', the *gerade Länge* of the *Frankfurter Verständigung*, for then L', B and H (or H') would have been mutually at right angles and have represented a true physical right six-face, and the product would have been its actual volume. This was not possible in our investigations, for the length projection L' was not provided by most craniologists. Thus LBH, or LBH', still less $\frac{1}{2}(L_H + L_L) \times B_r \times H_r$, cannot be looked upon as the volumes of right six-faces, closely embracing the skull. Such products are merely arbitrary functions of characters of the skull, and there is no theoretical justification for multiplying them by $\frac{4\pi}{3} \times \frac{1}{8}$ as a factor, on the basis of the volume of an ellipsoid. The skull may resemble an ellipsoid, but if so its capacity must be measured by the product of three mutually rectangular axes, and not by quantities like LBH which are not mutually rectangular.

Of course we may start from any function f of cranial external or internal variables, and its fitness will depend on how intense the correlation of C and f may be. That is the sole test of

* On these points see the paper by M. A. Lewenz and K. Pearson, "On the measurement of Internal Capacity from Cranial Circumferences," *Biometrika*, Vol. III. pp. 366—397, 1904. Cf. especially p. 377.

whether one f is or is not better than a second. Dealing with intraracial data for Egyptians, Series E, Pearson and Hoadley found a correlation of C and $L_t B_t H_t$ of ·8959; they accordingly judged that the product $(L_t B_t H_t) = P_t$ was better than $(LBH) = P$ which gave a correlation of ·8208 only. They did not go further than providing the intraracial reconstruction formula for Ancient Egyptians*:

$$(\male)\ C = \cdot 0004372 P_t + 169\cdot 57 \pm 34/\sqrt{n} \quad \text{.............................(v).}$$

In the same place they gave the external diametral formula

$$(\male)\ C = \cdot 0003495 P + 226\cdot 52 \pm 44/\sqrt{n} \quad \text{.............................(vi),}$$

indicating that the internal diametral formula was about 25 per cent. better than the external diametral formula.

If the reconstruction formula be looked upon as $C = AP + B$, then the "best" values for A and B are to be found from the correlational calculus. Neither A nor B ought to be "guessed."

In 1904 Lewenz and Pearson† reinvestigated the formulae for intraracial and interracial reconstruction, as a protest against a "guess" formula of Dr Beddoe‡. He had used an f, into which he introduced the three circumferential arcs, and the cephalic index. This he was of course quite at liberty to do, although the intraracial correlation of cephalic index and capacity is almost insignificant, being about $-\cdot 15$ for dolichocephalic and $+\cdot 10$ for brachycephalic races. But the gravamen of our charge against Dr Beddoe was that having chosen his f, instead of finding its correlation with C, and testing accordingly whether his f was better or worse than other people's, he "guessed" his A and B. He guessed $B = 0$ (!) and felt by trial and error his way to a suitable A§. Now we fear Dr Wagner has done something of the same nature. He has chosen his f, but he has not shown that his f has a higher correlation with C than other f's. Dr Wagner then guesses A as $\frac{4\pi}{3}\frac{1}{8}$ for the coefficient of his f, which has no theoretical justification, substitutes the *mean* values of his variates in his f, and determines a B for each of his four races so as to satisfy their mean capacity; he then averages these B's and provides what he treats as interracial formulae

$$\left.\begin{array}{l}(\male)\ C = \cdot 0005236 L_t \times \tfrac{1}{2}(B_1 + B_2) + H_t \times 222 \text{ cm.}^3 \\ (\female)\ C = \cdot 0005236 L_t \times \tfrac{1}{2}(B_1 + B_2) \times H_t + 192 \text{ cm.}^3\end{array}\right\} \quad \text{.....................(vii).}$$

Now this seems almost as difficult to accept from the mathematical standpoint as Dr Beddoe's procedure. We must not be misled by the fact that Dr Wagner has taken four *races*. The justification or want of it is precisely the same as if I had taken four skulls of a single race and based an intraracial formula on them by taking

$$C = \cdot 0005236 L_t \times \tfrac{1}{2}(B_1 + B_2) \times H_t + B$$

and choosing B to fit those four skulls! It does not matter that our author is here dealing with *means* and not individual cranial values. Until the correlation between his racial (mean) C and his racial (mean) product P_t is so high that four points will determine accurately his straight regression line his formula remains theoretically unjustified. He has made his A take the same value not only for both sexes, but for both types of formulae intraracial and interracial, for it is clear that he looks upon

$$C = \cdot 0005236 P_2 + \cdot 222$$

* Pearson and Hoadley: *Biometrika*, Vol. XXI (1929), p. 92.

† "Measurement of Internal Capacity from Cranial Circumferences," *Biometrika*, Vol. III, pp. 366—397.

‡ "De l'Évaluation de la signification de la capacité cranienne," *L'Anthropologie*, Tome XXV, 1903, pp. 267—294.

§ It is needless here to enter on other errors of Dr Beddoe; one only is of importance for our present purpose: he, like Dr Wagner, did not take care to use the same characters as were chosen by the authors of the formula.

as an intraracial formula for the Norwegian skull; nor does he make clear how he changes from the L_i, B_i and H_i for an individual skull to a product $\bar{L}_i\bar{B}_i\bar{H}_i$ of racial means, for the mean "ellipsoid" of many skulls would not be the "ellipsoid" of mean axes. Even Dr Wagner's theoretical justification for using the ellipsoidal coefficient, ·0005236, falls to the ground when he uses $\frac{1}{2}(B_f+B_i)$ for his intermediate axis, for if we give an ovoid form to the skull by using a maximum breadth and a lesser breadth nearer to one end, then the coefficient for the volume will not take the ellipsoidal value. All experience shows that the coefficient of the product term does vary from sex to sex and from race to race. Indeed, as I shall show, a better prediction formulae than Dr Wagner gets for his four races can be at once obtained by considering both coefficients in $C=AP_i+B$ as unknown and determined by least square or correlational methods.

But there is a point which must be regarded in all these product prediction formulae. We have seen that roughly the characters measured on the skull follow normal distributions. Accordingly the regression formula for C on any *linear* function of L, B, H will be planar, but it by no means follows that the regression formula for C on a product will be linear. No theoretical demonstration of this has so far been given*, and at any rate in the case of the product of the three circumferential arcs Lewenz and Pearson have shown by graphing the product to the capacity that for a wide range of races it is very unlikely that linearity exists†.

2. Let us look a little into the arithmetic of the formulae cited.

The actual result of Dr Wagner's discussion is that he applies for males the intraracial formula for Norwegians as an interracial formula, namely:

$$C=\cdot0005236\bar{L}_i\times\tfrac{1}{2}(\bar{B}_f+\bar{B}_i)\times\bar{H}_i+222 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{vii}^{\text{bis}}),$$

where the bars above the variables denote racial means, and

$\bar{L}_i=\frac{1}{2}(L_R+L_L)=$ mean of internal lengths = the same quantity as used by Hoadley and Pearson in 1929;

$\bar{B}_i=$ maximum internal breadth, agreeing with Hoadley and Pearson's measurements;

$B_f=$ maximum external frontal breadth. This was a character not originally selected for measurement in the Biometric Laboratory, for its terminals frequently fall on very complicated sutures, so that they are very doubtful and subject to large personal equation. Dr Wagner does not state how he dealt with much indented sutures. H_i with Dr Wagner is the distance from bregma to the auricular axis less the thickness of the parietal bone. This latter should be in the *direction of the external measurement* and *at the bregma*, but our author takes it on the side wall of the parietal bone, thus assuming that there is such a general character as the "thickness of the parietal bone." The Hoadley-Pearson H_i is the internal basion height, perpendicular to the Frankfort Plane. We will call this H_i' for the time being and we have the following values for two races, the Norwegian and Egyptians of the 20th to 30th Dynasties:

Race	$\bar{L}_i$	$\frac{1}{2}(\bar{B}_f+\bar{B}_i)$	$\bar{H}_i$	$\bar{P}_i$	$\bar{B}_i$	$\bar{H}_i'$	$\bar{P}_i'$
Norwegians	172·28	126·61	107·96	235,3003·617	134·05	127,07	293,4571·907
Egyptians	170·44	124·03‡	107·97§	228,2450·515	132·14	128·85	290,1952·175

* I am not overlooking the fact that if variations from the mean be substituted for L, B and H, then LBH can be expanded as a linear function to a *first* approximation; but in this case we might as well start with a linear function and accordingly discard any of the advantages supposed to arise from the use of the product function.

† See *Biometrika*, Vol. III. 1904, p. 387.

‡ $\bar{B}_f=115·92$.

§ $\bar{H}_i=$ Bregma auricular height − parietal thickness $=113·43-5·46=107·97$.

Now let us apply to these products the interracial male formula (vii) of Wagner (intraracial for the Norwegians) and the intraracial formula of Pearson and Hoadley for the Egyptians. We have:

	Norwegians	Egyptians
Wagner ...	1454 (−2)	1417 (−23)
Pearson-Hoadley	1453 (−3)	1438 (− 2)
Observed	1456	1440

These results suggest three points, namely that:

(i) There would not appear to be any advantage obtainable by introducing $\bar{B}_f$, a measure undoubtedly difficult to determine accurately and destroying the hypothesis that we are taking *internal* measurements.

(ii) There is no special advantage in using the ellipsoidal coefficient ·0005236.

(iii) We are as likely to get as good a result from the Pearson-Hoadley form of product P_i' as from Dr Wagner's P_i.

We will accordingly discard the unsatisfactory $\bar{B}_f$, and the still more unsatisfactory $\bar{H}_i$ with its measurement to the bregma and its inadequate bregmatic thickness, and see whether the Pearson-Hoadley P_i' does or does not give as good results for an interracial formula as Wagner's formula. He had the advantage of measuring the brain-box internally for five races in both sexes, using in all 191 male and 182 female skulls. All his series were short, some of them terribly short, in six of the ten cases under 25 crania! We took the largest series in the Biometric Laboratory, 729 male adult crania, because our purpose was to test how far internal measurements gave a better result than external measurements, freeing ourselves as far as possible from the errors of random sampling. And we found they did—an improvement resulted of about 25 °/₀. Now Dr Wagner's great merit is to have increased our supply of data by measuring internally five races to which we can add a sixth, our Egyptians series. I have not the boldness to suggest an interracial formula based on four, or even six races, but I do wish to show that a formula better fitting those six races can be found by the usual processes of statistical theory, than by "guessing" A and then averaging B. Before doing so I should like to point out that for racial comparisons, as far as cranial capacity is concerned, but little stress can be laid on racial estimates of capacity which differ from the observed by 25 to 50 cm.³ Thus consider four of Dr Wagner's shorter series and compare them with previously determined capacities for longer series:

Dr Wagner's Series		Race	Previous Values	
Capacity	Size of Sample		Capacity	Size of Sample
1418	10	Maori ...	1476	43
1278	13	Australians	1295	134
1526	18	Eskimos ...	1559	34
1473	63	Lapps ...	1522	66

(References are given on p. 143.)

It is obvious from these results that even if Dr Wagner's interracial formula gave exactly in these four races his observed capacities, those capacities would not agree within 58, 18, 33, and

49 cm.³ with the values obtained from larger samples. This may be due to the size of the samples, to differences in technique, or to the influence of local races; but, clearly, any interracial prediction formulae coming within 25 cm.³ to 50 cm.³ of each other may easily be equally valid, if we are determining a "racial" character, not seeking a short cut to find the mean capacity of a series of crania.

I propose now to consider in the first place three prediction formulae for cranial capacities on the basis of the six races for which internal cranial measurements have been taken, namely:

(*a*) Dr Wagner's formula.

(*b*) The "best" formula that could be obtained from Dr Wagner's data, by the use of correlational theory.

(*c*) The Pearson-Hoadley form of product, using all the available data.

We need the following values:

Race	$\bar{L}_i$	$\frac{1}{2}(\bar{B}_f+\bar{B}_i)$	$\bar{H}_i$	$\bar{P}_i$	$\bar{B}_i$	$\bar{H}_i'$	$\bar{P}_i'$
Norwegians	172·28	126·51	107·96	235,3003·617	131·05	127·07	293,4571·607
Lapps ...	168·05	130·33	108·71	238,0901·691	140·05	124·88	293,9101·061
Maori ...	171·21	121·78	109·19	237,0606·455	131·31	130·44	293,2407·999
Australians	163·77	117·31	106·46	204,5294·477	123·38	128·77	260,1910·220
Eskimos ...	176·53	122·36	114·92	248,2200·255	130·78	133·61	308,1509·744
Egyptians E	170·44	124·03*	107·07	228,2450·515	132·14	128·85	280,1052·175

From these we obtain for (*b*) by the method of correlation,

$$C = \cdot 00050241178\,\bar{P}_i + 130\cdot 345 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (b),$$

and for (*c*) by the same method,

$$C = \cdot 00051495732\,\bar{P}_i' - 61\cdot 083 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c),$$

while (*a*) is as before,

$$C = \cdot 0005236\,\bar{P}_i + 222\ \text{cm.}^3 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

From these formulae results the following table:

Race	Observed Capacity	Wagner's (*a*)	"Best" P_i Formula (*b*)	Pearson's P_i' Formula (*c*)
Norwegians ...	1456	1454 (− 2)	1460 (+ 4)	1450 (− 6)
Lapps ...	1473	1469 (− 4)	1475 (+ 2)	1452 (−21)
Maori ...	1418	1414 (− 4)	1417 (− 1)	1449 (+31)
Australians ...	1278	1293 (+15)	1287 (+ 9)	1279 (+ 1)
Eskimos ...	1520	1522 (− 4)	1532 (+ 6)	1527 (+ 1)
Egyptians E	1440	1417 (−23)	1420 (−20)	1433 (− 7)
Mean Root Square Residual		11·59	9·47	15·75

Equation (*b*) shows that using Wagner's value for P_i the ellipsoidal coefficient does not give the best values, nor does Wagner's value of the constant term. We improve the root mean square residual by about 9·4 per cent. by using a correct statistical procedure. We see further as a result of the above table that the Pearson-Hoadley P_i' gives a worse result than Dr Wagner's,—at least when we deal with the above six races. This is a misfortune, for its $\bar{L}_i$, $\bar{B}_i$ and $\bar{H}_i'$ are so much

* $\bar{B}_f$ for these Egyptians = 115·92 mm.

easier of determination than using both $\bar{B}_f$ and $\bar{B}_i$, and the complicated and theoretically very doubtful $\ddot{H}_i$. But it is clear that using the internal vertical basion height H_i, so easy to determine, does not provide a formula so good, judged by the square root mean square residual, as Dr Wagner's. There are points to be noted however in the formula (c). It gives the highest and lowest race capacities excellently, far better than (a) or (b); it fails peculiarly with the Maori capacity, but here it must be noted that Dr Wagner's measure of capacity is much below that of Scott, indeed by almost 50 cm.³. I defer a closer discussion of this point till we turn to external diametral product formulae. Meanwhile I will try to reach a formula better than Dr Wagner's in which all the quantities to be measured can be obtained in a fairly simple manner, and which avoids his $\bar{B}_f$ and $\ddot{H}_i$.

We have seen so far that the vertical basion height, while not bad for a component of an inter-racial formula, is not as good as Dr Wagner's $\ddot{H}_i$. Let us see what the vertical opisthion height $\bar{H}_i''$ will lead us to.

The table of needful measurements is the following:

Race	L_i	$\bar{B}_i$	H_i''	P_i''	Predicted Value	Deviation from Observed
Norwegians ...	172·28	134·05	129·28	298,5609·064	1476·5	+20
Lapps ...	168·05	140·05	127·60	300,3117·359	1485	+12
Maori ...	171·21	131·31	124·29	279,4236·212	1383	−35
Australians ...	163·77	123·38	129·08	260,8183·071	1291	+13
Eskimos ...	170·53	130·79	131·39	303,3347·507	1500	−26
Egyptians E...	170·44	132·14	130·05	294,2491·670	1455	+15

Obtaining the "best" formula by the method of least squares, we have:

$$C = \cdot 0004004 2125\ \bar{L}_i \bar{B}_i \ddot{H}_i'' + 12\cdot 310 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(viii)}.$$

Substituting the products P_i'' in the formula, we obtain the predictions given in the sixth column of the above table, which provide the deviations from the observed values given in the seventh column. These have a root mean square residual of 21·75 cm.³, not a very serious resulting error when we think of the amounts by which one sample of a race differs from a second, but worse than the formula which uses the vertical basion height, and very considerably worse than the "best" formula using Wagner's P_i. At first sight it would appear as if nothing had been obtained by investigating the results of using the vertical basion and vertical opisthion internal heights. But a careful examination of the two sets of deviations shows that if we use formula (c) in round numbers seven times and formula (viii) five times, i.e. weight them with $\frac{7}{12}$ and $\frac{5}{12}$ respectively, the large deviations will all be reduced to insignificant proportions. We reach by this procedure the formula:

$$C = (\cdot 00030039177\ H_i' + \cdot 00020434210\ \bar{H}_i'')\ \bar{L}_i \bar{B}_i - 30\cdot 503 \quad \ldots\ldots\ldots\ldots\ldots\text{(ix)}.$$

We have then the following results:

Race	Predicted Value	Deviation from Observed	
Norwegians ...	1461	+ 5	Root Mean Square Residual =6·10
Lapps ...	1466	− 7	
Maori ...	1421	+ 3	
Australians ...	1284	+ 6	
Eskimos ...	1516	−10	
Egyptians E...	1442	+ 2	

For practical purposes it is adequate to use

$$C = \frac{(\cdot 3001\, \bar{H}_i' + \cdot 2013\, \bar{H}_i'')\, \bar{L}_i \bar{B}_i}{1000} - 305\cdot 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (\text{ix}^{\text{bis}}),$$

where $\bar{H}_i'$ and $\bar{H}_i''$ are the mean internal vertical heights from basion and opisthion respectively, $\bar{L}_i$ is the mean of half the sum of the two maximum internal lengths, and $\bar{B}_i$ the maximum internal breadth.

Formula (ix$^{\text{bis}}$) will predict *for these six races* the cranial capacity with a mean residual almost half that of Wagner's, and about two-thirds of that of the "best" formula available when using Wagner's expression for the product. Are we then to say that we have obtained a satisfactory *interracial* formula? Personally, I should say most decidedly not. You cannot form a satisfactory interracial formula on the basis of six races, any more than an intraracial formula on six skulls*, even if they are wisely distributed. There is only one brachycephalic race—the Lapps—included, and no proper representation of Asia, Europe, America or Africa. Another group of six quite different races would probably give an entirely different equation. This is a problem which I would urge Dr Wagner to tackle; he would at least increase our data for internal cranial comparisons.

Pearson and Hoadley *demonstrated* for the first time that the internal diametral product did give a higher correlation with cranial capacity than the external, and they measured the amount of the difference on their series of Egyptians. Dr Wagner has not worked out this difference for any of his five series, but he has contributed to our material for internal measurement. This is a valuable step in the right direction, but till the data are increased six- to tenfold on considerably longer series than some he has used, we ought not to set about finding, still less speak of, an interracial formula. Indeed, before even this is attempted, there ought to be some standardisation of instrument used for internal measurement, and further a standardisation of technique in measurement, especially that of the capacity. I am inclined to think, for example, that Dr Wagner is getting on his series lower capacities than other craniologists would do in using apparently the same method. What is more, I do not believe many craniologists would accept his process of deducing the internal "ear-bregma" height H_i by subtracting from the external "ear-bregma" height, a quantity he terms the "thickness of the parietal bone," which he does not measure at the bregma. Even if he did measure it at the bregma, it does not follow that the external "ear-bregma" line cuts the cranial vault at right angles. He does not make use of the quantity he terms the glabellar thickness of the frontal bone, but I am inclined to think that his process does not measure this thickness. I hold, however, that standardisation of measurements is essential before more work is done, and that most probably the L_i, B_i and H_i', H_i'' (all measured by Dr Wagner, but the latter two not used) will be easier to accurately determine, and give better results than using B_f and H_i as finally adopted by him.

3. Lastly, I should like to say a few words on Dr Lee's and my external diametral product prediction formulae. They give as closely as it is possible to give a prediction formula with the $\bar{L}\bar{B}\bar{H}$ product *for the races on which they are based*. Thus (ii) give root mean square residuals $= 36\cdot 16$ cm.^{3} and $35\cdot 92$ cm.^{3}, and (iii) $22\cdot 17$ cm.^{3} and $35\cdot 92$ cm.^{3} respectively.

The same formulae applied to Dr Wagner's five races give:

(ii) $41\cdot 02$ cm.^{3} and $33\cdot 09$ cm.^{3}, as against $36\cdot 16$ cm.^{3} and $35\cdot 92$ cm.^{3};

(iii) $41\cdot 70$ cm.^{3} and $33\cdot 09$ cm.^{3}, as against $22\cdot 17$ cm.^{3} and $35\cdot 92$ cm.^{3}.

* The six skulls have an advantage over the six races. With caution their measurements may be accurately made, and the prediction formula follows by the usual statistical procedure. But with racial means a new factor enters into the problem, namely the size of the samples of each race, and this will affect the A and B constants of the prediction formulae, even if the individual crania of each race be measured with extreme accuracy.

While the female root mean square residuals have actually been decreased the male root mean square residuals, as we should anticipate, have been considerably increased, when we apply the formulae to the data for races on which they were not computed. But supposing our formulae were correctly applied, would a prediction with a root mean square residual of 41 cm.³ be valueless? Is it worth our while really striving at a root mean square residual of 9 or even 6 cm.³? I doubt it, if two observers, owing to difference of technique, to varying size of sample, or to the effect of local races, may differ by 20 to 60 cm.³ in their direct measures of capacity!

Thus far I have assumed that Dr Wagner has applied our formulae correctly; but he has not done so. He has replaced our $H = OH$ of the *Frankfurter Verständigung* by what he terms the "ear-bregma" distance. He does not state how he measures this or how he found his poria. But assuming these to be determined by the skull on the craniophor, our OH is not his "ear-bregma" height, as he supposes. It may in some races be as much as 2–3 mm. greater (e.g. in the Maori or the Tibetan skulls), or there may be practically equality as in the case of the type English skull. It all depends on the slope of the cranial roof from apex to bregma. 1—3 millimetres may not appear much regarded in itself, but it modifies most sensibly P the product. For example, taking the Lapps, if the OH exceed by 2·5 mm. the ear-bregma distance in males, then formula (ii) would predict the value 1466 instead of 1419, and be in error -7 instead of -54, while if the female OH exceeded by 0·8 mm. the ear-bregma distance, the predicted capacity would have been 1301 instead of 1291, showing no error instead of -7. I lay no stress on these illustrations; I only intend to indicate that the actual OH cannot be measured by the "ear-bregma" height; the difference depends on how the apex-bregma stretch of the crown slopes and how near together the apex and bregma are situated. We cannot therefore apply legitimately formulae (ii) and (iii), using the "ear-bregma" distance. We can only test what an external diametral formula will achieve by working one out for Wagner's data. I have worked out the best male prediction formulae for the above six races, when we take as our products of external measurements

$$P_t''' = LB(OH') \quad \text{and} \quad P_t^{\text{iv}} = LBH',$$

where OH' is not our vertical height OH, but the "ear-bregma" distance, and H' is not our vertical basion height H, but the basion bregma distance. All these external characters are provided by Wagner for his series.

We form the following table:

Race	L	B	OH'	P_t'''	H'	P_t^{iv}
Norwegians ...	187·70	142·34	113·28	302,6526·465	132·81	354,5313·723
Lapps ...	179·17	140·52	113·02	290,0909·720	129·05	338,7819·103
Maori ...	186·81	137·88	116·06	298,9390·527	136·63	351,9228·479
Australians ...	183·30	130·77	113·85	272,9000·553	135·92	325,8021·565
Eskimos ...	190·44	137·50	110·50	313,2104·5	139·39	305,1580·571
Egyptians E...	185·33	139·10	112·43	290,2757·030	133·51	344,7008·549

Now before going further we note:

(i) That the Lapp P_t''' and P_t^{iv} are less than those for the Norwegians. Accordingly, the predicted capacity will inevitably be less for the Lapps than for the Norwegians; the observed capacity for the former is greater than for the latter.

(ii) The Maori P_t''' and P_t^{iv} are greater than those for the Lapps. Accordingly, the predicted capacities for the Maori will be greater than for the Lapps; the observed capacity for the former is considerably less than for the latter.

(iii) The Egyptian E P_t''' and P_t^{iv} are less than those for the Maori, but the observed capacity of the latter is less than for the Egyptians.

These results indicate that the prediction formulae based on P_i'' and P_i^{iv} will not be very satisfactory in reproducing the capacities said to be observed. I obtained the following prediction formulae using:

OH', the "ear-bregma" length,

$$\bar{C}_i = \cdot 00057022174\ P_i'' - 272\cdot 514 \quad \text{.......................} (x);$$

H', the basion-bregma length,

$$\bar{C}_i = \cdot 000050513039\ P_i - 320\cdot 291 \quad \text{.......................} (xi).$$

We have the following results for males:

Race	Formula (x)	Deviation	Formula (xi)	Deviation
Norwegians ...	1471	+15	1472	+16
Lapps ...	1437	−30	1391	−82
Maori ...	1450	+32	1457	+39
Australians ...	1300	+22	1325	+47
Eskimos ...	1532	+ 6	1521	2
Egyptians E...	1400	−40	1421	−19
Root Mean Square Residual		27·88		42·90

From this table we can draw two conclusions:

(i) It confirms Dr Wagner's statement that it is better to base a prediction formula on the auricular height rather than on the basion height.

(ii) That the large values obtained by applying the Pearson-Lee formulae (apart from the error of using OH' for OH) arises in large part from applying them to a set of entirely different races from those on which they are based. If we take the best material for external measurement available in the data, namely, $\bar{L}$, $\bar{B}$, and $\bar{OH}'$, we have a root mean square residual of 27·88, a value as good as those obtained by applying our (ii) and (iii) to the data on which they were based, and far better than applying our (ii) and (iii) to Wagner's data. In other words, it is not legitimate to compare the relative efficacy of two formulae by applying one to the material on which it is based and the other to material totally different from that on which it is itself based.

I am far from asserting that we cannot get a better external formula than one with a root mean square residual of 28, and that we ought to use my internal product formula with a residual of 6, or even Dr Wagner's with a residual of about 12. There are many points to be considered about the data on which either type of formula is based. We cannot make comparisons on *internal* measurements, for they only exist at present for six series, but we can do so on external measurements. I have collected some to place beside Dr Wagner's data.

The table (p. 143) appears very instructive. We see that different recorders give distinctly different values for approximately the same racial group. Thus, take the Eskimo group; Fürst and Hansen are dealing with a series which has mean diameters undoubtedly smaller than Wagner's series. The relative $\bar{P}_i$'s would suggest a skull with 124 cm.³ less capacity than that of Wagner's series, yet their mean capacity is 2 cm.³ greater; it might be anticipated to be much less. Again, consider Hrdlička's series of Greenland Eskimo; his series, almost double that of Wagner's in number, suggests crania considerably smaller than those of Wagner, and the prediction formula suggests an average capacity 20 cm.³ less than Wagner's, but Hrdlička found a capacity 32 cm.³ *greater*.

Next turn to the Maori. Scott's series, more than double that of Wagner, suggests a smaller capacity by 20 cm.³ than Wagner's, yet his observed capacity is 58 cm.³ *greater* than Wagner's!

Race	Recorder	$\bar{C}$	$\bar{L}$	$\bar{B}$	H'	Γ_t	Predicted Capacity
Lapps	Wagner* (63)	1473	179·17	146·52	129·05	338,7819·103	1391
"	Schreiner† (151) ...	1465	178·3	147·0	128·1	335,7513·810	1376
"	Hällsten‡ (66)	1522	180·7	147·0	129·5	343,9895·550	1417
Australians (?)	Wagner* (13)	1278	183·30	130·77	135·92	325,8021·565	1325
" (Queensland)	Morant § (19)	1288	185·1	131·1	133·3	323,4739·113	1314
" (General) ...	Morant § (140)	1295	187·8	132·2	138·1	342,8030·796	1412
Maori	Wagner* (16)	1418	186·81	137·88	136·63	351,9228·479	1457
"	Scott‖ (43)	1476	185·5	140·1	137·6	357,6024·480	1486
Eskimos	Wagner* (18)	1520	190·44	137·56	139·39	365,1589·571	1524
" (Greenland)	Hrdlička¶ (34)	1559	190·5	135·9	139·5	361,1508·525	1504
"	Fürst and Hansen¶ (175)	1528	188·4	134·4	138·2	340,6486·272	1400

Considering the Australians, Morant's general series suggests by its diameters a series of crania larger by 83 cm.^{3} than Wagner's, and this is based not on 13 but 140 skulls, but the observed capacities only differ by 17 cm.^{3}. The Queensland and Wagner's series of about the same size suggest that the former falls short of the latter by 11 cm.^{3}, but the observed capacities differ in the inverse way by 10 cm.^{3}. This again suggests, it does not of course prove, that Wagner's value is too small.

Lastly, turning to the Lapp data, we note that Schreiner's and Wagner's, although differing much in predicted and observed values, give very close differences of observed and of predicted values, 8 cm.^{3} and 15 cm.^{3}. This is as it should be, if Wagner is a member of Schreiner's school. But if we turn to Hällsten's measurements, the latter's crania are larger for all diameters, and give a predicted value exceeding Wagner's by 26 cm.^{3}, yet the actually observed values are such that Hällsten's exceed Wagner's by 49 cm.^{3}.

Assuming, as I think we may, that the measured diameters will not differ largely by differences of personal equation in measurements, the above results do appear to indicate that Dr Wagner's technique in determining capacity gives results falling short of that of other observers; possibly, though I cannot prove it, they packed tighter. But what I do want to emphasise is this, that until we have standardisation in measurement of capacities, until we have such large series that errors of random sampling are negligible, and until we have such careful specification of the source of cranial series that we can define local races, various observers will give for the same "race" cranial capacities differing by any amount up to 60 cm.^{3}. In view of this, refined prediction formulae which provide a 6 cm.^{3} root mean square residual are scarcely to be preferred to rougher formulae with 20 to 30 cm.^{3} root mean square residuals. In other words, till the three conditions mentioned above are satisfied, I am still of the opinion that while internal measurements of the skull may lead us to interesting results as to the size characters of the brain, their value as leading to cranial capacity prediction is hardly worth the increased labour

* Memoir in this issue.

† *Zur Osteologie der Lappen*, Bd. II. S. 68. *Instituttet for Sammenlignende Kulturforskning*, Oslo, 1931.

‡ Hällsten: Matériaux pour servir à la connaissance des crânes des peuples Finnois. Bidrag till Kännedom af Finlands Natur och Folk *Finska Vetenskaps-Societeten*.

§ *Biometrika*, Vol. XIX. pp. 421, 429.

‖ *Biometrika*, Vol. XI. p. 98.

¶ *Annals of Eugenics*, Vol. I. pp. 261, 268.

of their determination. Overlooking for a moment the question of differences of technique, we may note that the standard error for capacity of a series of 13 crania is about 31 cm.³; of a series of 16, 28 cm.³; of a series of 18, 27 cm.³; of a series of 63, 14 cm.³; and of a series of 81, 13 cm.³; and further that even deviations 50 % greater than these in excess or defect will occur in 13 to 14 % of samples of these sizes. What, I ask, is the value of using a formula with a 12 or even 6 cm.³ residual error on the *observed* small samples to get a predicted racial mean capacity when our sampling deviations alone are of the above order?

One further point I would like to emphasise strongly. Pearson and Hoadley found in the case of 729 male Egyptian crania the internal length of the right hemisphere greater than that of the left hemisphere. This was so opposed to the statements of some earlier investigators, notably Broca, who had asserted that the localisation of certain important centres found in the left hemisphere led to or must lead to a greater size of the left hemisphere, that Dr Woo investigated this point by direct measurements on the frontal and parietal bones of a number of races. His conclusions for 887 Egyptian male crania showed larger external measurements on the right frontal and parietal bones. This evidence Dr Wagner (see his p. 116) dismisses somewhat light-heartedly. We may add that Dr Woo has found in the following series (i) English (120), (ii) Australian (100), (iii) Southern Chinese (80), (iv) Punjabis (80), (v) Kanakas (40), (vi) Congo Negroes (50), and (vii) Nigerian Negroes (45), for the three measurements (*a*) ophryon to stepanion, (*b*) minimum arc bregma to asterion, (*c*) minimum arc sphenion to lambda with various degrees of difference preponderance in the size of the *right* frontal and parietal bones. It is needless to remark that these are the districts in which the active centres are stated to be localised on the *left* side and so to give it its predominance. This predominance of the left side Dr Wagner tells us "is confirmed by certain other investigations which Hoadley and Pearson do not mention in their survey of the literature." He then refers to papers by Hrdlička (1907) and Connolly (1932). The former might have been referred to had we known of it, but hardly the latter, as our paper was published three years before Connolly's! In neither of these papers is the procedure exactly comparable with the measurements taken by Hoadley and Pearson, Dr Wagner, or Dr Woo, and for numbers Connolly's series are not comparable with those of Hoadley and Pearson. The contradictions are so great that one is almost compelled either to believe that there is really no predominance of either hemisphere, or to accept the impossible, namely that some investigators, when the skulls are in *norma basalis* with the frontal, or it may be the occipital, on their right, have wrongly designated their measurements, mistaking right for left hemisphere, or *vice versa*!

Finally with regard to the measurements made by Dr Wagner on the cranial contours published in this Journal and discussed by him on pp. 113—116 above, it is only fair to Dr Woo to remark that a similar investigation on the same contours was made two years ago by Dr Woo himself and will be found already published by the *Academia Sinica**. Personally I am doubtful as to this use of such contours, and certain that contours diagraphically drawn should not be included, for they do not represent points lying in one and the same plane, and accordingly the distances measured are not necessarily homologous lengths on left and right hand sides of a median line.

* "Evidence of the Asymmetry of the Human Skull. Derived from Contour Measurements," Ts'ai Yüan P'ei Anniversary Volume (Suppl. Vol. I. *Bulletin of the Institute of History and Philology, Academia Sinica*, Peiping, 1933). Dr Woo states that "where the differences are real and not chance ones, it is always the right side which is the greater," and again that "the main conclusions of the present enquiry are in perfect accord with those of my earlier study, both having shown that the normal human skull is not symmetrical since the right side is slightly larger than the left in several of its parts."

SOME PROBLEMS IN THE ANALYSIS OF REGRESSION AMONG k SAMPLES OF TWO VARIABLES

By B. L. WELCH, B.A.

I. Introduction.

In a series of papers J. Neyman and E. S. Pearson have developed a method based on a special application of the principle of likelihood, the object of which is to determine the most suitable test criteria to employ in testing certain statistical hypotheses.

E. S. Pearson and S. S. Wilks have extended the method to two variables. k samples, Σ_t, of two variables x and y being given, each sample was assumed to be drawn from a normal bivariate population π_t specified by parameters a_t, b_t, σ_{xt}, σ_{yt}, and ρ_t; i.e. the joint distribution of x and y for the population π_t was assumed to be of the form

$$p(x, y) = \frac{1}{2\pi\sigma_{xt}\sigma_{yt}\sqrt{1-\rho_t^2}} e^{-\frac{1}{2(1-\rho_t^2)}\left\{\frac{(x-a_t)^2}{\sigma_{xt}^2} - \frac{2\rho_t(x-a_t)(y-b_t)}{\sigma_{xt}\sigma_{yt}} + \frac{(y-b_t)^2}{\sigma_{yt}^2}\right\}}$$

Certain hypotheses about the population parameters were then considered*.

In problems of regression rather than correlation this initial assumption may not however be appropriate, and it is proposed in the present note to consider the application of these methods of approach to the case in which we are primarily concerned with the analysis of variation of y for a given value of x. In the type of problem under review the observations fall naturally into a number, say k, of groups, and within the tth group the deviations of y from the regression straight line $y = \beta_t x + \alpha_t$ are supposed normally distributed with standard deviation σ_t. Thus for the tth group the probability distribution of y for a given x is assumed to be of the form

$$p(y|x) = \frac{1}{\sqrt{2\pi}\,\sigma_t} e^{-\frac{1}{2\sigma_t^2}(y-\beta_t x-\alpha_t)^2} \quad \text{....................(1).}$$

In Fig. 1 a situation is shown where there are three groups, the sets of observations (x, y) for these being drawn for convenience so that they fall clear of one another.

The value of y that appears with any particular x in group t is regarded as a random deviate from a normal distribution whose mean is $\beta_t x + \alpha_t$ and standard deviation σ_t, where σ_t, β_t, and α_t are independent of x and are thus characters of the whole group t. Taking together all the values of x that can possibly occur in the group

* E. S. Pearson and S. S. Wilks: *Biometrika*, Vol. xxv, 1933, pp. 353—78.

a number of means $\beta_t x + \alpha_t$ are obtained which all lie on the line $Y = \beta_t X + \alpha_t$. These population regression lines are shown in Fig. 1 ($t = 1, 2, 3$). It should be noted that nothing has been said about the distribution of x, which need not even be continuous. Thus the array means do not necessarily lie continuously along the regression line. It follows that if, in a particular problem, the assumption of equation (1) can be made legitimately for certain x's in the group but not for others, then conclusions drawn from the criteria obtained below will still be valid within the restricted area defined by the permissible x's.

Our purpose is to apply the method of Neyman and Pearson to determine from the observed data criteria appropriate to test a variety of hypotheses regarding the group parameters σ_t, β_t, and α_t. One such hypothesis may be illustrated by con-

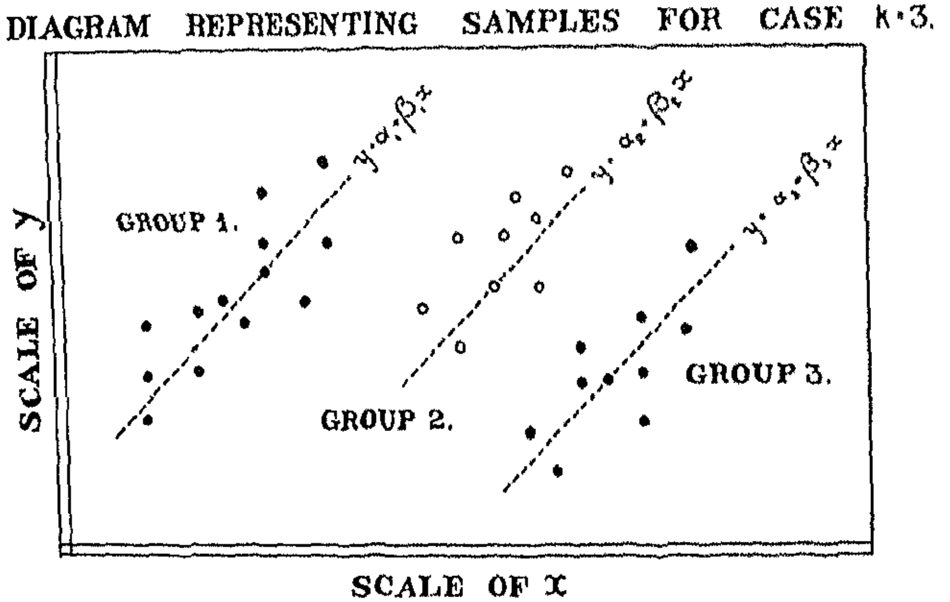

Fig. 1.

sidering the following example. An investigation into the heights of school-children in Glasgow indicated that within certain limits of age, the regression of height (y) on age (x) both for boys and for girls could be regarded as linear, and that for a given age the distribution of height could in each case be taken as normal with standard deviation the same for each age considered. Clearly here the distribution of x need not concern us. All that we need to know is the range of age within which the assumption of linearity can be made. Assuming that such relations hold for a number of different groups of children (they may belong to different schools or social classes or races) we might ask whether the same regression line can be used to predict height from age for all groups, and also whether the variance about this regression line can be regarded as being the same throughout. This is in effect asking whether σ_t, β_t, and α_t are the same for all t.

Another situation which may be tested is shown in Fig. 1 where $\beta_1 = \beta_2 = \beta_3$, but α_t is not constant. The regression lines here are parallel but not coincident.

The method employed below leads for each hypothesis considered to a different test criterion λ. The sampling distribution to which this criterion is referred is that which it would follow (were the hypothesis tested true) if in repeated sampling the values of x were kept fixed, so that the only variation is that of y. It will be seen, however, that this sampling distribution of λ is independent of the x's. Thus, provided the hypothesis tested concerns only relations between σ_t, β_t, and α_t which do not contain x, the condition of fixed x's can be removed and the test of significance can be regarded as being made for any set of x's subject only to equation (1) being satisfied.

II. THE METHOD.

The observations in each group may be described as forming a sample Σ_t $(t = 1, 2, \ldots k)$ of (x, y). As stated above, this sample will be regarded rather as a sample of y, in which each y corresponds to a given x. It will then be found convenient to speak of a population π_t from which Σ_t has been drawn, meaning by this that for each x in Σ_t the corresponding y is distributed normally about $\beta_t x + \alpha_t$ with standard deviation σ_t. π_t is then specified by the parameters σ_t, β_t, and α_t.

The probability function for the joint occurrence of the k samples Σ_t is obtained by multiplying together expressions of the type given in equation (1). If (x_{ti}, y_{ti}) is the ith individual of the sample Σ_t, n_t the number of individuals in that sample and $N = \sum\limits_1^k (n_t)$, then this function may be written

$$p = \left(\frac{1}{\sqrt{2\pi}}\right)^N \prod_1^k \left(\frac{1}{\sigma_t^{n_t}}\right) e^{-\frac{1}{2}\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where

$$\theta = \sum_{t=1}^{k} \sum_{i=1}^{n_t} \left\{\frac{(y_{ti} - \beta_t x_{ti} - \alpha_t)^2}{\sigma_t^2}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

It will be convenient to denote by A, B, C, D the following conditions:

A that included in equation (1), namely that

$$p(y \mid x) = \frac{1}{\sqrt{2\pi}\,\sigma_t} e^{-\frac{1}{2\sigma_t^2}(y - \beta_t x - \alpha_t)^2},$$

B that σ_t is the same for each group, i.e.

that $\sigma_t = \sigma \quad (t = 1, 2, \ldots k)$,

C that $\beta_t = \beta \quad (t = 1, 2, \ldots k)$,

D that $\alpha_t = \alpha \quad (t = 1, 2, \ldots k)$.

In addition it will be necessary to express algebraically the condition that the population means of the groups are collinear. If $\bar{x}_t$ and $\bar{y}_t$ are the means for the

n_t observations in the tth group and the symbol [ ] is used to denote expected or population value, then

$$[y_t] = \beta_t \bar{x}_t + \alpha_t \qquad (t = 1, 2, \ldots k),$$

and the condition that the point $(\bar{x}_t, [\bar{y}_t])$ lie on the line

$$y = lx + m$$

is that $$\beta_t \bar{x}_t + \alpha_t = l\bar{x}_t + m \qquad (t = 1, 2, \ldots k).$$

This will be called condition E.

Neyman and Pearson's method affords a simple rule for obtaining appropriate test criteria once two sets of conditions have been defined. These are: (*a*) the conditions which can be assumed to be satisfied, and (*b*) the conditions which define the hypothesis to be tested. For instance, suppose that in addition to the assumption implied in equation (1) we can assume that σ_t is the same for each group and we wish to test whether in addition all the β_t's are equal. Then, using the notation given above, the set of conditions (*a*) consists of A and B taken together and the set (*b*) of A, B, and C. The conditions (*a*) define a class Ω of admissible sets of populations π_t, and the conditions (*b*) define a sub-class ω, of Ω, to which the set π_t must belong if the hypothesis tested be true.

The maximum value of p (equation (2)), when the population parameters vary so that the set π_t belongs to Ω, is called p (Ω max.). The maximum value when the set π_t is restricted to ω is called p (ω max.). The likelihood of the hypothesis has then been defined to be

$$\lambda = \frac{p\,(\omega \text{ max.})}{p\,(\Omega \text{ max.})}.$$

The greater the value of λ the more likely is it judged that the set π_t is restricted to ω, i.e. that the hypothesis tested is true. The scale of λ on which judgment is based is obtained by referring λ to its sampling distribution when the hypothesis is true.

Much of the value of this method of approach lies in the fact that an essential first step in obtaining λ is the definition of the classes Ω and ω, this requirement encouraging clear thinking in the process of analysis.

III. Types of Hypotheses Considered and Notation Used.

The hypotheses that will be tested, regarding the regression of y upon x, may be divided into three classes analogous to those which Neyman and Pearson denoted by H, H_1, and H_2*. Firstly there is the hypothesis H, that σ_t is the same for every group, i.e. that the variance of y about the regression lines is the same throughout the whole range of data. This is the hypothesis that condition B above is true. Then there are several hypotheses of the H_2 type concerning the values of α_t and β_t (conditions C, D or E), in testing which H_1 is accepted as true. For each

* J. Neyman and E. S. Pearson, "On the Problem of k Samples," *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, Série A, 1931.

of these it is found that the present method leads to one of R. A. Fisher's tests in the analysis of variance. Finally there are hypotheses of the H type where nothing more is assumed than the condition A above and the hypothesis tested concerns both condition B and one or more of the conditions C, D, and E. The likelihood criteria for this type of hypothesis and for the H_1 hypothesis introduces into the analysis the geometric mean of sums of squares.

In developing the tests the following notation will be used. $\bar{x}_0$ and $\bar{y}_0$ are the means for the whole $N = \sum_t (n_t)$ observations, $\sum_i$ indicates summation for all observations (x_{ti}, y_{ti}) within the tth group, $\sum_t$ indicates summation for all groups, and $S = \sum_t \sum_i$.

$$\left.\begin{aligned} b_t &= \frac{\sum_i (x_{ti} - \bar{x}_t)(y_{ti} - \bar{y}_t)}{\sum_i (x_{ti} - \bar{x}_t)^2}, & b_a &= \frac{S(x_{ti} - \bar{x}_t)(y_{ti} - \bar{y}_t)}{S(x_{ti} - \bar{x}_t)^2}, \\ b_m &= \frac{S(\bar{x}_t - \bar{x}_0)(\bar{y}_t - \bar{y}_0)}{S(\bar{x}_t - \bar{x}_0)^2}, & b_0 &= \frac{S(x_{ti} - \bar{x}_0)(y_{ti} - \bar{y}_0)}{S(x_{ti} - \bar{x}_0)^2}, \end{aligned}\right\} \quad \ldots\ldots\ldots(4)$$

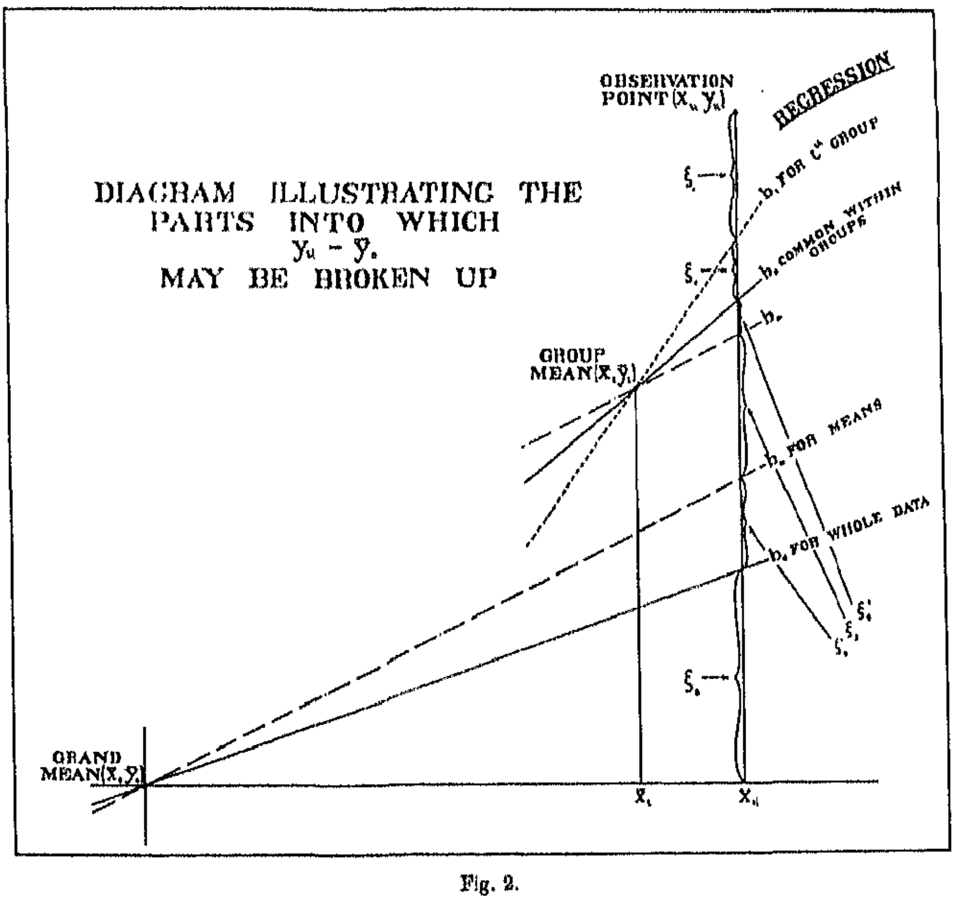

Fig. 2.

Thus b_t is the regression coefficient in the tth group, and b_a an average value obtained *from within groups on the assumption that the k values of b_t do not differ* significantly. b_m is the coefficient obtained from the weighted mean of the groups, and b_0 is obtained from pooling together the whole N observations. b_0 may also be regarded as a weighted average of b_a and b_m.

The corresponding regression lines are represented diagrammatically in Fig. 2, which shows also the parts into which $y_{ti} - y_0$ may be broken, as follows:

$$\left.\begin{array}{ll} \xi_1 = (y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t), & \xi_4' = (b_a - b_m)(x_{ti} - \bar{x}_t), \\ \xi_2 = (x_{ti} - \bar{x}_t)(b_t - b_a), & \xi_4'' = (b_m - b_0)(x_{ti} - \bar{x}_0), \\ \xi_3 = (\bar{y}_t - \bar{y}_0) - b_m(\bar{x}_t - \bar{x}_0), & \xi_5 = b_0(x_{ti} - \bar{x}_0). \end{array}\right\} \quad \text{........(5)}$$

The parts $\xi_1, \xi_2, \xi_3, \xi_4 (= \xi_4' + \xi_4'')$, and ξ_5 can be readily shown to be orthogonal, the product of any pair summed over the whole N observations vanishing. Hence, for example,

$$S(\xi_1 + \xi_2 + \xi_3 + \xi_4)^2 = S(\xi_1^2) + S(\xi_2^2) + S(\xi_3^2) + S(\xi_4^2) \quad \text{.........(6)}.$$

IV. The Test for H.

As an example of a hypothesis of the H type, take the inclusive hypothesis that the populations π_t have $\sigma_t = \sigma$, $\beta_t = \beta$, and $\alpha_t = \alpha$ $(t = 1, 2, \ldots k)$, i.e. that there is no differentiation whatever between the form of regression in the different groups. Suppose we wish to test this hypothesis assuming only the form of the probability functions.

Ω is defined by the condition A and p (Ω max.) occurs when

$$\beta_t = b_t; \; \alpha_t = \bar{y}_t - b_t\bar{x}_t; \; n_t\sigma_t^2 = \sum_i \{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2.$$

The conditions defining ω are A, B, C, and D. Thus to find p (ω max.) we write $\beta_t = \beta$, $\alpha_t = \alpha$, and $\sigma_t = \sigma$ in (2) and maximise with respect to σ, β, and α. This gives

$$\beta = b_0; \; \alpha = \bar{y}_0 - b_0\bar{x}_0; \; N\sigma^2 = S\{(y_{ti} - \bar{y}_0) - b_0(x_{ti} - \bar{x}_0)\}^2.$$

On substituting we find $\lambda = p(\omega \max.)/p(\Omega \max.)$, or using the more convenient criterion $L = \lambda^{2/N}$,

$$L = \frac{\sqrt[N]{\prod_1^k \left[\frac{1}{n_t}\sum_i \{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2\right]^{n_t}}}{\frac{1}{N} S\{(y_{ti} - \bar{y}_0) - b_0(x_{ti} - \bar{x}_0)\}^2} \quad \text{..................(7)},$$

i.e. the ratio of the weighted geometric mean of the variance about the regression lines b_t to the total variance about a line fitted to the whole data.

It will be seen from (5) and (6) that L may be written

$$L = \prod_t \left(\frac{N}{n_t}\right)^{\frac{n_t}{N}} \frac{\sqrt[N]{\prod_1^k \left[\left(\sum_i \xi_1^2\right)^{n_t}\right]}}{S(\xi_1^2) + S(\xi_2 + \xi_3 + \xi_4)^2}.$$

If the hypothesis H is true, it is known* that

$S(\xi_1+\xi_2+\xi_3+\xi_4)^2/\sigma^2$ is distributed as χ^2 with $N-2$ degrees of freedom,

$\sum_t(\xi_1^2)/\sigma^2$ is distributed as χ^2 with n_t-2 degrees of freedom,

and $S(\xi_2+\xi_3+\xi_4)^2/\sigma^2$ is distributed as χ^2 with $2k-2$ degrees of freedom.

It follows that L may be written in the form

$$L=\prod_t\left(\frac{N}{n_t}\right)^{\frac{n_t}{N}}\frac{\sqrt[N]{\prod_t(\chi_t^2)^{n_t}}}{\sum_t(\chi_t^2)+\chi_q^2},$$

where the $k+1$ values of χ^2 are independently distributed†.

Following the lines of Neyman and Pearson's proof for the case of a single variable it can be shown that the pth moment coefficient of L about zero is

$$\mu_p'=\frac{N^p}{\prod_1^k\left\{n_t^{\frac{pn_t}{N}}\right\}}\frac{\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-2}{2}+p\right)}\prod_t\left\{\frac{\Gamma\left(\frac{n_t-2}{2}+\frac{pn_t}{N}\right)}{\Gamma\left(\frac{n_t-2}{2}\right)}\right\}\qquad\qquad(8).$$

If the number of observations in each group is the same, i.e. $n_t=n$, then

$$\mu_p'=\frac{k^p\,\Gamma\left(\frac{N-2}{2}\right)}{\Gamma\left(\frac{N-2}{2}+p\right)}\left\{\frac{\Gamma\left(\frac{n-2}{2}+\frac{p}{k}\right)}{\Gamma\left(\frac{n-2}{2}\right)}\right\}^k\qquad\qquad(8').$$

As in the case of a single variable, it is probable that the sampling distribution of L can here again be represented approximately by the Type I distribution

$$p(L)=\frac{\Gamma(m_1+m_2)}{\Gamma(m_1)\Gamma(m_2)}L^{m_1-1}(1-L)^{m_2-1}\qquad\qquad(9),$$

the constants being calculated as follows by the method outlined by the above two writers‡:

$$m_1=\frac{\mu_1'(\mu_1'-\mu_2')}{\mu_2'-\mu_1'^2},\quad m_2=\frac{(1-\mu_1')(\mu_1'-\mu_2')}{\mu_2'-\mu_1'^2}\qquad\qquad(10),$$

where μ_1' and μ_2' are given by (8).

The probability that L is less than some specified value L_0 may then be found

* *A general theorem from which it follows that these sums of squares are distributed independently in* χ^2, is proved by R. A. Fisher in *Metron*, Vol. v. 1925. A proof following somewhat different lines is given by W. G. Cochran in *Proc. Camb. Phil. Soc.*, Vol. xxx. Part II. 1934.

† In the present case χ_t^2 and χ_q^2 have respectively n_t-2 $(t=1, 2, \ldots k)$ and $2k-2$ degrees of freedom: in Neyman and Pearson's case the degrees of freedom were n_t-1 and $k-1$ respectively. Their proof will be found in pp. 468—471, *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, Série A, 1931.

‡ *Loc. cit.* and also in *Biometrika*, Vol. xxiv, p. 415.

from the Incomplete B-function Tables*. If only the 5 % and 1 % points of L are wanted, they may be obtained by interpolating into R. A. Fisher's z-tables with

$$f_1 = 2m_2, \quad f_2 = 2m_1,$$

and then using the transformation $L = f_2/(f_2 + f_1 e^{2z})$, where the letters f_1 and f_2 are used to denote degrees of freedom in place of Fisher's n_1 and n_2.

V. The Test for H_1.

The class Ω is now defined by condition A, and ω by conditions A and B, i.e. the test is whether σ_t is constant for all groups. p (Ω max.) occurs when

$$\beta_t = b_t;\ \alpha_t = \bar{y}_t - b_t\bar{x}_t;\ n_t\sigma_t^2 = \sum_i \{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2.$$

p (ω max.) occurs when

$$\beta_t = b_t;\ \alpha_t = \bar{y}_t - b_t\bar{x}_t;\ N\sigma^2 = S\{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2.$$

On substituting and writing $L_1 = \lambda_1^{2/N}$ we have

$$L_1 = \frac{\sqrt[N]{\prod_t \left[\frac{1}{n_t}\sum_i \{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2\right]^{n_t}}}{\frac{1}{N} S\{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2} \quad \ldots\ldots\ldots\ldots\ldots(11),$$

i.e. the ratio of the weighted geometric mean of the variance about the regression lines b_t to the weighted arithmetic mean of the same. Our only concern is now with the parts ξ_1 (Fig. 2), and when H_1 is true L_1 can be written in the form

$$L_1 = \prod_t \left(\frac{N}{n_t}\right)^{\frac{n_t}{N}} \frac{\sqrt[N]{\prod_t (\chi_t^2)^{n_t}}}{\sum_t (\chi_t^2)},$$

where each χ_t^2 is distributed in the known χ^2 form with $n_t - 2$ degrees of freedom $(t = 1, 2, \ldots k)$.

The moments of L_1 can be found in the same way as those of Neyman and Pearson's single variate L_1, which is of the same form but with $n_t - 1$ degrees of freedom for χ_t^2. The pth moment of L_1 about zero is thus

$$\mu_p' = \frac{N^p}{\prod_t \left\{n_t^{\frac{pn_t}{N}}\right\}} \frac{\Gamma\left(\frac{N-2k}{2}\right)}{\Gamma\left(\frac{N-2k}{2} + p\right)} \prod_t \left\{\frac{\Gamma\left(\frac{n_t - 2}{2} + \frac{pn_t}{N}\right)}{\Gamma\left(\frac{n_t - 2}{2}\right)}\right\} \quad \ldots\ldots\ldots(12),$$

and if the number of observations in each group is the same ($n_t = n$), then

$$\mu_p' = \frac{k^p\,\Gamma\left(\frac{N-2k}{2}\right)}{\Gamma\left(\frac{N-2k}{2} + p\right)} \left\{\frac{\Gamma\left(\frac{n-2}{2} + \frac{p}{k}\right)}{\Gamma\left(\frac{n-2}{2}\right)}\right\}^k \quad \ldots\ldots\ldots\ldots\ldots\ldots(12').$$

* *Tables of the Incomplete Beta Function*, edited by Karl Pearson, *Biometrika* Office, University College, London.

As before, a Type I distribution is used as an approximation and m_1 and m_2 are calculated by substituting from (12) into (10). The Incomplete B-function Tables can then be used to give $p(L_1 < L_0)$.

In the case of a single variate tables have been given for certain points of the distribution of L_1 when the number of individuals in each sample is the same*. These tables may be used in the present test if they are entered with $n-1$ instead of n.

VI. THE TESTS FOR H_2.

The essential preliminary condition for testing hypotheses of the H_2 type is that σ_t should be the same for each group. If this is satisfied, θ of equation (3) can be written equal to ϕ/σ^2, where

$$\phi = S(y_{ti} - \beta_t x_{ti} - \alpha_t)^2.$$

Then
$$p = \frac{1}{(\sqrt{2\pi})^N \sigma^N} e^{-\frac{\phi}{2\sigma^2}}$$

Let ϕ (Ω min.) be the value of ϕ obtained by minimising ϕ for variation of the parameters such that the set of populations π_t belongs to Ω, and let $\phi(\omega$ min.) be similarly defined for ω. Then for $p(\Omega$ max.) we must have

$$\phi = \phi(\Omega \text{ min.}),$$

and also
$$\frac{dp}{d\sigma} = 0 = \left(\frac{1}{\sqrt{2\pi}}\right)^N e^{-\frac{\phi}{2\sigma^2}} \left[\frac{1}{\sigma^N}\frac{\phi}{\sigma^3} - \frac{N}{\sigma^{N+1}}\right],$$

i.e.
$$N\sigma^2 = \phi(\Omega \text{ min.}).$$

Thus
$$p(\Omega \text{ max.}) = \left(\frac{1}{\sqrt{2\pi}}\right)^N \left\{\frac{N}{\phi(\Omega \text{ min.})}\right\}^{\frac{N}{2}} e^{-\frac{N}{2}}.$$

Similarly
$$p(\omega \text{ max.}) = \left(\frac{1}{\sqrt{2\pi}}\right)^N \left\{\frac{N}{\phi(\omega \text{ min.})}\right\}^{\frac{N}{2}} e^{-\frac{N}{2}},$$

whence
$$L_2 = \lambda_2^{\frac{2}{N}} = \left[\frac{p(\omega \text{ max.})}{p(\Omega \text{ max.})}\right]^{\frac{2}{N}} = \frac{\phi(\Omega \text{ min.})}{\phi(\omega \text{ min.})},$$

or is the ratio of two minimum sums of squares.

To take a particular instance, suppose that σ_t is the same for each group and it is proposed to test whether β_t, the slope of the regression line, is the same within each group. The class Ω is defined by conditions A and B, and ω by conditions A, B, and C. For Ω, ϕ may be split up as follows:

$$\phi(\Omega) = S\{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2 + S\{(b_t - \beta_t)(x_{ti} - \bar{x}_t)\}^2 + S\{\bar{y}_t - \beta_t \bar{x}_t - \alpha_t\}^2,$$

* P. C. Mahalanobis, *Sankhyā, The Indian Journal of Statistics*, Vol. I. pp. 114 and 122. Additional rather fuller tables have been prepared for publication in the Department of Applied Statistics, University College, London.

the sums of squares having $N-2k$, k, and k degrees of freedom respectively. Similarly

$$\phi(\omega) = S\{(y_{ti} - y_t) - b_t(x_{ti} - x_t)\}^2 + S\{(b_t - b_a)(x_{ti} - x_t)\}^2$$
$$+ S\{(b_a - \beta)(x_{ti} - \bar{x}_t)\}^2 + S\{\bar{y}_t - \beta x_t - \alpha_t\}^2,$$

these sums of squares having $N-2k$, $k-1$, 1, and k degrees of freedom.

Minimising ϕ in each case with respect to the parameters at our disposal, is seen to be equivalent to making the terms containing those parameters vanish. Thus

$$L_2 = \frac{\phi(\Omega \text{ min.})}{\phi(\omega \text{ min.})} = \frac{S\{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2}{S\{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2 + S\{(b_t - b_a)(x_{ti} - \bar{x}_t)\}^2}.$$

When β_t is the same for each group, L_2 follows exactly the distribution of equation (9) with $m_1 = (N-2k)/2$ and $m_2 = (k-1)/2$, and the Incomplete B-function Tables can be used to obtain the exact significance level. If we are interested only in the 5 % and 1 % levels we can ask if $S(\xi_2)^2 = S\{(b_t - b_a)(x_{ti} - \bar{x}_t)\}^2$ is significantly large compared with $S(\xi_1^2) = S\{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2$*. These two sums of squares are independent and the L_2 test is thus equivalent to one of R. A. Fisher's tests in the analysis of variance. We may proceed in fact by calculating

$$z = \tfrac{1}{2}\log_e\left\{\frac{v_1/f_1}{v_2/f_2}\right\},$$

where

$$v_1 = S(\xi_2^2), \qquad v_2 = S(\xi_1^2),$$
$$f_1 = k-1, \text{ and } f_2 = N-2k.$$

The significance of z may then be tested by using R. A. Fisher's tables.

In the hypothesis tested, which may be denoted by $H_{2,1}$, we have been concerned only with the slope of the regression lines within the different groups, i.e. in testing whether β_t is constant. We may, however, be interested in testing a variety of other hypotheses regarding α_t and β_t which are all of the H_2 type, i.e. for all of which conditions A and B of p. 147 are assumed satisfied. In each case certain further conditions are assumed and others define the hypothesis to be tested. The resulting criteria of the likelihood method are precisely those which flow from the application of the analysis of covariance technique, but a brief indication in summarised form of the solution in the case of four further hypotheses may be of interest.

$H_{2,2}$. Finding after applying the test no reason to doubt $H_{2,1}$, we may assume that β_t is constant and test the further hypothesis that the regression lines for all groups are coincident. This is $H_{2,2}$.

$H_{2,3}$ and $H_{2,4}$. If $H_{2,2}$ is not satisfied we may still test the hypothesis that the population mean y's for each group lie on a straight line when plotted against the mean x's. Two cases arise:

(a) $H_{2,3}$; when it is not assumed that β_t is constant, i.e. when $H_{2,1}$ is not true.

(b) $H_{2,4}$; when it is assumed that $H_{2,1}$ is true.

* The meaning of the comparison is brought out by Fig. 2. We are asking if the parts ξ_2 are of significance having regard to the magnitude of the parts ξ_1.

$H_{2,5}$. The hypothesis that the regression lines for all groups are coincident may also be reached after testing first $H_{2,1}$ and then $H_{2,4}$, i.e. starting from the assumption that $H_{2,4}$ is true.

Reference should be made to Fig. 3, which shows diagrammatically the arrangement of the group regression lines when the various hypotheses are true. For $H_{2,2}$ and $H_{2,5}$ the regression lines should have been drawn to coincide but they have been side-stepped a little in order that the groups might be distinguished. The conditions determining the classes Ω and ω for each H_2 test are set out formally in the second and third columns of Table I.

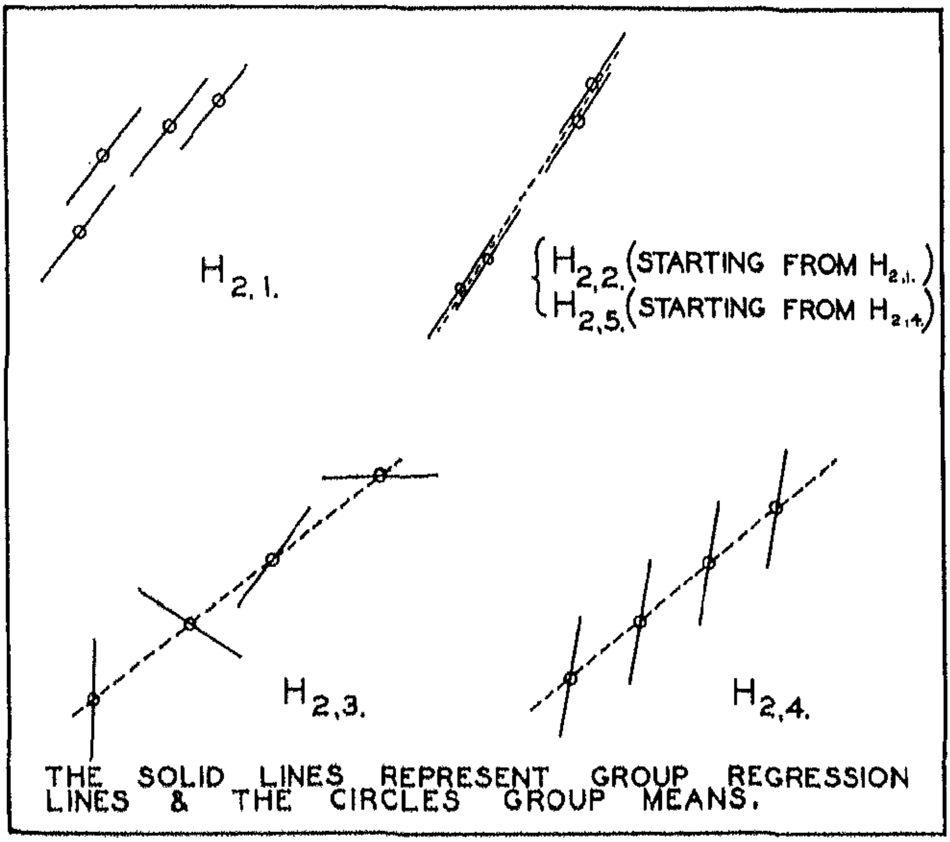

Fig. 3.

For each set of conditions the sum of squares $\phi = S\{y_{ti} - \beta_t x_{ti} - \alpha_t\}^2$ can be split up in a different way. For instance, when condition E holds, ϕ can in the first place be analysed into

$$\phi = S\{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2 + S\{(b_t - \beta_t)(x_{ti} - \bar{x}_t)\}^2 + S\{\bar{y}_t - \beta_t \bar{x}_t - \alpha_t\}^2.$$

The condition then permits the last sum of squares to be written

$$S\{\bar{y}_t - l\bar{x}_t - m\}^2,$$

and this breaks down further into

$$S\{\bar{y}_t - l\bar{x}_t - m\}^2 = S\{(\bar{y}_t - \bar{y}_0) - b_m(\bar{x}_t - \bar{x}_0)\}^2 + S\{(b_m - l)(\bar{x}_t - \bar{x}_0)\}^2 + S\{\bar{y}_0 - l\bar{x}_0 - m\}^2.$$

The algebra of the analysis of ϕ in other cases need not be set out here. The L_2 criteria for the different H_2 hypotheses have been shown to be given by

$$L_2 = \frac{\phi\,(\Omega \text{ min.})}{\phi\,(\omega \text{ min.})},$$

and it is seen that L_2 in each case can be written in the form $v_2/(v_1 + v_2)$, where v_1 and v_2 are independent sums of squares. The appropriate v_1 and v_2 and their respective degrees of freedom f_1 and f_2 are given in Table I*.

The exact significance level of L_2 can be found by using the Incomplete B-function Tables with $p = \frac{f_2}{2}$, and $q = \frac{f_1}{2}$ or alternatively, if we require only 5 % and 1 % levels of significance, we can use z, where

$$z = \tfrac{1}{2} \log_e \left\{\frac{v_1/f_1}{v_2/f_2}\right\}.$$

Something further should be added about the hypotheses $H_{2,3}$ and $H_{2,4}$ where the linearity of the means of the groups is considered. They differ from the others in that the relation being tested concerns not only the constants σ_t, β_t, and α_t but brings in also the mean x's. We are asking whether condition E can be true, i.e. whether $\beta_t \bar{x}_t + \alpha_t = l\bar{x}_t + m$ $(t = 1, 2, \ldots k)$. In the event of a positive answer being received to this question, it is not permissible to say that no matter what values $\bar{x}_t$ had been obtained the corresponding expected mean y's would be linear. Indeed if this were so, then it is obvious that we must have $\beta_t = l$ and $\alpha_t = m$, and thus have a common regression line for the whole data. It is however permissible to draw conclusions about the linearity of the means for any other sets of x's which lead to the same values $\bar{x}_t$, provided only that they satisfy the initial condition A. For we saw that the relation $\beta_t \bar{x}_t + \alpha_t = l\bar{x}_t + m$ enabled us to break down ϕ in such a way that it was possible to say that $v_1 = S\{(\bar{y}_t - \bar{y}_0) - b_m(\bar{x}_t - \bar{x}_0)\}^2$ was distributed as $\chi^2\sigma^2$ with $f_1 = k - 2$, and this takes no account of the way in which $\bar{x}_t$ is made up. It follows that the distribution of L_2, when condition E is satisfied, is independent of the partitioning of $\bar{x}_t$. Just as when testing relations which concern only σ_t, β_t, and α_t, it is possible to start by considering a fixed set of x's and then to remove this restriction, so in the present case owing to the nature of the distribution of the criterion we are able to extend our conclusions to any set of x's, subject only to the condition that $\bar{x}_t$ $(t = 1, 2, \ldots k)$ does not change, and that condition A remains satisfied.

* That v_1 and v_2 in each case are distributed as $\chi^2\sigma^2$ follows from the theorem already referred to. It will be found helpful to compare the sums of squares required for the different tests with the parts ξ_1, ξ_2 etc. shown in Fig. 2. A convenient form in which to calculate the sums of squares was set out by E. S. Pearson, in Appendix I to B. H. Wilsdon's paper, *Supplement to Journal of Royal Statistical Society*, Vol. 1. p. 180. The notation there used is the same as that of this paper.

TABLE I. *Conditions defining Ω and ω and sums of squares entering into tests for hypotheses H_2.*

Hypothesis	Conditions defining Ω	Conditions defining ω	v_1	f_1	v_2	f_2
$H_{2.1}$	A, B	A, B, C	$S\{(b_t-b_a)(x_{ti}-\bar{x}_t)\}^2=S(\xi_2^2)$	$k-1$	$S\{y_{ti}-\bar{y}_t-b_t(x_{ti}-\bar{x}_t)\}^2=S(\xi_1^2)$	$N-2k$
$H_{2.2}$	A, B, C	A, B, C, D	$S\{(\bar{y}_t-\bar{y}_0)+b_a(x_{ti}-\bar{x}_t)-b_0(x_{ti}-\bar{x}_0)\}^2=S(\xi_3+\xi_4)^2$	$k-1$	$S\{y_{ti}-\bar{y}_t-b_a(x_{ti}-\bar{x}_t)\}^2=S(\xi_1+\xi_2)^2$	$N-k-1$
$H_{2.3}$	A, B	A, B, E	$S\{(\bar{y}_t-\bar{y}_0)-b_m(\bar{x}_t-\bar{x}_0)\}^2=S(\xi_3^2)$	$k-2$	$S\{y_{ti}-\bar{y}_t-b_t(x_{ti}-\bar{x}_t)\}^2=S(\xi_1^2)$	$N-2k$
$H_{2.4}$	A, B, C	A, B, E	$S\{(\bar{y}_t-\bar{y}_0)-b_m(\bar{x}_t-\bar{x}_0)\}^2=S(\xi_3^2)$	$k-2$	$S\{y_{ti}-\bar{y}_t-b_a(x_{ti}-\bar{x}_t)\}^2=S(\xi_1+\xi_2)^2$	$N-k-1$
$H_{2.5}$	A, B, C, E	A, B, C, D	$(b_a-b_m)^2\dfrac{\{S(\bar{x}_t-\bar{x}_0)^2\}\{S(x_{ti}-\bar{x}_t)^2\}}{S(x_{ti}-\bar{x}_0)^2}=S(\xi_4^2)$	1	$S\{y_{ti}-\bar{y}_t-b_a(x_{ti}-\bar{x}_t)\}^2+S\{(\bar{y}_t-\bar{y}_0)-b_m(\bar{x}_t-\bar{x}_0)\}^2=S(\xi_1+\xi_2+\xi_3)^2$	$N-3$

TABLE II. *Summary of tests of crushing strength performed on normal concrete (y) and dry consistence mortar (x).*

Unit: lbs. per sq. in. Sums of squares and products given to nearest 1000.

Source	(1) Sum of Squares (y)	(2) Sum of Products (x and y)	(3) Sum of Squares (x)	(4) Regression Coefficient Col. (2) ÷ Col. (3)	(5) Col. (1) − Col. (4) × Col. (2)*
1 day ($t=1$)	3 323	7 368	16 781	$b_1=\cdot 43907$	88
3 days ($t=2$)	9 541	13 972	21 942	$b_2=\cdot 63677$	644
7 days ($t=3$)	7 920	11 326	20 327	$b_3=\cdot 55719$	1 609
28 days ($t=4$)	5 926	8 598	17 798	$b_4=\cdot 48307$	1 772
Within Ages	26 710	41 264	76 848	$b_a=\cdot 536951$	4 554
Between Ages	104 831	113 087	123 517	$b_m=\cdot 915560$	1 293
Whole Series	131 541	154 351	200 365	$b_0=\cdot 770349$	12 637

* Column (5) contains the sums of squares of the deviations from the respective regression lines, calculated most conveniently by noting that

$$\sum_i\{(y_{ti}-\bar{y}_t)-b_t(x_{ti}-\bar{x}_t)\}^2=\Sigma(y_{ti}-\bar{y}_t)^2-b_t\sum_i(y_{ti}-\bar{y}_t)(x_{ti}-\bar{x}_t),$$

and similarly for b_a, b_m and b_0.

If the question at issue is not whether the means $\bar{y}_t$ are linear but simply whether they differ significantly, then instead of condition E we shall have to test whether $\beta_t \bar{x}_t + \alpha_t = m$ $(t = 1, 2, \ldots k)$. The sum of squares $S(\bar{y}_t - \beta_t \bar{x}_t - \alpha_t)^2$ will become

$$S(\bar{y}_t - m)^2 = S(\bar{y}_t - \bar{y}_0)^2 + S(\bar{y}_0 - m)^2$$

and the L_2 criteria will be modified accordingly, $S(\bar{y}_t - \bar{y}_0)^2$ with $f_1 = k - 1$ being used instead of $S\{(\bar{y}_t - \bar{y}_0) - b_m(\bar{x}_t - \bar{x}_0)\}^2$ with $f_1 = k - 2$. The restriction on the x's is the same as before.

VII. The Hypotheses H.

Only one test of the H variety was treated in § IV. Without going into much detail, it is of interest to note that other hypotheses of this type can be considered in much the same fashion as were the hypotheses H_2. The criterion L each time takes the same form:

$$L = \prod_t \binom{N}{n_t}^{\frac{n_t}{N}} \frac{\sqrt[N]{\prod_t (\chi_t^2)^{n_t}}}{\sum_t (\chi_t^2) + \chi_q^2} \qquad \text{.........................(13).}$$

χ_q^2 is the only part that changes, and as it does the moments of equation (8) will be modified. In fact, if f is the number of *degrees of freedom* of χ_q^2, then the moments of L about zero, where there are n observations in each group, will be

$$\mu_p' = \frac{k^p \Gamma\left(\frac{N - 2k + f}{2}\right)}{\Gamma\left(\frac{N - 2k + f}{2} + p\right)} \left\{\frac{\Gamma\left(\frac{n-2}{2} + \frac{p}{k}\right)}{\Gamma\left(\frac{n-2}{2}\right)}\right\}^k \qquad \text{..............(14).}$$

The complete hypothesis considered above was a particular case where f was $2k - 2$. As one further instance of an H test, suppose that no other assumption than condition A is made and we wish to test whether conditions B and C hold, i.e. whether $\sigma_t = \sigma$ and $\beta_t = \beta$. This differs from the $H_{2,1}$ test in that constancy of σ_t is not now assumed, but is one of the elements to be tested. The L criterion is seen to be

$$L = \frac{\sqrt[N]{\prod_1^k \left[\frac{1}{n_t} \sum_i \{(y_{ti} - \bar{y}_t) - b_t(x_{ti} - \bar{x}_t)\}^2\right]^{n_t}}}{\frac{1}{N} S\{(y_{ti} - \bar{y}_t) - b_a(x_{ti} - \bar{x}_t)\}^2}.$$

Putting L into the form (13), we have

$$\chi_q^2 = \frac{S\{(b_t - b_a)(x_{ti} - \bar{x}_t)\}^2}{\sigma^2},$$

which has $k - 1$ degrees of freedom. The moments of L follow by putting $f = k - 1$ in (14), and then the 5 % and 1 % levels may be obtained by the method outlined on p. 152.

VIII. ILLUSTRATIVE EXAMPLE.

The data which will be used to illustrate the tests described above was collected at the Building Research Station. From the point of view of the H_2 tests, it has already been discussed by Mr B. H. Wilsdon. The problem concerns the relationship between crushing strength of concrete and mortar, and the particular data considered here are those on which his Table I (c)* is based. Corresponding tests were performed on dry mortar (x) and normal concrete (y) made from eight different cements ($n=8$, $i=1, \ldots 8$), and the results fall into four groups ($k=4$, $t=1, \ldots 4$) corresponding to four different ages of mortar and concrete. The relevant information is summarised in Table II.

First it may be asked whether the regression of concrete strength upon mortar strength can be represented by the same regression line for all four ages, and also at the same time whether the variance about that regression line is the same for each age. This is testing the complete hypothesis H. To calculate L, insert into (7) the values from Col. (5) of Table II, i.e. for $\sum_i \{(y_{ti}-\bar{y}_t)-b_t(x_{ti}-\bar{x}_t)\}^2$ ($t=1, 2, 3, 4$), 88, 644, 1609 and 1772 respectively, and for $S\{(y_{ti}-\bar{y}_0)-b_0(x_{ti}-\bar{x}_0)\}^2$, 12637. It is found that $L=\cdot 2008$.

From (8) the moments of L are

$$\text{Mean}=\cdot 7038874, \qquad \mu_2=\cdot 0128115, \qquad \text{i.e. } \sigma=\cdot 1132.$$

Following the method given for approximating to the 5 % and 1 % levels of L, we have from (10)

$$m_1=9\cdot 967092, \quad m_2=4\cdot 19297, \quad f_2=19\cdot 9342, \quad f_1=8\cdot 3859,$$

whence interpolating in R. A. Fisher's z-tables,

$$\begin{cases} z_{\cdot 05}=\cdot 4435, & z_{\cdot 01}=\cdot 6302, \\ L_{\cdot 05}=\cdot 4947, & L_{\cdot 01}=\cdot 4026. \end{cases}$$

L is clearly significantly small and the composite hypothesis must be rejected.

We may now investigate more closely where the heterogeneity lies. To test only whether the variances are equal, i.e. the hypothesis H_1, the appropriate criterion is that of equation (11). Substituting the values from Table II, we have $L_1=\cdot 6168$.

The lower 5 % and 1 % levels of L_1 may be obtained from the tables already referred to, or otherwise directly from (12) we have

$$\text{Mean}=\cdot 8798590, \quad \mu_2=\cdot 0078003, \quad \text{i.e. } \sigma=\cdot 08832,$$

giving $$m_1=11\cdot 0437, \quad m_2=1\cdot 5080.$$

* *Loc. cit.* p. 184. The figures given in the present paper are not exactly the same as those in Mr Wilsdon's table owing to the fact that the original figures as they were given to me were later altered. The discrepancies do not materially affect the conclusions reached.

Instead of proceeding by calculating f_1 and f_2 and interpolating into the z-tables, the exact significance level of L_1 may be found by evaluating

$$B_{\cdot 6168}(11{\cdot}0437,\ 1{\cdot}5080).$$

We obtain the chance that $L_1 < {\cdot}6168$ to be ${\cdot}0126$. It is thus unlikely that σ_t is constant for all ages.

Since the variances may differ there is doubtful justification in proceeding to test the hypotheses H_2, but the results are given in Table III for the sake of illustration*. z in each case is $\frac{1}{2}\log_e\left\{\frac{v_1/f_1}{v_2/f_2}\right\}$, where v_1 and v_2 are given in Table I and may be calculated from the figures of Table II.

TABLE III. *Summary of H_2 tests.*

Hypothesis	z	f_1	f_2	$z_{\cdot 05}$	$z_{\cdot 01}$	Significance of z
$H_{2,1}$	Negative	3	24	·5508	·7757	Not significant
$H_{2,2}$	1·3856	3	27	·5427	·7631	Significant
$H_{2,4}$	·0718	2	27	·6051	·8513	Doubtful
$H_{2,5}$	1·7585	1	29	·7155	1·0139	Significant

These tests would allow us to accept the equality of the regression coefficients for each group and perhaps the linearity of the means. The equality of the within groups regression coefficient (β) and the regression coefficient of the means (l), i.e. the hypothesis of a common regression line for the whole data, would have to be rejected.

In conclusion I should like to express my thanks to Dr E. S. Pearson for suggesting this problem to me and for the help he has given me at every stage of the work.

* The reader may be referred to a sampling experiment carried out by E. S. Pearson in the case of a single variable, which suggested that the H_2 test was not affected even by quite large changes in σ_t from group to group. See *Biometrika*, Vol. xxi. p. 340, 1929.

ON AN IMPORTANT CLASS OF STATISTICAL HYPOTHESES.

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1. Introductory.

Dr J. Neyman suggested to me as a topic for my thesis for a doctor's degree a problem which may be stated as follows.

Assume that it is known that each of the n independent variates

$$x_1, x_2, \dots x_n \qquad \dots\dots(1)$$

follows the Gaussian law of frequency,

$$p(x_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i - m_i)^2}{2\sigma^2}} \qquad \dots\dots(2),$$

with a common standard deviation, σ. The mean values, m_i, of the variates (1) are connected with s unknown parameters

$$a_1, a_2, \dots a_r, a_{r+1}, \dots a_s \qquad (s < n) \dots\dots(3)$$

by means of the formulae

$$m_i = c_{1i}a_1 + c_{2i}a_2 + \dots + c_{si}a_s \qquad (i = 1, 2, \dots n) \quad \dots(4),$$

where the coefficients c_{jk} are known and form a non-singular matrix

$$(c_{jk})_{\substack{j=1,2,\dots s\\ k=1,2,\dots n}} \qquad \dots\dots(5).$$

The problem to be solved consists in testing the statistical hypothesis, H, that r known linear and independent functions of the parameters (3) have some definite given values, for instance that

$$b_{1i}a_1 + b_{2i}a_2 + \dots + b_{si}a_s = B_i^0 \qquad \text{for } i = 1, 2, \dots r \leqslant s \dots\dots(6),$$

where the coefficients b and the constant terms B^0 are known numbers.

It will be convenient to call a statistical hypothesis of the above type a linear hypothesis.

Different aspects of the problem considered have been already treated by many authors, as by A. A. Markoff (1), "Student" (2), R. A. Fisher (3, 4), J. Neyman and E. S. Pearson (5, 6) and by J. Neyman (7). The points of view of these authors were different. I shall treat the problem in its general form following the lines indicated by Neyman and Pearson (8, 9 and 10). It will be seen that some of the calculations I shall have to perform and some of the results I shall obtain are identical with those already published by the authors quoted above.

2. Examples.

Before I proceed to the solution of the problem it will perhaps be useful to illustrate its generality.

Example 1. Suppose that $s=1$, and that the equations (4) reduce to the following:

$$m_i = a_1 \qquad (i=1,2,\dots n) \quad \dots\dots\dots\dots(7).$$

In this case the hypothesis, H, reduces itself to that of "Student," namely that

$$a_1 = a_1^0 \quad \dots\dots\dots\dots(8).$$

In fact the variates (1) may be considered as forming a sample for one unknown univariate normal population, π, and the hypothesis to test is that the mean of the population π is equal to a_1^0.

Example 2. Suppose that $s=2$, while the equations (4) are

$$m_i = a_1 \qquad \text{for } i=1,2,\dots r \quad \dots\dots\dots\dots(9),$$

and

$$m_j = a_2 \qquad \text{for } j=r+1,\, r+2,\dots n \quad \dots\dots\dots\dots(10).$$

The hypothesis, H, that

$$a_2 - a_1 = B_1^0 = 0 \quad \dots\dots\dots\dots(11)$$

(corresponding to the equations (6)) may now be formulated as follows. The variates (1) may be considered as forming two samples,

$$x_1, x_2, \dots x_r \quad \dots\dots\dots\dots(12)$$

and

$$x_{r+1}, x_{r+2}, \dots x_n \quad \dots\dots\dots\dots(13),$$

from two normal univariate populations π_1 and π_2, with common unknown variance σ^2, and the hypothesis tested consists in the assumption that the means of the populations π_1 and π_2 are equal. This is what we may call the generalised hypothesis of "Student."

Example 3. Assume $s>2$,

$$n = n_1 + n_2 + \dots + n_s \qquad (n_i > 1) \quad \dots\dots\dots\dots(14),$$

and suppose that the equations (4) are

$$\left.\begin{array}{lll} & m_i = a_1 & \text{for } i=1,2,\dots n_1, \\ \text{and} & m_j = a_2 & \text{for } j=n_1+1,\, n_1+2,\dots n_1+n_2, \\ & \dots\dots\dots\dots\dots\dots & \\ \text{generally} & m_i = a_k & \text{for } \sum_{j=1}^{k-1} n_j < i \leqslant \sum_{j=1}^{k} n_j, \end{array}\right\} \quad \dots\dots\dots\dots(15),$$

where $k=1,2,\dots s$.

Consider the hypothesis, H, which may be expressed by means of $s-1$ equations of the form (6):

$$a_k - a_s = B_k^0 = 0 \qquad \text{for } k=1,2,\dots s-1 \quad \dots\dots\dots\dots(16).$$

It will be easily seen that the variates (1) may now be considered as forming s samples from s normal univariate populations $\pi_1, \pi_2, \dots \pi_s$, with a common standard deviation σ. The hypothesis to test is, that the means of all the populations are equal. This is the case considered by Neyman and Pearson (hypothesis H_2 in their notation) and also by R. A. Fisher in his analysis of variance. However it will be seen that the statement of the problem and the point of view of the solution will be different from those of Fisher.

Example 4. Put $s = 2$ and assume the equations (4) to be of the form

$$m_i = a_1 + a_2 y_i \qquad (i = 1, 2, \ldots n) \ldots\ldots\ldots\ldots (17),$$

where the numbers y_i are known.

We may now consider three important hypotheses:

H_1, assuming that $\quad a_1 = a_1^0 \quad \ldots\ldots\ldots\ldots (18),$

H_2, assuming that $\quad a_2 = a_2^0 \quad \ldots\ldots\ldots\ldots (19),$

H_3, assuming that $\quad a_1 + a_2 Y = X \quad \ldots\ldots\ldots\ldots (20),$

where a_1^0, a_2^0, X and Y are given numbers.

The hypotheses H_1, H_2 and H_3 are connected with certain problems of correlation theory. Consider a population π of individuals with characters x and y and suppose that for each value of y the character x is normally distributed about its mean, say,

$$m_y = a_1 + a_2 y \quad \ldots\ldots\ldots\ldots (21),$$

with the standard deviation σ, independent of the value of y. The population π may be considered as a set of populations, say π_y, each of them being determined by the value of the character y, common to all its elements. The variates (1) may now be considered as a sample drawn from the population π. If besides the values of (1) we know also the corresponding values of the character y,

$$y_1, y_2, \ldots y_n \quad \ldots\ldots\ldots\ldots (22),$$

where some of the y's may be equal, but at least two of them should be different, then we may consider each value of x as belonging to an individual, drawn at random from the partial population π_y.

The hypothesis H_1 assumes that the constant term in the regression equation (17) has a certain definite value. This hypothesis, in its application to some problems of agricultural experimentation, has been considered by J. Neyman(11).

The hypothesis H_2 concerns the value of the regression coefficient of x on y. Its test is to be found in the book by R. A. Fisher(12).

Finally the hypothesis H_3 has been dealt with by H. Schultz(13) and—in a more general way—by Miss K. Iwaszkiewicz and concerns the error of the ordinate of a fitted regression line. The result obtained by K. Iwaszkiewicz is published—with a suitable reference—in a paper by J. Neyman(7) showing its importance to some agricultural problems*.

The hypotheses considered in Example 4 may be easily generalised for the cases when the regression of x on y is not linear. This hypothesis in its general form is treated in Fisher's book(12).

Example 5. Put $s = 2r$ and assume

$$n = n_1 + n_2 + \ldots + n_r \qquad (r > 1) \quad \ldots\ldots\ldots\ldots (23).$$

The equations (4) will be written

$$m_i = a_{2j-1} + a_{2j} y_i \quad \text{for} \sum_{k=1}^{j-1} n_k < i \leqslant \sum_{k=1}^{j} n_k \qquad (j = 1, 2, \ldots r) \quad \ldots\ldots (24).$$

* The method of testing the hypothesis H_3 is given in § 8.

In applications we have sometimes to consider the hypothesis H_1 expressed by means of $r-1$ equations

$$a_{2k} - a_{2r} = B_k^0 = 0 \qquad (k = 1, 2, \dots r-1) \quad \dots(25),$$

and also the hypothesis H_2, that

$$\left.\begin{aligned} a_{2k-1} - a_{2r-1} = B^0_{2k-1} = 0 \\ a_{2k} - a_{2r} = B^0_{2k} = 0 \end{aligned}\right\} \qquad (k = 1, 2, \dots r-1) \quad \dots(26).$$

These hypotheses may be expressed in statistical terms as follows. Consider r populations $\pi_1, \pi_2, \dots \pi_r$ of individuals having some characters x and y. Suppose that the regression of x on y in each population π_i is known to be linear. Suppose further that, whatever the population π_i and whatever the specified value of y, the variance of x in the corresponding y-array is known to have the same (though unknown) value σ^2. The hypothesis H_1 consists in the assumption that the coefficients of regression of x on y in all r populations have equal values. The hypothesis H_2 is more stringent and assumes that not only the coefficients of regression but also the constant terms in the regression equations have a common value in all populations $\pi_1, \pi_2, \dots \pi_r$, thus that the regression lines of x on y are in all populations identical.

As far as I know these hypotheses were first considered by Miss K. Iwaszkiewicz (14), who showed their importance in agricultural experimentation and in some industrial research problems.

Further examples could be easily given, but I think that those which have been described above, together with those given in §§ 5 and 8, are sufficient to show the generality of the problem.

It is here assumed that, before applying the results of this paper to any group of experimental data, a test has been applied to show that the assumption of the standard deviation of the x's being constant is actually satisfied *.

3. Simplification of the Problem.

Before I proceed to the solution of the problem, I shall reduce it to a simpler form.

It may be shown that the problem of testing the hypothesis H, expressed by r independent linear equations with regard to the parameters a with known coefficients, can be reduced to that of testing the hypothesis, say H_1, ascribing to r of the parameters a some definite values. In fact, the equations (6) being independent, we may solve them with regard to some r parameters a. As the order in which the parameters are numbered is of no importance, we may assume that the solution gives

$$a_k = a_k(B_1^0, B_2^0, \dots B_r^0, a_{r+1}, a_{r+2}, \dots a_s) \qquad (k = 1, 2, \dots r) \quad \dots(27).$$

* This must either have been previously ascertained or else it must be done on the data themselves. In the case of bivariate populations this certainly would require that there should be enough observations of the x's with the same y to determine with reasonable accuracy the array variance.

Substituting the above values of a_k ($k = 1, 2, \ldots r$) into the equations (4), we get similar linear equations, say,

$$m_i = c_{1i}' B_1^0 + c_{2i}' B_2^0 + \ldots + c_{si}' a_s \quad \ldots\ldots\ldots\ldots\ldots\ldots(28),$$

with known coefficients c'. Now it may be assumed that the means m_i are linear functions of the form (28) of s parameters

$$B_1, B_2, \ldots B_r, a_{r+1}, a_{r+2}, \ldots a_s \quad \ldots\ldots\ldots\ldots\ldots\ldots(29)$$

and that the hypothesis H determines the values of r of these parameters, namely,

$$B_k = B_k^0 \qquad (k = 1, 2, \ldots r).$$

Thus if we are able to test any hypothesis H_1, we shall be able to test also the hypothesis of the type H.

4. SOLUTION BASED ON THE PRINCIPLE OF LIKELIHOOD.

I shall now give a solution of the problem of testing the hypothesis H_1, following the lines indicated by J. Neyman and E. S. Pearson (8), that is to say the solution based on the principle of likelihood (in the sense of the word used by Neyman and Pearson—not by R. A. Fisher). This is as follows. The first stage consists in stating Ω, or the set of admissible simple hypotheses.

We shall include in the set Ω any hypothesis, ascribing to the unknown parameters $\sigma, a_1, a_2, \ldots a_s$ any definite values, with the only restriction that $\sigma > 0$.

The second stage consists in defining the set ω of simple hypotheses belonging to the composite hypothesis, H_1, to be tested.

The set ω will contain any hypothesis which may be described by

$$a_1^0, a_2^0, \ldots a_r^0, a_{r+1}, a_{r+2}, \ldots a_s, \sigma \quad \ldots\ldots\ldots\ldots\ldots\ldots(30),$$

where the a_i^0's are specified by the hypothesis tested, H_1, while $a_{r+1}, a_{r+2}, \ldots a_s$ and $\sigma > 0$ are arbitrary.

Further proceedings are as follows. We suppose that the observations provide us with values of the variates $x_1, x_2, \ldots x_n$. We have to calculate the simultaneous frequency function of these variates, determined by a simple hypothesis contained in Ω. Let it be $p(x_1, x_2, \ldots x_n)$. Obviously $p(x_1, x_2, \ldots x_n)$ will depend upon the values of parameters a_i and σ. We keep the x's constant, and choose out of the set Ω the simple hypothesis for which $p(x_1, x_2, \ldots x_n)$ is maximum. This absolute maximum of $p(x_1, x_2, \ldots x_n)$ will be denoted by p_Ω. Next we keep again x's constant, and choose out of the hypotheses included in ω the one for which $p(x_1, x_2, \ldots x_n)$ is maximum. This will be a relative maximum and will be denoted by p_ω.

According to the principle mentioned, the test of the hypothesis H_1 should be based on the values of the likelihood (in the sense of Neyman and Pearson), namely,

$$\lambda = \frac{p_\omega}{p_\Omega} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(31).$$

If for any set of observational data λ is small, this means that the set of admissible hypotheses contains some of them, from the point of view of which the

facts observed are much more probable than from the point of view of the hypothesis tested. In these circumstances we are intuitively inclined to reject the tested hypothesis. In order to judge whether the value of λ is "small" or "large," we consider the probability P of rejecting unjustly the hypothesis tested, that is to say the probability of rejecting H_1, when it is true. In most practical cases we may agree upon the value of P which may be considered as "small." For instance, we may accept the view that it will do no great harm if we reject the hypothesis unjustly not oftener in the long run than say once in a hundred times. This arbitrarily chosen value of the probability of an error in rejecting the hypothesis H_1 has been termed by Neyman the confidence coefficient ϵ. The choice of ϵ determines the limit between "small" and "large" values of λ. Denote by

$$P\{(\lambda \leqslant \lambda_0) \mid H_1\}$$

the probability, determined by the hypothesis H_1, that the observations will give us the value of $\lambda \leqslant \lambda_0$, where λ_0 is any number between zero and unity. Choose now λ_0, so that we have

$$P\{(\lambda \leqslant \lambda_0) \mid H_1\} = \epsilon \quad \text{.............................(32)},$$

and consider it as a limit between the values of λ "small" and "large." Obviously, if we agree to reject the hypothesis tested when $\lambda \leqslant \lambda_0$, the probability of rejecting H_1 unjustly will be

$$P = \phi(H_1) P\{(\lambda \leqslant \lambda_0) \mid H_1\} \leqslant P\{(\lambda \leqslant \lambda_0) \mid H_1\} = \epsilon \quad \text{........(33)},$$

where $\phi(H_1) \leqslant 1$ denotes the unknown probability *a priori* of the hypothesis H_1.

Thus the test of the hypothesis H_1 requires the knowledge of the probability

$$P\{(\lambda \leqslant \lambda_0) \mid H_1\},$$

or of the frequency distribution of λ, determined by the hypothesis tested, H_1.

Having thus described the problem, we shall proceed to its solution.

The simultaneous frequency function of the variates (1) will be

$$p(x_1, x_2, \dots x_n) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-\frac{\sum_{i=1}^{n}(x_i - m_i)^2}{2\sigma^2}} \quad \text{..................(34)}.$$

The process of maximising (34) with regard to the a_i and σ is simpler, if we try to maximise $\log p(x_1, x_2, \dots x_n)$ instead of (34). Taking into account the equations (4) connecting the m's with the parameters a, we may write

$$\frac{\partial \log p}{\partial \sigma} = -\frac{n}{\sigma} + \frac{\chi^2}{\sigma^3} = 0 \quad \text{..........................(35)},$$

$$\frac{\partial \log p}{\partial a_j} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - c_{1i}a_1 - c_{2i}a_2 - \dots - c_{si}a_s)\, c_{ji} = 0 \quad \text{........(36)},$$

where

$$\chi^2 = \sum_{i=1}^{n}(x_i - m_i)^2 = \sum_{i=1}^{n}(x_i - c_{1i}a_1 - c_{2i}a_2 - \dots - c_{si}a_s)^2 \quad \text{........(37)}.$$

Obviously the equations (36) are those which we should get if we wanted to minimise χ^2. The possibility of their solution is guaranteed by the condition concerning the matrix (5).

Denote by nS_a^2 the minimum value of χ^2 (37). This will be obtained by solving (36) with regard to the a's. Denote by

$$\alpha_1, \alpha_2, \ldots \alpha_s \quad \text{.......................(38)}$$

the solutions of (36). It is easily seen that each α is a linear function of the x's. nS_a^2 is obtained from (37) by substituting α's instead of the a's. The equation (35) will now reduce to the following:

$$\sigma^2 = S_a^2 \quad \text{.......................(39).}$$

It is easy to prove that by substituting the α's instead of the a's, and S_a instead of σ in (34), we actually get the maximum value of this function. It will be

$$p_\Omega = \left(\frac{1}{S_a\sqrt{2\pi}}\right)^n e^{-\frac{n}{2}} \quad \text{.......................(40).}$$

The process of finding p_ω is quite similar and we get, say,

$$p_\omega = \left(\frac{1}{\sqrt{S_a^2 + S_b^2}\sqrt{2\pi}}\right)^n e^{-\frac{n}{2}} \quad \text{.......................(41),}$$

where $n(S_a^2 + S_b^2)$ stands for the minimum value of the sum of squares χ^2 in (37), calculated under conditions that

$$a_1 = a_1^0,\ a_2 = a_2^0,\ \ldots\ a_r = a_r^0 \quad \text{.......................(42),}$$

while other parameters remain arbitrary.

The likelihood is now easily found:

$$\lambda = \left(\frac{S_a^2}{S_a^2 + S_b^2}\right)^{\frac{n}{2}} = \left[1 + \left(\frac{S_b}{S_a}\right)^2\right]^{-\frac{n}{2}} * \quad \text{.......................(43).}$$

We see that the value of λ depends upon the value of, say,

$$Z = \frac{S_b}{S_a} \quad \text{.......................(44).}$$

Thus instead of considering λ as the appropriate test and rejecting the hypothesis when $\lambda \leqslant \lambda_0$, we may use Z or Z^2 and reject the hypothesis H_1 whenever $Z \geqslant Z_0$, where Z_0 must be so chosen that

$$P\{(Z \geqslant Z_0) \mid H_1\} = \epsilon \quad \text{.......................(45).}$$

To get the expression of the left-hand side of (45) we have to start from (34), where instead of the m's we have to put their expressions in terms of the a's appropriate to the hypothesis H_1.

Simple calculations show that there exists the identity

$$\chi^2 = \sum_{i=1}^{n} (x_i - m_i)^2 = nS_a^2 + nS_0^2 \quad \text{.......................(46),}$$

where $$nS_0^2 = \sum_{i=1}^{n} [c_{1i}(\alpha_1 - a_1^0) + c_{2i}(\alpha_2 - a_2^0) + \ldots + c_{si}(\alpha_s - a_s^0)]^2 \quad \text{......(47),}$$

while $$nS_a^2 = \sum_{i=1}^{n} (x_i - c_{1i}\alpha_1 - c_{2i}\alpha_2 - \ldots - c_{si}\alpha_s)^2 \quad \text{.......................(48).}$$

* This formula shows that the condition stated in (8) on p. 161, that the number s of parameters should be smaller than the number n of variables, is essential. In fact, if it were $s = n$, then S_a^2 would be zero and so would λ. Thus any result of the observation would lead to the rejection of the hypothesis if the test were upon the principle of likelihood.

It is to be noticed that S_a^2 is entirely independent of $a_{r+1}, a_{r+2}, \ldots a_s$. Thus we may write

$$p(x_1, x_2, \ldots x_n) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{-n\frac{S_a^2+S_0^2}{2\sigma^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(49).$$

Now introduce new variates, say,

$$E_1, E_2, \ldots E_s, y_{s+1}, y_{s+2}, \ldots y_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(50),$$

connected with the x's by means of the formulae

$$x_i = m_i + c_{1i}E_1 + c_{2i}E_2 + \ldots + c_{si}E_s + c_{s+1,i}y_{s+1} + \ldots + c_{ni}y_n \ldots\ldots(51),$$

where c_{ki} for $k \leqslant s$ are the given coefficients of the formulae (4) and for $k > s$ are so chosen that

$$\sum_{i=1}^n c_{ki}c_{ji} = 0 \qquad \text{if } k \neq j \text{ and } j > s \quad \ldots\ldots\ldots\ldots\ldots\ldots(52),$$

and

$$\sum_{i=1}^n c_{ji}^2 = 1 \qquad \text{for } j > s \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(53).$$

It is easily seen that it is always possible to find coefficients $c_{s+k,i}$ satisfying the above conditions, provided the matrix (5) is not singular.

In order to obtain the simultaneous frequency function of the new variables (50), we have to substitute (51) into (49) and to multiply the result into the Jacobian of the transformation. This Jacobian, say G, is certainly not zero. In fact it is easily seen that owing to the special choice of $c_{s+k,i}$, satisfying (52) and (53), the square G^2 is equal to the square of the matrix (5) and thus is positive, not zero.

Owing to (4) the equation (51) may be written in the form

$$x_i = c_{1i}(E_1 + a_1^0) + c_{2i}(E_2 + a_2^0) + \ldots + c_{si}(E_s + a_s^0) + c_{s+1,i}y_{s+1} + \ldots + c_{ni}y_n \quad \ldots\ldots(54).$$

Multiplying each of these equations into c_{ji} $(j = 1, 2, \ldots s)$ and summing, we see that the terms in y's are cancelling, owing to (52), and further that the sums $E_k + a_k$ are satisfying the same linear equations in terms of the x's as the a's, namely the equations (36). We thus have the identities

$$E_k = a_k - a_k^0 \qquad \text{for } k = 1, 2, \ldots r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(55),$$

and

$$E_k = a_k - a_k \qquad \text{for } k = r+1, r+2, \ldots s \ldots\ldots\ldots\ldots\ldots\ldots(56).$$

Using this we have

$$x_i - c_{1i}a_1 - c_{2i}a_2 - \ldots - c_{si}a_s = c_{s+1,i}y_{s+1} + \ldots + c_{ni}y_n \quad \ldots\ldots\ldots(57).$$

Squaring (57) and summing we get, owing to (48), (52) and (53),

$$nS_a^2 = \sum_{k=s+1}^n y_k^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(58),$$

while

$$nS_0^2 = \sum_{i=1}^n (c_{1i}E_1 + c_{2i}E_2 + \ldots + c_{si}E_s)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(59).$$

If we substitute (51) into (34) and multiply the result by G, we get

$$p(E_1, E_2, \ldots E_s, y_{s+1}, y_{s+2}, \ldots y_n) = \text{const.} \times e^{-n\frac{S_a^2+S_0^2}{2\sigma^2}} \quad \text{........(60)},$$

and it is easily seen that, owing to the fact that S_a^2 is dependent only upon y and S_0^2 only upon E's, the variates y_i are independent of the E's, and besides are independent among themselves. The expression nS_a^2 being a sum of $n-s$ independent normal variates, it is known that the distribution of S_a follows the law

$$p(S_a) = \text{const.} \times S_a^{n-s-1} e^{-\frac{nS_a^2}{2\sigma^2}} \quad \text{.......................(61)}.$$

It follows that the simultaneous distribution of $S_a, E_1, E_2, \ldots E_s$ is given by

$$p(S_a, E_1, E_2, \ldots E_s) = \text{const.} \times S_a^{n-s-1} e^{-n\frac{S_a^2+S_0^2}{2\sigma^2}} \quad \text{...........(62)}.$$

Further on we shall use this result.

Our problem being to find the distribution of S_b/S_a, we have now to consider the process of calculating S_b.

I have already pointed out that nS_a^2 is entirely independent of the a's. Thus when calculating the minimum value of the sum χ^2 in (46) with regard to the arbitrary parameters $a_{r+1}, a_{r+2}, \ldots a_s$, we are really trying to minimise the sum of squares nS_0^2 in (47), where the a's and the a_i^0's are to be considered constant and the a_{r+k}'s are arbitrary. Thus S_b^2 will be the minimum of S_0^2. It will be a positive definite quadratic form in r variates $E_1, E_2, \ldots E_r$. As these have been found to be independent of the y's and thus of S_a, S_b is also independent of S_a, and, as is easily seen, follows the distribution

$$p(S_b) = \text{const.} \times S_b^{r-1} e^{-\frac{nS_b^2}{2\sigma^2}} \quad \text{.......................(63)}.$$

Therefore the simultaneous distribution of S_a and S_b is given by the product

$$p(S_a, S_b) = \text{const.} \times S_a^{n-s-1} S_b^{r-1} e^{-n\frac{S_a^2+S_b^2}{2\sigma^2}} \quad \text{..............(64)}.$$

Introducing instead of S_b the new variate

$$S_b = S_a Z \quad \text{..................................(65)},$$

we get

$$p(S_a, Z) = \text{const.} \times S_a^{n-(s-r)-1} e^{-nS_a^2\frac{1+Z^2}{2\sigma^2}} Z^{r-1} \quad \text{...........(66)}.$$

Now on integrating with regard to S_a within the limits 0 and ∞, we get

$$p(Z) = \text{const.} \times Z^{r-1} (1+Z^2)^{-\frac{n-(s-r)}{2}} \quad \text{.................(67)},$$

and the problem may be considered as solved. The tests obtained are those invented and applied in many cases by R. A. Fisher.

The expressions

$$\frac{nS_a^2}{n-s}, \quad \frac{nS_b^2}{r} \quad \text{................................(68)}$$

are considered by him as independent estimates of the common variance.

The method of testing is reduced to the calculation of

$$z = \tfrac{1}{2}\log\frac{S_b^2(n-s)}{S_a^2 r} = \log Z + \tfrac{1}{2}\log\frac{n-s}{r} \quad \text{.................(69)},$$

and to referring to the table of values of z corresponding to the "confidence" coefficient $\epsilon = \cdot 05$ and $\epsilon = \cdot 01$, given by R. A. Fisher, where n_1 and n_2 correspond to my r and $n-s$ respectively.

Another method of course would be to substitute in (67), say,

$$u = (1 + Z^2)^{-1} \quad \text{................................(70)},$$

and in obtaining
$$p(u) = \frac{1}{B\left(\frac{n-s}{2}, \frac{r}{2}\right)} u^{\frac{n-s}{2}-1}(1-u)^{\frac{r}{2}-1} \quad \text{.................(71)},$$

and in testing whether
$$(1+Z^2)^{-1} \leqslant u_0 \quad \text{................................(72)},$$

where u_0 is such as to have
$$\int_0^{u_0} p(u)\,du = \epsilon \quad \text{................................(73)}.$$

The value of u_0 may be found from the Incomplete Beta-Function Tables*.

Of course if $r = 1$, then the distribution of Z reduces itself to that of "Student's" t.

It is interesting that the tests of R. A. Fisher are consequences of the principle of likelihood, stated above.

I think it will be useful to finish this section by summarising the mechanical process of testing any hypothesis of the type H and in illustrating it by one example.

The steps in testing a hypothesis H are as follows:

1. State the set of admissible hypotheses Ω and test whether each of them is describing the distribution of any variate x_i ($i = 1, 2, \ldots n$) by means of the formula

$$p(x_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x_i - m_i)^2}{2\sigma^2}} \quad \text{..........................(74)},$$

where σ is independent of i, and m_i may be represented by a linear function of certain $s < n$ unknown parameters a.

2. Express the hypothesis H tested by means of independent linear equations (let r be their number) with regard to the parameters a. If this is possible, then:

3. Calculate the minimum value of

$$\chi^2 = \sum_{i=1}^{n}(x_i - m_i)^2 \quad \text{.............................(75)}$$

which it may have for any set of values of the parameters a, corresponding to admissible hypotheses contained in Ω. This minimum value will be nS_a^2. Divide the minimum obtained by the number of degrees of freedom, equal to n, the

* K. Pearson, *Tables of the Incomplete Beta Function*. *Biometrika* Office, 1934.

total number of observations, minus s, the number of parameters a. The ratio will be

$$\frac{nS_a^2}{n-s} \qquad (76).$$

4. *Calculate the minimum value that χ^2 may have for values of the parameters a, satisfying the r linear equations, expressing the hypothesis to be tested.* The minimum will be $n(S_a^2+S_b^2)$. *Subtract nS_a^2*, obtained previously, and divide by the number of degrees of freedom, equal to r, the number of independent equations expressing H. The result will be

$$\frac{nS_b^2}{r} \qquad (77).$$

5. Calculate z, according to (69), and refer to Fisher's tables.

5. EXAMPLE: THE PROBLEM PROPOSED BY K. IWASZKIEWICZ.

Obviously the above five steps must be divided into two parts. Everything up to and including the determination of formulae giving S_a^2 and S_b^2 may be regarded as theoretical work. The practical test will then consist of numerical calculations leading to z, for which different formulae simplifying the arithmetic may be given.

The example I shall consider is that of the problem proposed by Miss K. Iwaszkiewicz, to test the hypothesis H whether the regression coefficients of x on y in k different populations, from which we have samples, are equal. Suppose that the sample, Σ_i, from the ith population, π_i, consists of n_i individuals, for which the values of x and y are respectively

$$\left.\begin{matrix} x_{i1}, x_{i2}, \ldots x_{in_i} \\ y_{i1}, y_{i2}, \ldots y_{in_i} \end{matrix}\right\} \qquad (78).$$

We consider each population π_i as being subdivided into partial populations, say π_{iy}, according to the values of y, appropriate to the individuals of π_{iy}. Each individual of the sample Σ_i may be considered as being randomly drawn from a separate population π_{iy}.

We consider as given that the regression of x on y in each population π_i is linear. This is equivalent to the assumption that the mean, m_{iy}, of the population π_{iy} is equal to

$$m_{iy} = a_{i1} + a_{i2}y \qquad (79),$$

where a_{i1} and a_{i2} are some unknown parameters, the number of which is $s = 2k$.

If there is evidence that the variances of the x's within single partial populations π_{iy} have equal values σ^2, we may proceed further.

The hypothesis to be tested may be expressed in

$$r = k - 1 \qquad (80)$$

independent linear equations in terms of the parameters a, namely,

$$a_{i2} - a_{k2} = 0 \qquad (i = 1, 2, \ldots k-1) \qquad (81).$$

We consider the sum of squares, χ^2, which may be written

$$\chi^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - a_{i1} - a_{i2} y_{ij})^2 \quad \text{.......................(82).}$$

The process of minimising this sum is easy, and we find

$$nS_a^2 = \sum_{i=1}^{k} n_i \sigma_{x_i}^2 (1 - r_i)^2 \quad \text{.........................(83),}$$

where σ_{x_i} and r_i mean respectively the standard deviation of x's and the coefficient of correlation between x and y in the sample Σ_i.

Now we shall have to assume that the parameters a satisfy the equations by which the hypothesis H is expressed. Denote by a_2 the common value of a_{i2}. Obviously $nS_a^2 + nS_b^2$ will be the minimum of the sum, say,

$$\chi_1^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - a_{i1} - a_2 y_{ij})^2 \quad \text{.......................(84),}$$

with regard to all possible values of the parameters a_2 and a_{i1}, for $i = 1, 2, \ldots k$. After easy algebra we have

$$n(S_a^2 + S_b^2) = \sum_{i=1}^{k} n_i \sigma_{x_i}^2 - \frac{\left(\sum_{i=1}^{k} n_i r_i \sigma_{x_i} \sigma_{y_i}\right)^2}{\sum_{i=1}^{k} n_i \sigma_{y_i}^2} \quad \text{..............(85),}$$

where σ_{y_i} means the standard deviation of y's in Σ_i. Some further easy algebra gives us the result

$$nS_b^2 = \frac{\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} n_i n_j \sigma_{y_i}^2 \sigma_{y_j}^2 (\alpha_i - \alpha_j)^2}{\sum_{i=1}^{k} n_i \sigma_{y_i}^2} \quad \text{.................(86),}$$

where

$$\alpha_i = r_i \frac{\sigma_{x_i}}{\sigma_{y_i}} \quad \text{...................................(87)}$$

means the regression coefficient of x on y in the sample Σ_i. The above formula is certainly not the most convenient one for calculation, but it shows the connection of the test with the differences in the regressions within single samples. The formulae which are the most convenient for calculation are, as usual, the less elegant ones, and may be easily deduced directly from the process of minimising (82) and (84).

6. The Method of Testing the Hypothesis H, based on the Best Critical Regions.

We shall now treat the same problem from the point of view of the theory of Best Critical Regions*. If we consider the data of the observations

$$x_1, x_2, \ldots x_n \quad \text{..................................(88)}$$

as co-ordinates of a point Σ (the "sample point") in the n-dimensioned space (the sample space W), then any method of testing the hypothesis H consists in a rule

* Neyman and Pearson, "On the Problem of the most efficient Tests of Statistical Hypotheses." (See reference 9.)

of rejecting H, whenever the sample point lies within a certain region, say w, in the sample space. The region w will be called the critical region; the supplementary region $W-w$ the region of acceptance. If the hypothesis H is a composite one to be tested, thus leaving one or more parameters determining the simultaneous distribution of the x's unspecified, then the appropriate critical region w should be "similar" to the sample space with regard to these unspecified parameters. I shall now explain this concept having in view the particular hypothesis H, considered in previous sections.

The hypothesis H does not specify the parameters

$$\sigma, a_{r+1}, a_{r+2}, \ldots a_p \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(89).$$

We shall say that a region w is similar to W with regard to these parameters and that it has the power ϵ, if the probability, say $P\{(\Sigma \epsilon w) \mid H\}$*, determined by any simple hypothesis belonging to H, that the sample point Σ will fall within w is equal to ϵ, *whatever be the values of the parameters* (89), unspecified by H.

In order to define the best critical region, we have to consider an alternative simple hypothesis, say h_1. We shall say that w_0 is the best critical region with *regard to h_1, having the power ϵ, if*

(1) w_0 is similar to W with regard to (89) and has the power ϵ, and

(2) if the probability determined by the alternative hypothesis h_1 that the sample point Σ will be contained in w_0 is larger than (or at least equal to) the same probability corresponding to any other region, w, satisfying (1).

We see that the process of finding the best critical regions must be preceded by finding the regions similar to W.

Neyman and Pearson gave the method of determining the most general regions similar to the sample space, the application of which is possible under the following conditions:

(*a*) The simultaneous probability function $p\{(x_1, x_2, \ldots x_n) \mid H\}$ should possess the derivatives of all orders with regard to all parameters unspecified by H for every set of values of $x_1, x_2, \ldots x_n$.

(*b*) If a is a parameter unspecified by the hypothesis, then denoting

$$\phi_a = \frac{\partial \log p}{\partial a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(90),$$

we should have, say,

$$\phi_a' = \frac{\partial \phi_a}{\partial a} = A + B\phi_a \quad \ldots\ldots\ldots\ldots\ldots\ldots(91),$$

where A and B are independent of the x's.

(*c*) The hypothesis H does not specify the values of $s-r+1$ parameters. Assume that these parameters are written in any order and form a series, say,

$$a_1, a_2, \ldots a_{s-r+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(92).$$

* The notation used in this formula, $\Sigma \epsilon w$, means: the point Σ is an element of w. It is generally used in the theory of sets. The letter ϵ standing here for "element" should not be confused with ϵ denoting the power of the region w.

Denote by ϕ_i the function defined as in (90),

$$\phi_i = \frac{\partial \log p}{\partial \alpha_i} \qquad \ldots\ldots(93),$$

and consider the hypersurfaces

$$\phi_j = C_j = \text{const.} \qquad (j = 1, 2, \ldots s - r + 1) \ldots(94).$$

Fix the values of the parameters α and C and consider the intersection of the $i-1$ hypersurfaces (94) corresponding to $j = 1, 2, \ldots i-1$. This intersection will be denoted by $S(\alpha_i; C_1, C_2, \ldots C_{i-1})$. Keep the values of α's constant and change the C's in an arbitrary way. Doing so we shall obtain a family of intersections $S(\alpha_i; C_1, C_2, \ldots C_{i-1})$. This family will be denoted by $F(\alpha_i)$. The condition (c) requires that it should be possible to find such an order of parameters unspecified by the hypothesis tested, that each family $F(\alpha_i)$, for $i = 2, 3, \ldots s - r + 1$, should be independent of α_i. This condition means that given any two different values of α_i, say α_i' and α_i'', any surface $S(\alpha_i'; C_1, \ldots C_{i-1})$ included in the family $F(\alpha_i')$ should be included also in $F(\alpha_i'')$, corresponding perhaps to some values of the C's different from those to which it corresponded in $F(\alpha_i')$.

If the above conditions (*a*), (*b*) and (*c*) are satisfied, then the method of finding the similar regions may be applied and the best critical region is constructed in the following way.

Consider the intersection of $s - r + 1$ surfaces (94) corresponding to fixed values of parameters (compatible with the hypothesis tested H) and to some fixed values of the constants C. This will be denoted by $W(C_1, C_2, \ldots C_{s-r+1})$. Denote by $w_0(C_1, C_2, \ldots C_{s-r+1})$ the part of the hypersurface $W(C_1, C_2, \ldots C_{s-r+1})$ defined by the inequality

$$p\{(x_1, x_2, \ldots x_n) \mid h_1\} \geqslant p\{(x_1, x_2, \ldots x_n) \mid H\}\, k(C_1, C_2, \ldots C_{s-r+1}) \quad \ldots(95),$$

where $k(C_1, C_2, \ldots C_{s-r+1})$ must be found for any system of the C's so as to satisfy the condition

$$\iint \ldots \int_{w_0(C_1, \ldots C_{s-r+1})} p\{(x_1, \ldots x_n) \mid H\}\, dw(C_1, \ldots C_{s-r+1})$$
$$= \epsilon \iint \ldots \int_{W(C_1, \ldots C_{s-r+1})} p\{(x_1, \ldots x_n) \mid H\}\, dW(C_1, \ldots C_{s-r+1}) \quad \ldots\ldots(96).$$

The best critical region w_0 is then built up of regions $w_0(C_1, \ldots C_{s-r+1})$ corresponding to all possible values of constants $C_1, C_2, \ldots C_{s-r+1}$.

We have now to consider whether the probability function $p\{(x_1, \ldots x_n) \mid H\}$ satisfies the conditions (*a*), (*b*) and (*c*).

Ad (*a*). The condition (*a*) is obviously satisfied.

Ad (*b*). Turning to our previous notation concerning the parameters, we shall have

$$\log p\{(x_1, x_2, \ldots x_n) \mid H\}$$
$$= -n \log \sqrt{2\pi} - n \log \sigma - \frac{\sum_{i=1}^{n} (x_i - c_{1i}a_1 - \ldots - c_{ri}a_r - c_{r+1,i}\, a_{r+1} - \ldots - c_{si}a_s)^2}{2\sigma^2}$$

$$\ldots\ldots(97),$$

$$\phi_\sigma = \frac{\partial \log p}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} (x_i - \sum_{j=1}^{s} c_{ji} a_j)^2 \quad \text{.............(98)},$$

$$\phi_\sigma' = \frac{\partial \phi_\sigma}{\partial \sigma} = \frac{n}{\sigma^2} - \frac{3}{\sigma^4} \sum_{i=1}^{n} (x_i - \sum_{j=1}^{s} c_{ji} a_j)^2$$

$$= -\frac{2n}{\sigma^2} - \frac{3}{\sigma}\phi_\sigma \quad \text{..................................(99)}.$$

Thus ϕ_σ satisfies the equation (91). Consider now, say,

$$\phi_{aj} = \frac{\partial \log p}{\partial a_j} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \sum_{l=1}^{s} c_{li} a_l) c_{ji} \quad \text{..................(100)}.$$

Obviously $$\phi_{aj}' = \frac{\partial \phi_{aj}}{\partial a_j} = -\frac{1}{\sigma^2} \sum_{i=1}^{n} c_{ji}^2 \quad \text{...........................(101)},$$

and satisfies (91) as it is independent of the x's, and B may be assumed to be zero.

Ad (c). The hypothesis (c) is also satisfied. To show this we write the parameters, which are not specified by the hypothesis H, in the following order:

$$\alpha_1 = a_{r+1};\ \alpha_2 = a_{r+2};\ \ldots \alpha_{s-r} = a_s;\ \alpha_{s-r+1} = \sigma \quad \text{...............(102)}.$$

For every $2 \leqslant i \leqslant s-r$ the hypersurface $S(\alpha_i;\ C_1, C_2, \ldots C_{i-1})$ is the intersection of the hyperplanes, the equations of which are obtained by equating (100) to arbitrary constants, thus equivalent to the following equations:

$$\sum_{i=1}^{n} c_{ji} x_i = C_j \qquad (j = r+1, r+2, \ldots r+i-1) \text{...(103)}.$$

These equations being independent of $\alpha_i = a_{r+1}$, so is the family $F(\alpha_i)$. Now put $i = s-r+1$. The hypersurface $S(\alpha_{s-r+1};\ C_1, C_2, \ldots C_{s-r})$ is now the intersection of hyperplanes (103) for $j = r+1, r+2, \ldots s$, which are independent of $\alpha_{s-r+1} = \sigma$. This means that $F(\alpha_{s-r+1})$ is independent of $\alpha_{s-r+1} = \sigma$.

Thus all conditions which are sufficient for the applicability of the Neyman-Pearson method are satisfied and we may proceed to the construction of the best critical region w_0.

Consider then hypersurfaces $W(C_1, \ldots C_{s-r+1})$, formed by the intersection of hyperplanes (103) for $j = 1, 2, \ldots s-r$ and of the hypersphere

$$\phi_\sigma = C_{s-r+1} \quad \text{................................(104)}.$$

According to the general theory the best critical region with regard to the alternative hypothesis h_1 consists of parts $w(C_1, \ldots C_{s-r+1})$ of the hypersurfaces $W(C_1, \ldots C_{s-r+1})$ determined by the inequality

$$p\{(x_1, x_2, \ldots x_n) \mid h_1\} \geqslant p\{(x_1, x_2, \ldots x_n) \mid H\}\, k(C_1, C_2, \ldots C_{s-r+1}) \text{...(105)},$$

where $k(C_1, C_2, \ldots C_{s-r+1})$ must be chosen so as to satisfy the equation

$$\iint \cdots \int_{w(C_1, \ldots C_{s-r+1})} p\{(x_1, \ldots x_n) \mid H\}\, dw(C_1, \ldots C_{s-r+1})$$

$$= \epsilon \iiint_{W(C_1, \ldots C_{s-r+1})} p\{(x_1, \ldots x_n) \mid H\}\, dW(C_1, \ldots C_{s-r+1}) \quad \text{........(106)}$$

on every hypersurface $W(C_1, \dots C_{s-r+1})$, corresponding to every possible system of values of the constants $C_1, \dots C_{s-r+1}$.

The solution is somewhat simpler if we introduce new variables instead of (88).

Denote by
$$a_1^0, a_2^0, \dots a_r^0, a_{r+1}, \dots a_s, \sigma \quad \dots\dots(107)$$
the values of parameters corresponding to the hypothesis H, and by
$$a_1', a_2', \dots a_r', a_{r+1}, \dots a_s, \sigma \quad \dots\dots(108)$$
those corresponding to the alternative h_1. The best critical region and the integral over this region from $p\{(x_1, \dots x_n) \mid H\}$ being independent of the values of the last $s-r+1$ parameters (107), we may assume that their values are the same as in (108).

Now instead of the x's we introduce new variates
$$E_1, E_2, \dots E_s, y_{s+1}, y_{s+2}, \dots y_n \quad \dots\dots(109),$$
connected with the x's by means of the formulae (54). Instead of considering the probability function $p\{(x_1, \dots x_n) \mid H\}$ we shall have to consider the function (60), namely,
$$p\{(E_1, E_2, \dots E_s, y_{s+1}, y_{s+2}, \dots y_n) \mid H\} = \text{const.}\, e^{-n\frac{S_a^2 + S_0^2}{2\sigma^2}} \quad \dots(60\ bis),$$
where
$$nS_a^2 = \sum_{k=s+1}^{n} y_k^2 \quad \dots\dots(58\ bis),$$
and
$$nS_0^2 = \sum_{i=1}^{n} (c_{1i}E_1 + c_{2i}E_2 + \dots c_{si}E_s)^2 \quad \dots\dots(59\ bis).$$

Having regard to (55) and (56) we conclude that the transformation used will lead to the following formula:
$$p\{(E_1, E_2, \dots E_s, y_{s+1}, y_{s+2}, \dots y_n) \mid h_1\} = \text{const.} \times e^{-n\frac{S_a^2 + S_1^2}{2\sigma^2}} \quad \dots(110),$$
where S_a^2 is the same as in (60 *bis*) but where, instead of S_0^2 given by (59 *bis*), we shall have S_1^2 determined by the formula
$$nS_1^2 = \sum_{i=1}^{n} (c_{1i}E_1' + c_{2i}E_2' + \dots + c_{ri}E_r' + c_{r+1}E_{r+1} + \dots + c_{si}E_s)^2 \dots(111),$$
if
$$E_k' = E_k - e_k \qquad (k = 1, 2, \dots r) \quad \dots\dots(112)$$
and
$$e_k = a_k' - a_k^0 \qquad (k = 1, 2, \dots r) \quad \dots\dots(113).$$

We shall make another linear transformation, introducing instead of E_{r+1}, $E_{r+2}, \dots E_s$ new variables, say $\psi_{r+1}, \psi_{r+2}, \dots \psi_s$, which will transform nS_0^2 into
$$nS_0^2 = \sum_{ij=1}^{r} R_{ij}E_iE_j + \sum_{j=r+1}^{s} \psi_j^2 \quad \dots\dots(114).$$

The same transformation will give us
$$nS_1^2 = \sum_{ij=1}^{r} R_{ij}E_i'E_j' + \sum_{j=r+1}^{s} \psi_j^2 \quad \dots\dots(115).$$

It will be seen that the quadratic form

$$\sum_{ij=1}^{r} R_{ij} E_i E_j = nS_0^2 \quad \text{..............................(116)}$$

is positive and definite and represents the minimum value of nS_0^2 with regard to all possible systems of values of $a_{r+1}, a_{r+2}, \ldots a_s$.

The coefficients R_{ij} may be obtained from the coefficients c_{ij} as follows*. Denote by

$$g_{ij} = \sum_{t=1}^{n} c_{it} c_{jt} = g_{ji} \quad \text{..............................(117)}$$

and consider the determinant, say,

$$G = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1s} \\ g_{21} & g_{22} & \cdots & g_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ g_{s1} & g_{s2} & \cdots & g_{ss} \end{vmatrix} \quad \text{..........................(118).}$$

Denote further by G_{ij}' the minor of G, corresponding to g_{ij}, and let

$$G_{ij} = \frac{G_{ij}'}{G} \quad \text{..................................(119).}$$

* This is an application of the following general theorem concerning the linear transformations, which we shall have to use again below.

If

$$\sum_{ij=1}^{s} g_{ij} x_i x_j \quad \text{..(1)}$$

is a positive definite form and the functions

$$y_p = b_{p1} x_1 + b_{p2} x_2 + \ldots + b_{ps} x_s \qquad (p = 1, 2, \ldots r < s) \quad \text{...........(2)}$$

are independent, then it is always possible to find another $s - r$ linear function, say,

$$y_q = b_{q1} x_1 + b_{q2} x_2 + \ldots + b_{qs} x_s \qquad (q = r+1, r+2, \ldots s) \quad \text{.........(3),}$$

such that together with (3) they form an independent system. Moreover, if the system of equations (2) and (3) is solved with regard to the x's and the solutions are substituted into (1) then we get

$$\sum_{ij=1}^{s} g_{ij} x_i x_j = \sum_{ij=1}^{r} T_{ij} y_i y_j + \sum_{t=r+1}^{s} y_t^2 \quad \text{..............................(4).}$$

The coefficients T_{ij} in (4) may be calculated from the coefficients g_{ij} in (1) and b_{pt} in (2). Consider the determinant

$$G = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1s} \\ g_{21} & g_{22} & \cdots & g_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ g_{s1} & g_{s2} & \cdots & g_{ss} \end{vmatrix} \quad \text{..............................(5),}$$

and denote by G_{ij} the elements of its reciprocal, say, G^{-1}. That is to say, G_{ij} will be equal to the minor of G corresponding to g_{ij}, divided into G. Consider further the determinant, say,

$$T = \begin{vmatrix} t_{11} & t_{12} & \cdots & t_{1r} \\ t_{21} & t_{22} & \cdots & t_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ t_{r1} & t_{r2} & \cdots & t_{rr} \end{vmatrix} \quad \text{..............................(6),}$$

where

$$t_{pq} = \sum_{ij=1}^{s} b_{pi} b_{qj} G_{ij} \quad \text{..............................(7).}$$

Now T_{pq} will be the element of the determinant T^{-1}, reciprocal to T. In other words T_{pq} is equal to the minor of T, corresponding to t_{pq}, divided into T.

Consider finally the determinant, say,

$$R = \begin{vmatrix} G_{11} & G_{12} & \dots & G_{1r} \\ G_{21} & G_{22} & \dots & G_{2r} \\ \dots & \dots & \dots & \dots \\ G_{r1} & G_{r2} & \dots & G_{rr} \end{vmatrix} \quad \dots\dots\dots\dots(120),$$

and let R_{ij}' be its minor corresponding to G_{ij}. Then we shall have

$$R_{ij} = \frac{R_{ij}'}{R} \quad \dots\dots\dots\dots(121).$$

The probability functions of the new variables

$$E_1, E_2, \dots E_r, \psi_{r+1}, \psi_{r+2}, \dots \psi_s, y_{s+1}, y_{s+2}, \dots y_n \dots\dots\dots\dots(122),$$

determined by the two hypotheses H and h_1 will have the previous forms (60 *bis*) and (110) respectively, with the exception that the expressions of nS_0^2 and nS_1^2 will now be (114) and (115).

We shall now transform to new variables the equations (103) and (104). It is easily seen that the equations (103) are equivalent to the following ones:

$$\frac{\partial (nS_0^2)}{\partial a_j} = \text{const.} \qquad (j = r+1, r+2, \dots s) \dots(123);$$

where nS_0^2 should have the form (59 *bis*), and where

$$E_j = a_j - \alpha_j \qquad (j = 1, 2, \dots s) \dots\dots\dots\dots(124).$$

But we may also use the form (114). The relationships between the ψ's and the E's may be written

$$\psi_j = \sum_{t=1}^{s} b_{jt} E_t \qquad (j = r+1, r+2, \dots s) \dots(125).$$

If we substitute these formulae into (114), we shall get identically (59 *bis*),

$$nS_0^2 = \sum_{ij=1}^{r} R_{ij} E_i E_j + \sum_{j=r+1}^{s} \left(\sum_{t=1}^{s} b_{jt} E_t \right)^2 \dots\dots\dots\dots(126).$$

The equations (103) transformed to the new variables may now be obtained by differentiating (126) with regard to a_j $(j = r+1, r+2, \dots s)$, and keeping in view (124) and (125). We have

$$\frac{\partial (nS_0^2)}{\partial a_j} = -2 \sum_{i=r+1}^{s} \psi_i b_{ij} \ \text{ for } j = r+1, r+2, \dots s \quad \dots\dots(127).$$

Thus the new form of the equations (103) will be

$$\sum_{i=r+1}^{s} b_{ij} \psi_i = C_j \qquad (j = r+1, r+2, \dots s) \dots\dots(128),$$

C_j being an arbitrary constant. The equations (128) depend only upon ψ's. They may be solved with regard to the ψ's and thus replaced by an equivalent system

$$\psi_{r+1} = D_{r+1}, \psi_{r+2} = D_{r+2}, \dots \psi_s = D_s \dots\dots\dots\dots(129),$$

where the D's are again arbitrary constants.

Turn now to equation (104). This is equivalent to

$$n(S_a^2 + S_0^2) = \text{const.} \quad \text{.........................(130).}$$

Owing to equations (114) it may be replaced by

$$n(S_a^2 + S_0^2) = \psi_0 = D_0 \quad \text{.........................(131),}$$

D_0 being an arbitrary constant.

The problem of finding the best critical region is now reduced to that of determining the hypersurface, say $w(D_0, D_{r+1}, D_{r+2}, \ldots D_s)$, which is a part of the intersection of (129) and (131) on which

$$p\{(E_1, \ldots E_r, \psi_{r+1}, \ldots \psi_s, y_{s+1}, \ldots y_n)|h_1\}$$
$$\geqslant p\{(E_1, \ldots E_r, \psi_{r+1}, \ldots \psi_s, y_{s+1}, \ldots y_n)|H\}\, k(D_0, D_{r+1}, \ldots D_s) \quad \text{......(132),}$$

k being subject to the condition of satisfying (106). Taking the logarithms of both sides of (132) we shall get

$$-\frac{1}{2\sigma^2}\left[nS_a^2 + \sum_{ij=1}^{r} R_{ij} E_i' E_j' + \sum_{t=r+1}^{s} \psi_t^2\right]$$
$$\geqslant -\frac{1}{2\sigma^2}\left[nS_a^2 + \sum_{ij=1}^{r} R_{ij} E_i E_j + \sum_{t=r+1}^{s} \psi_t^2\right] + k \quad \text{........(133),}$$

and after some reduction

$$\sum_{ij=1}^{r} R_{ij}(E_i E_j - E_i' E_j') \geqslant k\sigma^2 = k_1 \quad \text{(say)} \quad \text{..............(134).}$$

Owing to the relation $\qquad E_i' = E_i - \epsilon_i$ ....(135),

we have $\qquad E_i E_j - E_i' E_j' = \epsilon_i E_j + \epsilon_j E_i - \epsilon_i \epsilon_j$ ....(136).

Thus (134) *may be written in an equivalent form:*

$$2\sum_{ij=1}^{r} R_{ij}\epsilon_i E_j - \sum_{ij=1}^{r} R_{ij}\epsilon_i\epsilon_j \geqslant k_1 \quad \text{......................(137),}$$

or

$$\xi_1 = \sum_{ij=1}^{r} R_{ij}\epsilon_i E_j \geqslant k_2 \quad \text{(say)} \quad \text{......................(138).}$$

The region $w(D_0, D_{r+1}, \ldots D_s)$ is the part of the intersection of (129) and (131), *say* $W(D_0, D_{r+1}, \ldots D_s)$, *determined by the inequality* (138), *where* k_2 *should be* so chosen that the integral of the probability function corresponding to H over $w(D_0, D_{r+1}, \ldots D_s)$ should be equal to ϵ multiplied into the integral of the same function over $W(D_0, D_{r+1}, \ldots D_s)$. *It will be seen that the problem of determining* $w(D_0, D_{r+1}, \ldots D_s)$ is that of finding such a value k_2, that the relative probability (determined by H) of (138), given (129) and (131), may be equal to ϵ:

$$P\{(\xi_1 \geqslant k_2)|(H), \ (\psi_0 = D_0), \ (\psi_{r+1} = D_{r+1}), \ldots (\psi_s = D_s)\} = \epsilon \quad \text{......(139).}$$

This last problem would be easily solved if we could determine the simultaneous frequency function, say, $p(\xi_1, \psi_0, \psi_{r+1}, \ldots \psi_s)$ of variates $\xi_1, \psi_0, \psi_{r+1}, \ldots \psi_s$.

As all these variates depend only upon

$$S_a, E_1, E_2, \ldots E_s \quad \text{............................(140),}$$

we may start with the respective frequency function given by the formula (62),

$$p(S_a, E_1, E_2, \ldots E_s) = \text{const.} \times S_a^{n-s-1} e^{-n\frac{S_a^2 + S_0^2}{2\sigma^2}} \quad \text{........(141),}$$

where nS_0^2 has the form (59 *bis*). Introducing $\psi_{r+1}, \psi_{r+2}, \dots \psi_s$ as new variables, by means of the formulae (125) we get $p(S_a, E_1, \dots E_r, \psi_{r+1}, \dots \psi_s)$, having the same form as (141) except that nS_0^2 has now the expression (114),

$$p(S_a, E_1, \dots E_r, \psi_{r+1}, \dots \psi_s) = \text{const.} \times S_a^{n-s-1} e^{-n\frac{S_a^2+S_b^2+S_c^2}{2\sigma^2}} \quad \dots(142),$$

where

$$nS_c^2 = \sum_{i=r+1}^{s} \psi_i^2 \quad \dots\dots(143).$$

Our next step will consist in changing the system of variables,

$$S_a, E_1, \dots E_r, \psi_{r+1}, \dots \psi_s,$$

into the new one,

$$\psi_0, \psi_1, \psi_2, \dots \psi_s \quad \dots\dots(144),$$

by means of the formulae

$$\left.\begin{aligned} &\psi_i = \sum_{j=1}^{r} R_{ij} E_j \quad \text{for} \quad i = 1, 2, \dots r \\ &\psi_i = \psi_{i'} \quad \text{for} \quad i \text{ or } i' = r+1, r+2, \dots s \\ &\psi_0 = nS_a^2 + nS_b^2 \end{aligned}\right\} \quad \dots\dots(145).$$

It is easily seen that

$$nS_b^2 = \sum_{pq=1}^{r} G_{pq} \psi_p \psi_q \quad \dots\dots(146).$$

The Jacobian of the transformation being

$$\frac{\partial(S_a, E_1, \dots E_r, \psi_{r+1}, \dots \psi_s)}{\partial(\psi_1, \dots \psi_s, \psi_0)} = \left[\frac{\partial(\psi_1, \dots \psi_s, \psi_0)}{\partial(S_a, E_1, \dots E_r, \psi_{r+1}, \dots \psi_s)}\right]^{-1} = (2n\, S_a\, R)^{-1} \quad \dots\dots(147),$$

we find

$$p(\psi_0, \psi_1, \dots \psi_s) = \text{const.} \times (\psi_0 - nS_b^2)^{\frac{n-s-2}{2}} e^{-\frac{\psi_0 + nS_c^2}{2\sigma^2}} \quad \dots\dots(148).$$

The last step in this series of transformations will consist in introducing ξ_1, a new variable, and also in representing nS_b^2 in (146) as a sum of squares. This will be done by applying again the theorem of algebra I have given in the footnote on p. 177.

It will be seen that substituting (145) in (138) we get

$$\xi_1 = \sum_{i=1}^{r} e_i \psi_i \quad \dots\dots(149).$$

Applying the theorem just mentioned we may add to this equation another $r-1$ of them,

$$\xi_i = \sum_{i=1}^{r} e_{ij} \psi_j \qquad (i = 2, 3, \dots r) \quad \dots\dots(150),$$

such that they will form a system of independent equations, and if used to transform (146) to the new variables $\xi_1, \xi_2, \dots \xi_r$ this will give, say,

$$nS_b^2 = Z_{11}^2 \xi_1^2 + \sum_{i=2}^{r} \xi_i^2 \quad \dots\dots(151),$$

where

$$Z_{11}^{-2} = \sum_{ij=1}^{r} R_{ij} e_i e_j \quad \dots\dots(152).$$

The Jacobian of the transformation of $\psi_0, \psi_1, \ldots \psi_s$ into the new system

$$\xi_1, \xi_2, \ldots \xi_r, \psi_{r+1}, \ldots \psi_s, \psi_0$$

being obviously constant and different from zero, we get

$$p(\xi_1, \xi_2, \ldots \xi_r, \psi_0, \psi_{r+1}, \ldots \psi_s) = \text{const.} \times \left(\psi_0 - Z_{11}^2 \xi_1^2 - \sum_{t=2}^{r} \xi_t^2\right)^{\frac{n-s-2}{2}} e^{-\frac{\psi_0 + nS_e^2}{2\sigma^2}} \quad \ldots\ldots(153).$$

The required probability function of $\xi_1, \psi_0, \psi_1, \ldots \psi_s$ is now easily found by integrating for ξ_i $(i = 2, 3, \ldots r)$ within the extreme limits of these variates, compatible with the fixed values of the others. The extreme limits for ξ_r will be, say,

$$-L_r \leqslant \xi_r \leqslant +L_r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(154),$$

if

$$L_r = \sqrt{\psi_0 - Z_{11}^2 \xi_1^2 - \sum_{t=2}^{r-1} \xi_t^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(155).$$

Putting $\xi_r = L_r u$ we shall get

$$\int_{-L_r}^{+L_r} p(\xi_1, \ldots \xi_r, \psi_0, \psi_1, \ldots \psi_s)\, d\xi_r = \text{const.}$$

$$\times \left(\psi_0 - Z_{11}^2 \xi_1^2 - \sum_{t=2}^{r-1} \xi_t^2\right)^{\frac{n-s-1}{2}} e^{-\frac{\psi_0 + nS_e^2}{2\sigma^2}} \int_{-1}^{+1} (1-u^2)^{\frac{n-s-2}{2}} du \quad \ldots\ldots(156),$$

where the integral with regard to u may be included in the constant factor. Obviously the process of integration may be repeated till we are left with

$$p(\xi_1, \psi_0, \psi_1, \ldots \psi_s) = \text{const.} \times (\psi_0 - Z_{11}^2 \xi_1^2)^{\frac{n-s+r-3}{2}} e^{-\frac{\psi_0 + nS_e^2}{2\sigma^2}} \quad \ldots(157).$$

Denoting the left-hand side of (139) for short by $P(k_2)$, we shall now get

$$P(k_2) = \frac{\int_{k_2}^{L} p(\xi_1, \psi_0, \psi_1, \ldots \psi_s)\, d\xi_1}{\int_{-L}^{+L} p(\xi_1, \psi_0, \psi_1, \ldots \psi_s)\, d\xi_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(158),$$

where $-L$ and $+L$ are the extreme limits of variation of ξ_1 compatible with the fixed value of ψ_0, namely,

$$L = \frac{\sqrt{\psi_0}}{Z_{11}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(159).$$

Equating (158) to ϵ and performing the easy algebra, we get

$$\int_{k_2}^{L} (\psi_0 - Z_{11}^2 \xi_1^2)^{\frac{n-s+r-3}{2}} d\xi_1 = \epsilon \int_{-L}^{+L} (\psi_0 - Z_{11}^2 \xi_1^2)^{\frac{n-s+r-3}{2}} d\xi_1 \quad \ldots(160).$$

Putting

$$\xi_1 = Lv, \quad k_2 = Lv_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(161),$$

v being a new variable, we reduce equation (160) to the following:

$$\int_{v_0}^{1} (1-v^2)^{\frac{n-s+r-3}{2}} dv = \epsilon \int_{-1}^{+1} (1-v^2)^{\frac{n-s+r-3}{2}} dv = \epsilon B\left(\frac{1}{2}, \frac{n-s+r-1}{2}\right) \quad \ldots\ldots(162).$$

From this equation v_0 may be found, and it appears to be an absolute constant, depending only upon ϵ. Taking into account (161), (159), (152) and the last of formulae (145) we get

$$k_2 = v_0 \sqrt{\sum_{ij=1}^{r} R_{ij} e_i e_j}\sqrt{nS_a^2 + nS_b^2} \quad \text{..................(163)}.$$

The inequality (138) determining the region $w(D_0, D_{r+1}, \ldots D_s)$ may be written in the form

$$\frac{1}{\sqrt{nS_a^2+nS_b^2}}\,\frac{\sum\limits_{ij=1}^{r} R_{ij} e_i E_j}{\sqrt{\sum\limits_{ij=1}^{r} R_{ij} e_i e_j}} \geqslant v_0 \quad \text{..................(164)}.$$

As this inequality is independent of the values $D_0, D_{r+1}, \ldots D_s$, we see that it determines the best critical region w_0, provided v_0 is found to satisfy (162).

7. Is there a Single Best Critical Region with Regard to the Whole Class of Alternative Hypotheses?

Having thus found the general solution of the problem of the best critical region of the hypothesis H with regard to a chosen alternative h_1, we have now to consider whether this region is changed if instead of the hypothesis h_1 we take into consideration some other alternative simple hypothesis, say h_2. This question is very important for the following reasons. It has been shown by Neyman and Pearson that the tests based upon the best critical regions could not be bettered even if the *à priori* probability law of all unknown parameters were known. Therefore, in cases where for every admissible alternative hypothesis h the best critical region is the same, the problem of testing the hypothesis H may be considered as entirely solved: no better test can be found. If, however, the best critical regions with regard to different alternative hypotheses are different, then there is no "best" test of the hypothesis H. Whatever be the test, it could be bettered if the *à priori* probability law were known. There are intuitive reasons for believing that in these cases a "good" test is provided by the principle of likelihood.

The best critical region with regard to the simple alternative hypothesis h_1 being defined by the inequality (164), we have to consider if and under what conditions this inequality is independent of h_1.

It will be perhaps useful to recall the significance of all the symbols in (164):

$$nS_a^2 + nS_b^2 \quad \text{................................(165)}$$

denotes the minimum of the sum of squares

$$\sum_{i=1}^{n}(x_i - m_i)^2 = \sum_{i=1}^{n}(x_i - \sum_{j=1}^{s} c_{ji} a_j)^2 \quad \text{....................(166)},$$

calculated under the assumptions (1) that the x's are constant, (2) that the parameters

$$a_j = a_j^0 \qquad (j = 1, 2, \ldots r) \quad \text{...........(167)}$$

have the values specified by the hypothesis tested, and (3) that the remaining parameters a_j for $j=r+1, r+2, \dots s$ are arbitrary. It is seen that (165) does not depend upon the alternative hypothesis. Symbols R_{ij} for $i, j = 1, 2, \dots r$ are also independent of the hypothesis h_1. According to formula (116),

$$nS_b^2 = \sum_{ij=1}^{r} R_{ij} E_i E_j \qquad \dots\dots(168),$$

and the R's are the coefficients in the expression of nS_b^2 in terms of the E's and may be calculated in terms of the given coefficients c_{ij}. The meaning of the E's is given by the formula (55), i.e.

$$E_k = a_k - a_k^0 \qquad (k=1, 2, \dots r) \dots\dots(169),$$

where a_k is a linear function of the x's giving the minimum value of the sum (166) if substituted for α_k, for $k=1, 2, \dots s$. Obviously E_k for $k=1, 2, \dots s$ does not depend upon the alternative hypothesis h_1. The only symbols depending upon h_1 in (164) are the ϵ's. In fact according to (113),

$$\epsilon_i = a_i' - a_i^0 \qquad (i=1, 2, \dots r) \dots\dots(170),$$

where a_i^0 is the value of the parameter a_i specified by H, and a_i' that specified by the alternative hypothesis h_1.

We see therefore that the question whether the best critical region may be independent of h_1 reduces to the question whether by some transformation the ratio

$$\frac{\sum_{ij=1}^{r} R_{ij}\epsilon_i E_j}{\sqrt{\sum_{ij=1}^{r} R_{ij}\epsilon_i\epsilon_j}} \qquad \dots\dots(171)$$

may be made independent of the ϵ's. This appears to be possible only if $r=1$. In fact the numerator of (171) being a linear function of the ϵ's, we may hope to be able to cancel them only if the denominator were a rational polynomial in ϵ's. Now comparing the expression under the square root in (171) with (168), we see that it is a special value of the positive definite quadratic form nS_b^2 corresponding to

$$E_i = \epsilon_i \qquad i=1, 2, \dots r \dots\dots(172),$$

and thus may be represented by a sum of at least r squares. Therefore the denominator in (171) is rational only if $r=1$. We have thus the following interesting general result:

If the hypothesis H considered in this paper is either,

(1) *specifying the values of more than one of the parameters a_i $(i=1, 2, \dots s)$, or*

(2) *specifying the values of more than one linear function of the same parameters, then there is no common best critical region with regard to all alternatives h_1.*

It will be seen that there will be no common best critical region for instance when we test the following hypotheses:

(1) that the means of $k > 2$ sampled populations are equal;

(2) that the regression coefficients in $k > 2$ sampled populations are equal;

(3) that the regression lines in two (or more) sampled populations are identical. In fact, even if we compare only two regression lines, the hypothesis that they are identical can only be expressed by $r=2$ equations.

In all these cases the best test available seems to be that provided by the principle of likelihood, discussed in § 4.

Let us now turn to the case $r=1$ and consider whether there is a common best critical region with regard to all alternative hypotheses. In this case the ratio (171) reduces to

$$\frac{R_{11}\epsilon_1 E_1}{\sqrt{R_{11}}\,|\epsilon_1|} \quad \text{.................................(173)},$$

and the inequality defining the best critical region (164), to

$$\frac{\sqrt{R_{11}}\,E_1}{\sqrt{nS_a^2+R_{11}E_1^2}}\,\frac{\epsilon_1}{|\epsilon_1|} \geqslant v_0 \quad \text{or} \quad \frac{\sqrt{R_{11}}\,E_1}{\sqrt{n}\,S_a}\,\frac{\epsilon_1}{|\epsilon_1|} \leqslant \frac{v_0}{\sqrt{1-v_0^2}} \quad \text{.........(174)}.$$

It will be seen that in the most general case there will be no common best critical region even in this case. In fact, if

$$\epsilon_1 = a_1' - a_1^0 \quad \text{.................................(175)}$$

may be for some admissible hypotheses positive, and for some others negative, then in the first case the best critical region will be defined by the inequality

$$\frac{\sqrt{R_{11}}\,E_1}{\sqrt{nS_a^2+R_{11}E_1^2}} \geqslant v_0 \quad \text{..........................(176)},$$

and in the second case by the inequality

$$\frac{\sqrt{R_{11}}\,E_1}{\sqrt{nS_a^2+R_{11}E_1^2}} \leqslant -v_0 \quad \text{..........................(177)}.$$

In many cases, however, we may assume that the class of alternative hypotheses consists only of such hypotheses which ascribe a definite sign to the expression (175). In such cases there is a common best critical region, determined either by (176) or by (177).

As the left-hand side of the inequalities (173) and (174) may be written in the form

$$\frac{\sqrt{n}\,S_b}{\sqrt{nS_a^2+nS_b^2}} \quad \text{.......................... .....(178)},$$

we see that the test may be based on the value of the ratio

$$Z=\frac{S_b}{S_a} \quad \text{..................................(179)},$$

which, as was mentioned in § 4, follows "Student's" probability law. In this case the best critical region is determined by the inequality

$$Z\,\frac{\epsilon_1}{|\epsilon_1|} \geqslant z_0 \quad \text{.................................(180)},$$

where z_0 should be found from the table of "Student's" integral to satisfy the condition

$$\int_{z_0}^{\infty}(1+z^2)^{-\frac{n-s+1}{2}}dz \Big/ \int_{-\infty}^{+\infty}(1+z^2)^{-\frac{n-s+1}{2}}dz = \epsilon \quad \text{........(181).}$$

We may combine this with the results of Markoff, who proved that the expression, say,

$$\sigma_{a_1}^2 = \frac{nS_a^2}{n-s}\frac{1}{R_{11}} \quad \text{............................(182),}$$

is the estimate of the standard error squared of the variate a_1. Taking into account (169), we may now define the best critical region by means of the inequality

$$(a_1 - a_1^0)\frac{\epsilon_1}{|\epsilon_1|} \geqslant z_0\sqrt{n-s}\,\sigma_{a_1} \quad \text{.......................(183).}$$

The above result may be used in two ways. Sometimes we may know the expression for σ_{a_1}. Then the best critical region may be found directly from (183). In other cases we do not know the value of σ_{a_1} and it may be desirable to calculate it for some purpose. Then we may do so by calculating nS_a^2 or the minimum of the sum of squares

$$\Sigma(x_i - m_i)^2 \quad \text{..............................(184),}$$

considering all s parameters a as variable. Next we minimise the same sum with regard only to $s-1$ parameters, whilst one of them, $a_1 = a_1^0$, is kept constant. The result is $n(S_a^2 + S_b^2)$. Substituting, we obtain

$$nS_b^2 = R_{11}(a_1 - a_1^0)^2 \quad \text{..........................(185).}$$

As a_1 must have been previously found in the process of getting nS_a^2, we may now calculate R_{11} and substitute it into (182) which will give us $\sigma_{a_1}^2$. This process will be illustrated below.

8. FURTHER EXAMPLES.

In this section I will give two further examples of linear hypotheses. In the first of them there exists a common best critical region with regard to all alternative hypotheses and in the other there does not.

Example 1. Denote by x and y the possible yields per acre given by some two varieties of cereals A and B. These yields depend largely upon the conditions of growth. If the two varieties are sown under similar conditions on adjacent plots, then both of them are similarly influenced and the yields thus obtained must be correlated. I shall suppose that experiments carried out in different experimental stations and possibly in different years gave the results

$$x_1, x_2, \ldots x_n, \quad \text{Mean} = \bar{x}, \quad \text{S.D.} = \sigma_x \quad \text{.................(186),}$$

$$y_1, y_2, \ldots y_n, \quad \text{Mean} = \bar{y}, \quad \text{S.D.} = \sigma_y \quad \text{.................(187).}$$

I shall further assume as known that the regression of y on x is linear*. The

* This has been found in many cases studied by M. Gorski and K. Iwaszkiewicz. See *Roczniki Nauk Rolniczych i Leśnych*, t. XXVIII. pp. 211 *et seq.* and t. XXXI. pp. 277 *et seq.* See also W. Staniszkis, "Ergebnisse dreijähriger (1927—29) vergleichender Versuche mit Leinsorten," *Wyniki Doswiadczen Polowych*, No. 6, Warsaw, 1934.

hypothesis, H, to be tested consists in the assumption that in conditions where the variety A is able to give a specified yield of X tons per acre, the average yield of the variety B does not exceed the value Y, which is also specified. Usually it will probably be assumed that $Y=X$.

If after having applied the test we reject the hypothesis H, this will be equivalent to the assumption that the experimental data give us sufficient evidence that in conditions when the yield of A is X, the variety B is better.

Let us now present the hypothesis H under the form of a linear hypothesis. We start by noticing that the assumption about the linearity of regression of y on x means that if we fix any value of x and consider the population, say π_x, of all possible y's, then the mean of this population, say m_x, will be connected with the fixed value of x by means of the formula

$$m_x = a_1 + a_2 x \quad \text{..............................(188)},$$

where a_1 and a_2 are independent of x. This equation, which must hold for any value of x, and therefore for any value of this variable given in the sample, is equivalent to my equation (4).

The rôle of the variates (1) is now played by the yields of the variety B, denoted by y. The hypothesis to test is expressed by the equation

$$m_X = a_1 + a_2 X = Y \quad \text{..........................(189)},$$

while the alternative hypotheses are

$$m_X' = a_1 + a_2 X > Y \quad \text{..........................(190)}.$$

The equation (189) corresponds to the equation (6) of the theory. If there is evidence that the standard deviation in each array of y is independent of x, then it is easily seen that the hypothesis considered here is a linear hypothesis. The number of unknown parameters $s=2$. These are the coefficients a_1 and a_2 of the regression equation (188). The hypothesis itself is expressed by means of only one equation (189). Therefore $r=1$. As the alternatives satisfy the inequality (190), we conclude that there is a common best critical region with regard to all alternatives. It may be defined by the inequality (183). In our case a_1^0 means the hypothetical value Y of the ordinate m_X of the regression line. a_1 is its value as obtained by method of least squares,

$$a_1 = \bar{y} + r\frac{\sigma_y}{\sigma_x}(X - \bar{x}) \quad \text{..........................(191)},$$

where r means the correlation coefficient between (186) and (187). ϵ_1 will be the difference between the value of the ordinate m_X as specified by the alternative hypotheses and that specified by the hypothesis tested. Thus $|\epsilon_1| = \epsilon_1 > 0$. The value of σ_{a_1} may be calculated by the formula of Miss K. Iwaszkiewicz (see Neyman (7)) or again it may be directly calculated following the method described in the last paragraphs of the preceding section. For the sake of illustration we shall follow this latter method.

We consider the sum of squares

$$n\chi^2 = \Sigma (y_i - a_1 - a_2 x_i)^2 \quad \text{.........................(192)}$$

and find its minimum with regard to a_1 and a_2. This is known to be

$$nS_a^2 = n\sigma_y^2 (1 - r^2) \quad \text{.............................(193).}$$

Next we assume that a_1 and a_2 fulfil the equality (189), substitute in (192)

$$a_1 = Y - a_2 X \quad \text{..............................(194),}$$

and minimise, say, $$n\chi'^2 = \Sigma \{y_i - Y - a_2 (x_i - X)\}^2 \quad \text{....................(195)}$$

with regard to a_2. *Easy algebra gives the result:*

$$n(S_a^2 + S_b^2) = n\left\{\sigma_y^2 (1 - r^2) + \frac{\sigma_x^2}{\sigma_x^2 + (X - \bar{x})^2}\left(\bar{y} + r\frac{\sigma_y}{\sigma_x}(X - \bar{x}) - Y\right)^2\right\} \quad \text{......(196).}$$

Comparing (196) with (193), (185), and (191), we find

$$R_{11} = \frac{n\sigma_x^2}{\sigma_x^2 + (X - \bar{x})^2} \quad \text{........................(197).}$$

Now (182) gives $$\sigma_{a_1}^2 = \frac{\sigma_y^2 (1 - r^2)}{n - 2}\left(1 + \frac{(X - \bar{x})^2}{\sigma_x^2}\right) \quad \text{...................(198)}$$

and finally the best critical region is determined by the inequality

$$\bar{y} + r\frac{\sigma_y}{\sigma_x}(X - \bar{x}) - Y \geq z_0 \sqrt{n - 2}\, \sigma_{a_1} \quad \text{................(199),}$$

where z_0 satisfies the condition (181).

Example 2. *The hypothesis about the linearity of regression.* Suppose that the observations gave us n pairs of values of two variables x and y. These pairs are grouped according to different values of x, which are

$$x_1,\ x_2,\ \ldots\ x_s \quad \text{................................(200).}$$

The values of y which correspond to the common value of x, say x_t, will be denoted by

$$y_{t1},\ y_{t2},\ \ldots\ y_{tn_t}, \quad \textit{Mean} = \bar{y}_t, \quad \sum_{t=1}^{s} n_t = n \quad \text{..............(201).}$$

I shall suppose that it is known that the standard deviation of the population of all y's corresponding to a fixed value of x is independent of the value of x and that the distribution of y's in the arrays is normal. The hypothesis, H, to test is *that the regression of y on x is linear.*

We shall start by showing that H is what I call a linear hypothesis. Denote by m_x the population mean of y's corresponding to a fixed value of x. That is to say, m_x will be the mean of an array of y's in the population sampled. The hypothesis H to be tested is expressed by the equation

$$m_x = A + Bx \quad \text{...............................(202),}$$

where A and B are some constant numbers and x has any value. This equation may be considered as corresponding to the equations (6) in the theory. In order

to write the equations corresponding to (4) we shall denote by m_{ij} the population mean of the variate y_{ij}. As this corresponds to the value x_i of x, we shall sum up our knowledge concerning m_{ij} by writing

$$m_{ij} = m_i \quad \text{.................................}(203)$$

for any $j = 1, 2, \dots n_i$ and $i = 1, 2, \dots s$. Thus we see that the population means of all variables y are linearly expressed in terms of s unknown parameters m_i. Here we see that a test of the hypothesis H is possible only if there is at least one array in which we have at least two observations. In fact, if for any i we had $n_i = 1$, then we should have $s = n$, contrary to the assumption of the theory that the number of parameters is less than the number of observations.

Now, as the rôle of the unknown parameters is played by the m_i's, we shall have to transform the equation (202) in order to bring it into correspondence with the equations (6), which are linear with regard to parameters and then contain only known numbers b and B. Substituting in (202) $x = x_1$, $x = x_2$ and $x = x_i$ for $i = 3, 4, \dots s$, we get

$$\left.\begin{array}{l} m_i - A - Bx_i = 0 \\ m_1 - A - Bx_1 = 0 \\ m_2 - A - Bx_2 = 0 \end{array}\right\} \quad \text{........................}(204),$$

a system of linear equation which allows the elimination of A and B. The result is given by

$$\begin{vmatrix} m_i, & 1, & x_i \\ m_1, & 1, & x_1 \\ m_2, & 1, & x_2 \end{vmatrix} = 0, \quad \text{for } i = 3, 4, \dots, s \dots (205),$$

or $$m_i(x_2 - x_1) + m_1(x_i - x_2) + m_2(x_1 - x_i) = 0, \quad \text{for } i = 3, 4, \dots s \quad \dots(206).$$

Now we see another condition which the observations must satisfy to allow the test of the hypothesis H. The number of the arrays, or s, must be at least equal to 3. Otherwise the equations (206), which express the hypothesis H, could not have been written. If $s \geqslant 3$, then the number, r, of the equations expressing the hypothesis H is $r = s - 2$.

It is easily seen that in the case considered there is no common best critical region with regard to all alternatives. In fact, even if $s = 3$ and thus $r = 1$, the class of alternative hypotheses would certainly include those which ascribe a value of m_3 such that

$$m_3(x_2 - x_1) > -m_1(x_i - x_2) - m_2(x_1 - x_i) \quad \text{..............}(207),$$

and also those for which

$$m_3(x_2 - x_1) < -m_1(x_i - x_2) - m_2(x_1 - x_i) \quad \text{..............}(208).$$

Therefore the test to be applied is that following from the principle of likelihood, defined by (72) or by

$$\frac{S_b^2}{S_a^2} \geqslant \frac{1}{u_0} - 1 \quad \text{...............................}(209),$$

where u_0 must be found from the Tables of the Incomplete Beta Function so as to satisfy the condition

$$B_{u_0}\left(\frac{n-s}{2}, \frac{r}{2}\right) = \epsilon B\left(\frac{n-s}{2}, \frac{r}{2}\right) \quad \text{.....................(210).}$$

Now S_a^2 and S_b^2 are found as follows. We consider the sum of squares

$$n\chi^2 = \sum_{i=1}^{s} \sum_{j=1}^{n_i} (y_{ij} - m_i)^2 \quad \text{.........................(211),}$$

and minimise it with regard to all m_i for $i = 1, 2, \ldots s$. This gives us

$$nS_a^2 = \sum_{i=1}^{s} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2 \quad \text{.........................(212).}$$

Next we assume that the hypothesis tested is true and minimise (211) with regard to what is not specified by the hypothesis. For this purpose we could use *the equations (209) and consider m_1 and m_2 as being not specified by H.* But obviously we may use instead the expression

$$m_i = A + Bx_i \quad \text{for} \quad i = 1, 2, \ldots s \quad \text{.....................(213),}$$

and consider A and B as numbers which are not specified by H. The minimum of (211) calculated under the conditions (213) is then, as it is well known,

$$n\,(S_a^2 + S_b^2) = n\sigma_y^2 (1 - r^2) \quad \text{.......................(214);}$$

where σ_y means the standard deviation of all y's in the sample, and r the correlation between x and y in the sample. Thus

$$nS_b^2 = n\sigma_y^2 (1 - r^2) - \sum_{i=1}^{s} \sum_{j=1}^{n_i} (y_{ij} - y_i)^2 \quad \text{.................(215),}$$

and the critical region is determined by the inequality, say,

$$u = \frac{\sum_{i=1}^{s} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2}{n\sigma_y^2 (1 - r^2)} \leqslant u_0 \quad \text{.........................(216).}$$

9. SUMMARY OF RESULTS.

1. The problem considered in the present paper consists in testing a composite hypothesis H, which may be described as a "linear hypothesis concerning the means of n normal variates with common standard deviation." It is supposed that the mean, m_t, of each of the n normal variates, x_t, having the same unknown standard deviation σ, are linear functions with known coefficients of s parameters a_i $(i = 1, 2, \ldots s)$. The values of parameters are not known and the hypothesis H to be tested consists in the assumption, either (1) that some $r \leqslant s$ of these parameters have some specified values, say $a_1^0, a_2^0, \ldots a_r^0$, the remaining parameters being unspecified, or (2) that some $r \leqslant s$ independent linear functions, θ_i, of these parameters with known coefficients have some definite values, say $\theta_1^0, \theta_2^0, \ldots \theta_r^0$.

2. It was possible to give a general solution of the above problem, based upon the principle of likelihood, proposed by J. Neyman and E. S. Pearson.

3. A general expression has been given for the best critical region for the hypothesis H with regard to any simple alternative hypothesis h_1.

4. It has been shown that in cases where the number r of the parameters a_i (or of their linear functions) which are specified by the hypothesis H exceeds one, then there is no best critical region common to all possible alternative hypotheses. In those cases therefore the method of testing the hypothesis H must be based upon the principle of likelihood—unless some other "good" test is invented.

5. If $r = 1$ and if the class of admissible hypotheses is limited to those ascribing to the parameter, say a_1, or to the linear function, say θ_1 (which are specified by H) values which are larger (or smaller) than the values specified by H, then there exists a best critical region, common to the whole class of the alternatives. In these cases the solution obtained of the problem of testing H could not be bettered even if the probability law *à priori* of all unknown parameters were known.

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EIN BEITRAG ZU UNTERSUCHUNGEN UEBER ZWEIDIMENSIONALE VERTEILUNGEN VON MASSENPUNKTEN BEI ZUFALLSARTIG BEDINGTEN BEWEGUNGEN*.

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Einleitung.

In dem Grundproblem der vorliegenden Arbeit handelt es sich um die Bestimmung einer Verteilungsdichte, wenn von einem Flächenelement mit dem inneren Punkt Q—dem Quellpunkt—ausgehend N Massenpunkte sich über eine Ebene in folgender Weise verteilen: Durch Q mögen m verschiedene Geraden $G_1, G_2, G_3, \ldots, G_m$ gehen, die m Bewegungsrichtungen repräsentieren. Auf jeder dieser Geraden gelte ein Wahrscheinlichkeitsgesetz $v_r(s_r)$, das ist die Wahrscheinlichkeitsdichte für die Länge einer Bewegung auf G_r oder parallel dazu. $v_r(s_r)\,ds_r$ ist also die Wahrscheinlichkeit, um auf einer Parallelen zu G_r von einem als Anfangspunkt genommenen Punkt P_{r-1} bis zu einem Punkt P_r zu kommen, der (von P_{r-1} ausgemessen) zwischen s_r und s_r+ds_r liegt, dabei ist s_r irgendein Parameter, der eineindeutig den Punkten der Geraden G_r zugeordnet ist. Ein Massenpunkt M_i bewege sich nun zunächst von Q ausgehend längs G_1 gemäss der dortigen Verteilungsfunktion $v_1(s_1)$ bis zu einem P_1, darauf von P_1 ausgehend parallel G_2 bis zu einem Punkt P_2 gemäss $v_2(s_2)$ usw. Allgemein bewege sich M_i von P_{r-1} ausgehend parallel G_r bis zu einem Punkt P_r gemäss der auf G_r geltenden Wahrscheinlichkeitsdichte $v_r(s_r)$. Nach m solchen Schritten, deren Reihefolge beliebig ist, da es sich um einfache Vektoraddition handelt, befindet sich M_i an einem Punkt $P_m=P$. Der analoge Vorgang finde für alle N Massenpunkte statt. Gefragt ist dann nach der Dichte $\Phi_1(P)$, d.h. nach der Anzahl der Massenpunkte pro Flächeneinheit an einem Punkt P der Ebene und zwar bei grossem m und N.

Es sind dabei alle m Richtungen als gleich wahrscheinlich angenommen; ist das jedoch nicht der Fall, so lässt sich dieser auf den obigen zurückführen, indem man die $v_r(s_r)$ entsprechend ändert. Sei z.B. im extremen Fall die Bewegung parallel einer bestimmten Geraden G_r unmöglich, so kann man das ausdrücken, indem man setzt:

$$\lim_{ds_r \to 0} v_r(s_r)\,ds_r = 1 \text{ für den Punkt } Q \text{ (oder den entsprechenden Punkt } P_{r-1}),$$

$$v_r(s_r) = 0 \text{ für alle anderen Punkte.}$$

Aus der Verteilung $\Phi_1(P)$ ergibt sich nun eine weitere, indem von *allen* Flächenelementen $d\!f$ mit einem inneren Punkt $\bar{Q}$ $\Phi_1(\bar{Q})\,d\!f$ Massenpunkte in

* Dissertation zur Erlangung der Doktorwürde der Mathematisch-Naturwissenschaftlichen Fakultät der Hamburgischen Universität.

derselben Weise ausgehen wie vorher die N Massenpunkte von Q aus. Die Verteilungsdichte der $\Phi_1(\bar{Q})\,df$ Massenpunkte um $\bar{Q}$ ist also

$$\Phi_1(\bar{Q})\,df\,\Phi_1(\bar{Q}P):N,$$

dabei ist $\Phi_1(\bar{Q}P)$ gleich $\Phi_1(P)$ bezogen auf $\bar{Q}$ statt Q als Anfangspunkt. Die Ueberlagerung aller dieser Verteilungen ergibt dann

$$\Phi_2(P)=\frac{1}{N}\int\Phi_1(\bar{Q}P)\,\Phi_1(Q)\,df.$$

Allgemein erhält man die n-te Verteilung durch die Rekursionsformel

$$\Phi_n(P)=\frac{1}{N}\int\Phi_1(\bar{Q}P)\,\Phi_{n-1}(\bar{Q})\,df,$$

wobei die Integrale beidemal über die ganze Ebene zu erstrecken sind.

Nach Lösung dieses Grundproblems soll dann die Verteilung berechnet werden, wenn sich von allen Punkten einer begrenzten Fläche mit der Dichte N die Massenpunkte in der obigen Weise über die Ebene ausbreiten, wobei mit Rücksicht auf die Anwendungen einige Spezialfälle in bezug auf die Art der Begrenzung näher ausgeführt werden sollen.

Ueber die biologische Bedeutung der folgenden Ausführungen möchte ich an dieser Stelle noch einige Erläuterungen geben. Es handelt sich in biologischer Hinsicht bei der vorliegenden Arbeit im wesentlichen um folgende zwei hauptsächlich interessierende Fragen:

1. Wie viele Moskitos befinden sich durchschnittlich pro Flächeneinheit in verschiedenen Entfernungen von einem stark Moskito-verseuchten Gebiet, wenn nur in diesem eine Vermehrungsmöglichkeit für jene vorhanden ist? Dabei möge die Ausbreitung der Insekten nur durch zufälliges Hin- und Herfliegen zustande kommen. Von irgend welchen grösseren bewussten oder absichtlichen "Zielflügen" und von äusseren Hemmnissen sei also zunächst abgesehen. Hier angenommene Verhältnisse liegen z.B. annähernd in dem Steppengelände des Wolgagebietes vor. Dort grenzen an die Sumpfniederungen der Wolga, die stark mit Moskitos bevölkert sind, verhältnismässig grosse Steppengebiete, die den Moskitos eigentlich nur in der Nähe menschlicher Siedlungen, durch die dort vorhandenen Tiertränken oder Wasserlachen, eine Brutmöglichkeit bieten. Werden diese nun desinfiziert (z.B. durch Bestreuen mit Arsenstaub), so kann eine Malaria-Uebertragung nur durch die aus den Wolganiederungen eingestreuten Moskitos zustande kommen. Für die Bekämpfung der Malaria ist es in einem solchen Fall daher ausserordentlich nützlich zu wissen, wie weit eine beträchtliche Anzahl dieser Moskitos in das Steppengelände eindringt. Ein beliebig weites Eindringen ist des allmählichen Absterbens wegen nicht möglich, da ausserhalb des Flussgürtels wohl Gelegenheit zur Eiablage, nicht aber, oder in verschwindend geringem Masse, Möglichkeiten zur Vollendung der Entwicklung vorhanden sind. Auf die Wichtigkeit dieser Frage für die Malariabekämpfung wurde bereits von Sir Ronald Ross hingewiesen, der auch schon eine Verteilungsfunktion zu berechnen versuchte, worauf später noch zurückzukommen sein wird.

In diesem Zusammenhang möchte ich darauf hinweisen, dass die Ausführungen dieser Arbeit sich nicht auf solche Tierarten beziehen, die bei ihren "Flügen" infolge irgendeines Instinkts absichtlich—wie z.B. die Bienen bei ihren täglichen Flügen zur Honigsuche—an den Ausgangspunkt zurückkehren, sondern ein Zurückkommen an den Ursprung sei reiner Zufall. In obigem Fall ist z.B. wegen der Möglichkeit der Eiablage in der Steppe ein regelmässiges Zurückfliegen der Moskitos an die Brutplätze des Flussgebietes selbst dann nicht wahrscheinlich wenn für jene ein sogenannter "Heimatsinn" anzunehmen wäre, wobei als Heimat im Fall der Moskitos Gegenden mit Brutmöglichkeiten anzusehen wären. Die Möglichkeit des Vorhandenseins eines "Heimatsinns" ist im übrigen bei den meisten Moskitoarten noch durchaus ungeklärt.

2. Wie wird sich die Ausbreitung (Infiltration) einer Tierart in ein zunächst unbesiedeltes Gebiet, das an ein besiedeltes grenzt, abspielen? Dabei ist wieder die wichtigste Voraussetzung, dass die Ausbreitung nur durch zufälliges Hin- und Herfliegen erfolgt. Weiter sei angenommen, dass *alle* Punkte der Ebene als "Plätze mit Vermehrungsmöglichkeiten" oder als "habitats" (in dem Sinne der noch zu besprechenden Arbeit von Herrn Prof. K. Pearson) möglich sind oder zumindest, dass die Entfernungen der einzelnen "habitats" sehr klein sind gegenüber der durchschnittlichen Länge eines Fluges. Dieses Problem ist natürlich nur als ein dynamisches zu betrachten, denn der stationäre Endzustand wird in dem hier angenommenen einfachsten Fall der ortsunabhängigen Vermehrungsmöglichkeit, örtlich konstant sein. Es kann sich also nur darum handeln, den zeitlichen Verlauf einer oben beschriebenen Besiedlung eines zunächst leeren Gebietes darzustellen, und zwar, wie gesagt, eines Gebietes, das an fast allen Punkten Siedlungsmöglichkeiten bietet. Auf den Fall, dass solche Siedlungsmöglichkeiten nur an vereinzelten Stellen, deren Entfernungen in die Grössenordnung einer Fluglänge oder mehr kommen, vorhanden sind, sind also die folgenden Ergebnisse nicht oder höchstens als erste grobe Annäherung anwendbar.

1. *Voraussetzungen und Ergebnisse der älteren Arbeiten.* (*Ronald Ross, Lord Rayleigh, Karl Pearson.*)

In diesen Arbeiten handelt es sich um ein ähnliches wie das in der Einleitung diskutierte Problem. Die wesentlichen biologischen Voraussetzungen, wie sie in der Arbeit von Herrn Professor Pearson angegeben wurden (s. Lit.-Verz. Nr. (3), S. 3 und 4), seien im folgenden kurz dargestellt.

Ueber eine genügend grosse Fläche seien gleichförmig sogen. "habitats"—d.h. für Mücken: Plätze mit Brut- und Lebensmöglichkeiten—verteilt. Die durchschnittliche Fluglänge l einer Mückenart von einem "habitat" zu einem neuen möge als "flight" bezeichnet werden. Diese "flights" sind zu unterscheiden von den "flitters," die ein blosses Hin- und Herfliegen in der Nachbarschaft der "habitats" zwecks Nahrungssuche etc. bedeuten.

Die Frage des Grundproblems lautet dann: Wie gross ist die Anzahl der Mücken pro Flächeneinheit—also die Dichte $\Phi_n(P)$—an einem Punkt P in der

Entfernung r von Q, wenn von Q ausgehend N Mücken n "flights" der Länge l (= durchschnittlicher Abstand benachbarter "habitats") ausführen? Dabei soll es sich um sogen. "Random Migration" handeln, d.h. die Verteilung soll durch zufälliges Hin- und Herfliegen entstehen. Die einzelnen Flugstrecken (von einem "habitat" zum anderen) seien jedoch gradlinig zurückgelegt und wie beim ersten Flug seien von einem Ausgangspunkt aus alle Richtungen gleich wahrscheinlich. Beim ersten Fluge verteilen sich also N Mücken in erster Näherung gleichmässig um Q auf einen schmalen Kreisring mit dem Radius l. Beim zweiten Flug wiederholt sich das von jedem Flächenelement des ersten Kreisringes, und die Verteilungen auf den nun entstehenden Kreisringen überlagern sich und ergeben die Verteilung nach dem zweiten Flug. In analoger Weise erhält man alle weiteren Verteilungen.

Sir Ronald Ross führt die Frage auf ein lineares Problem zurück. Er macht anscheinend die Annahme (das ist in seiner Arbeit jedoch nicht genau angegeben), dass nach einem Fluge von Q aus—bei Projektion der einzelnen Flüge auf die X-Achse—$\frac{1}{2}N$ der Mücken in Q geblieben sind, während je $\frac{1}{4}N$ sich im Abstand l links bzw. rechts von Q befinden. Von diesen drei Punkten wiederholt sich jetzt dieser Vorgang, nur dass statt N jetzt $\frac{1}{2}N$ bzw. $\frac{1}{4}N$ Mücken sich verteilen, so dass nach dem zweiten Fluge in Q: $\frac{6}{16}N$ und rechts und links von Q im Abstand $1l$ je $\frac{4}{16}N$ und im Abstand $2l$ je $\frac{1}{16}N$ Mücken sich befinden. Das gibt dann nach n Flügen die bekannte Verteilung der Binomialkoeffizienten von 2^{2n} multipliziert mit $N(\frac{1}{2})^{2n}$. Es befinden sich also nach n Flügen im Abstand kl von Q insgesammt

$$\left.\begin{array}{ll} 2\binom{2n}{n-k}N(\frac{1}{2})^{2n} & \text{für } k>0 \\ \binom{2n}{n-k}N(\frac{1}{2})^{2n} & \text{für } k=0 \end{array}\right\}\qquad\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{I}).$$

Wie aus der Wahrscheinlichkeitsrechnung bekannt gilt nun für grosse n und $n \gg k$

$$\binom{2n}{n-k}\frac{2N}{2^{2n}} \sim \frac{2N}{\sqrt{n\pi}}e^{-\frac{k^2}{n}}\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{II}).$$

Das muss also, da in Wirklichkeit eine kontinuierliche Verteilung besteht, die Anzahl Mücken sein, die sich nach dem n-ten Fluge auf einem Kreisring der Breite l, der den Kreis mit dem Radius kl umgibt, befinden. Bedeutet also $\Phi_n(r)$ die Dichte im Abstand r von Q, so muss sein:

$$\begin{aligned}\frac{2N}{\sqrt{n\pi}}e^{-\frac{k^2}{n}} &= \int_0^{2\pi}\int_{(k+\delta-1)l}^{(k+\delta)l}\Phi_n(r)\,r\,dr\,d\phi \qquad \text{wobei } 0\leqslant\delta\leqslant 1 \\ &= 2\pi\int_{(k+\delta-1)l}^{(k+\delta)l}\Phi_n(r)\,r\,dr \\ &= 2\pi\Phi_n([k+\epsilon]\,l)\int_{(k+\delta-1)l}^{(k+\delta)l} r\,dr \qquad \text{wobei } \delta-1\leqslant\epsilon\leqslant\delta \\ &= 2\pi l^2(k+\delta-\tfrac{1}{2})\,\Phi_n([k+\epsilon]\,l)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{III}),\end{aligned}$$

also muss die Dichte für ein r, dass zwischen $(k-1)\,l$ und $(k+1)\,l$ liegt, sein

$$\Phi_n([k+\epsilon]\,l)=\frac{Ne^{-\frac{k^2}{n}}}{l^2(k+\delta-\frac{1}{2})\,\pi\sqrt{\pi n}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{IV}).$$

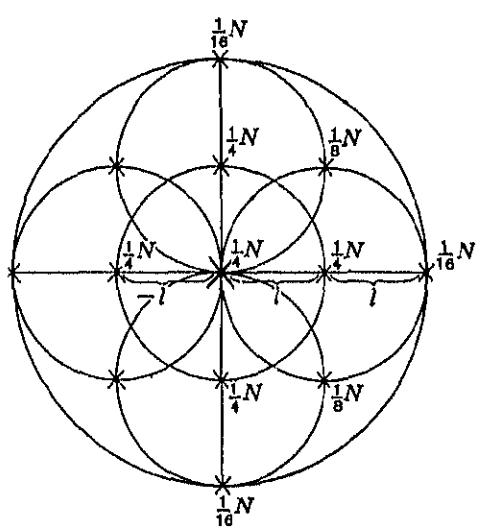
Diese Reduktion des zweidimensionalen Problems auf ein eindimensionales ist jedoch wohl reichlich willkürlich, und wieso dieses Resultat mit den späteren von Lord Rayleigh (V) und K. Pearson (VI) für grosse n übereinstimmt, wie Ronald Ross behauptet, ist wohl kaum einzusehen, denn jene erhalten für grosse n:

$$\Phi_n(r)=\frac{N}{nl^2\pi}e^{-\frac{r^2}{nl^2}},$$

also
$$\Phi_n([k+\epsilon]\,l)=\frac{N}{nl^2\pi}e^{-\frac{(k+\epsilon)^2l^2}{nl^2}}.$$

Da nun sowohl δ wie ϵ nur zwischen -1 und $+1$ variieren können, geht dieser letzte Ausdruck mit $n\to\infty$ etwa wie $1:n$ gegen 0, während (IV) nur wie $1:\sqrt{n}$ gegen 0 geht. Der absolute Unterschied zwischen diesen beiden Resultaten ist allerdings für grosse n nur klein, da ja beide gegen 0 gehen; relativ unterscheiden sie sich jedoch der Grössenordnung nach etwa um den Faktor $\frac{1}{k}\sqrt{\frac{n}{\pi}}$, der mit wachsendem n also beliebig gross werden kann, und nur bei $k\sim\sqrt{\frac{n}{\pi}}$ ist eine ziemliche Uebereinstimmung zu erwarten.

Etwas näher kommt der gestellten Aufgabe das von K. Pearson zitierte Resultat einiger Arbeiten von Lord Rayleigh über akustische Probleme (s. Lit.-Verz. Nr. (2)). Hierbei sind *zwei* Richtungen (x- und y-Achse) für den von einem Punkt ausgehenden gradlinigen Flug l als gleich wahrscheinlich angenommen. Ebenfalls sollen die Wahrscheinlichkeiten für einen Flug nach der positiven und negativen Seite der x- bzw. y-Achse hin gleich sein. Für grosse N sind also folgende Verteilungen zu erwarten:



Nach dem *ersten* Flug befinden sich in den 4 Punkten (×), siehe S. 195:

$$x = 0, \quad y = \pm l; \quad x = \pm l, \quad y = 0,$$

je $N:4$ und in allen andern Punkten der Ebene $0 \times N$ Mücken.

Von diesen vier Punkten verteilen sich nun die $N:4$ wie vorher die N Mücken von Q aus, so dass nach dem *zweiten* Fluge sich in den Punkten (×):

$$\begin{array}{lr} x = 0, \quad y = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots & N:4, \\ x = 0, \quad y = \pm 2l; \quad x = \pm 2l, \quad y = 0 \quad \ldots & \text{je } N:16, \\ x = \pm l, \quad y = \pm l \ldots \quad \ldots \quad \ldots \quad \ldots & \text{je } N:8 \end{array}$$

Mücken befinden, wie man leicht aus obiger Skizze entnimmt. Analog erhält man alle weiteren Verteilungen. Wenn $n \to \infty$ geht und gleichzeitig l gegen 0, so dass nl^2 endlich bleibt, ergibt sich eine Verteilung mit der Dichte

$$\Phi_n(r) = \frac{N}{\pi n l^2} e^{-\frac{r^2}{nl^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{V}),$$

die jedoch nur für grosse n und $r^2 \ll nl^2$ angenähert gilt.

Die exakte Lösung dieses allgemeinen Problems — also die der Verteilung bei Annahme einer *konstanten* Fluglänge mit Gleichwahrscheinlichkeit für *alle* Richtungen — hat dann Professor Karl Pearson 1906 gegeben (s. Lit.-Verz. Nr. (3)). Vorher hatte I. C. Kluyver einen allgemeinen Ansatz mit Hilfe Besselscher Funktionen aufgestellt, der nicht zum Ziele führte, jedoch die Lösung einiger spezieller Probleme und den Beweis für die Richtigkeit der Rayleigh'schen Lösung für grosse n erbrachte (s. Lit.-Verz. Nr. (4)). K. Pearson erhielt als Lösung ebenfalls ein Integral über ein Produkt Besselscher Funktionen (näheres über Besselsche Funktionen s. Lit.-Verz. Nr. (5), (6)), nämlich:

$$\Phi_n(r) = \frac{N}{2\pi} \int_0^\infty u J_0(ur) \{J_0(ul)\}^n du \quad \ldots\ldots\ldots\ldots\ldots\ldots (\text{VI}),$$

was sich durch geeignete Umformung in folgende Reihe entwickeln lässt:

$$\Phi_n(r) = N(\nu_0 \omega_0 + \nu_2 \omega_2 + \nu_4 \omega_4 + \ldots) \quad \ldots\ldots\ldots\ldots\ldots (\text{VII}),$$

wobei

$$\omega_{2s} = -\frac{(-\beta)^{s+1}}{\pi n l^2} \frac{d^s}{d\beta^s} \left(\frac{1}{\beta} e^{\frac{1}{\beta}}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (\text{VIII}),$$

mit

$$\beta = -\frac{nl^2}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{IX}),$$

und

$$\nu_{2s} = (-1)^s \frac{\pi n^{s+1} l^{2(s+1)} \int_0^\infty \omega_{2s} \Phi_n(r)\, e^{\frac{r^2}{nl^2}} r\, dr}{N \int_0^\infty \omega_{2s} r^{2s+1} dr} \quad \ldots\ldots\ldots\ldots (\text{X}).$$

Die bestimmten Integrale in diesem Ausdruck lassen sich auswerten, man erhält z.B. für

$$\nu_0 = 1, \quad \nu_2 = 0, \quad \nu_4 = -\frac{1}{4n}, \quad \nu_6 = -\frac{1}{9n^2},$$

$$\nu_8 = \frac{6n - 11}{12 n^3}, \quad \nu_{10} = \frac{50n - 57}{1800 n^4} \ldots.$$

Die Konvergenz der Reihe der ω-Funktionen ist nun für $n < 7$ sehr schlecht. Für diesen Fall haben K. Pearson und J. Blakeman die Verteilungsfunktion auf graphischem Wege gefunden. Für grosse n geht

$$\Phi_n(r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{V}\,a),$$

gegen $N\omega_0$, das ist die Rayleigh'sche Lösung.

Den bisher besprochenen Methoden lag, wie gesagt, die Annahme einer *konstanten* Fluglänge zugrunde. Vielfach wird in der Natur diese Annahme nicht berechtigt sein, sondern häufig werden sich schon nach der ersten Verteilung von einem Zentrum aus in *allen* Entfernungen die entsprechenden Lebewesen vorfinden, vorausgesetzt dass kein Punkt durch besonders günstige oder ungünstige Lebensbedingungen ausgezeichnet ist. In diesem Fall ist es nun durchaus nicht evident, dass nach n Flügen dieselbe Verteilung resultiert, wenn man nach einem Fluge von einem Zentrum aus annimmt: (1) Alle Mücken befinden sich auf einem schmalen Kreisring mit dem Radius l, oder (2) wenn mann annimmt, irgendeine Verteilung mit derselben *durchschnittlichen* Fluglänge l erstreckt sich über die ganze umliegende Fläche. Während beim ersten Fall die Verteilungen nur von Flächenelementen des Kreisringes ausgehen, die sich dann überlagern, findet dieser Vorgang in Wirklichkeit vielleicht von allen Flächenelementen einer ganzen Umgebung des Zentrums statt, und es ist nicht gesagt, dass die Verteilungen ausgehend von Flächenelementen ausserhalb bzw. innerhalb des Kreises mit dem Radius l sich derart superponieren, als ob sie alle von den Flächenelementen des Kreisringes mit dem Radius der durchschnittlichen Fluglänge l ausgingen.

2. *Die Grundverteilung nach dem ersten Fluge um ein gegebenes Zentrum bei* variabler *Fluglänge.*

In dieser Arbeit soll mit möglichst wenigen und plausiblen Voraussetzungen die Verteilungsfunktion bei variabler Fluglänge gefunden werden. Ich gehe dabei gemäss der in der Einleitung ausgeführten Problemstellung von folgenden Voraussetzungen aus, um die Verteilung nach dem ersten Fluge zu erhalten:

1. Es seien m verschiedene Richtungen (wobei nachdem $m \to \infty$) möglich, in die eine Mücke fliegt. Die Richtungen seien dargestellt durch m Geraden durch den Ursprung.

2. Die Fluglängen einer bestimmten Richtung G_r unterliegen einem Wahrscheinlichkeitsgesetz $v_r(s_r)$*.

3. Während des Fluges wechselt die Mücke m-mal ihre Richtung und durchfliegt in der jeweiligen Richtung eine gemäss $v_r(s_r)$ zu erwartende Strecke.

4. Die Wahrscheinlichkeitsdichten $v_r(s_r)$ auf den Geraden G_r sind ausserhalb eines Kreises mit dem Radius K um den Ursprung Null.

* $v_r(s_r)$ braucht im einzelnen nicht bekannt zu sein.

5. Die $v_r(s_r)$ mögen für alle r einer Lipschitzbedingung genügen, d.h. die zu den einzelnen $v_r(s_r)$ gehörigen Zahlen M_r besitzen eine obere Schranke M; es gilt also für beliebige Punkte der Geraden G_r:

$$\left| \frac{v_r(s_{r1}) - v_r(s_{r2})}{s_{r1} - s_{r2}} \right| < M_r < M.$$

Die letzten beiden Voraussetzungen erwähne ich nur, da sie beim Beweis benutzt werden, während sie für die eigentliche Problemstellung unwesentlich sind und nur die Art der sonst beliebigen Funktionen $v_r(s_r)$ etwas einschränken.

Unser Problem ist dann identisch mit der Frage nach der Wahrscheinlichkeitsdichte $w_m(QP)$ eines Punktes P (des Endpunktes des von Q aus abgetragenen Vektors $\overrightarrow{QP}$), den ich erhalte, wenn ich bilde:

$$\overrightarrow{Q}P_1 + \overrightarrow{Q}P_2 + \overrightarrow{Q}P_3 + \ldots + \overrightarrow{Q}P_m = \overrightarrow{Q}P,$$

wobei die Punkte P auf den Geraden G_r gemäss $v_r(s_r)$ variieren.

Fast dasselbe Problem ist nun für den k-dimensionalen Raum von H. Pollaczek-Geiringer (s. Lit.-Verz. Nr. (7)) behandelt worden, nur variieren dort die Punkte P_r auf geschlossenen Kurven C_r statt wie bei uns auf den Geraden G_r. Für den Beweis ist davon aber nur von Bedeutung, dass man durch die Annahme geschlossener Kurven in einem endlichen Gebiet bleibt, was in unserm Fall durch die zusätzliche Festsetzung 4 (S. 197) $v_r(s_r) = 0$ ausserhalb des Kreises K (d.h. nichts anderes als dass es für die Längen bei gradlinigem Flug eine obere Schranke K gibt) gewährleistet ist. Ausserdem könnte man sich die Geraden G_r ausserhalb des Kreises K in geschlossene Kurven übergehend denken, ohne an dem Problem etwas zu ändern.

Ich möchte mich deshalb hier auf den prinzipiellen Gang der Herleitung beschränken. Einzelheiten können aus der zitierten Arbeit entnommen werden. Gefragt ist zunächst nach der Wahrscheinlichkeit, dass die Vektorsumme

$$y_1(s_1) + y_2(s_2) + \ldots + y_m(s_m)$$

zwischen y und $y + dy$ liegt, wenn der Endpunkt von $y_r(s_r)$ auf G_r gemäss der Wahrscheinlichkeitsdichte $v_r(s_r)$ variiert.

Wenn $$y = \sqrt{m}\,u + b_m \quad \ldots\ldots (1),$$

ist die Wahrscheinlichkeit dafür, dass gilt

$$\sqrt{m}\,u + b_m \leqslant \sum_{r=1}^{m} y_r(s_r) \leqslant \sqrt{m}\,(u + du) + b_m \quad \ldots\ldots (2)$$

gegeben durch

$$J_m(u) = \int_{(m)} \ldots \int v_1(s_1)\, v_2(s_2) \ldots v_m(s_m)\, ds_1 ds_2 \ldots ds_m \quad \ldots\ldots (3),$$

das m-fache Integral erstreckt über den Teil eines m-dimensionalen Würfels in dem (2) gilt.

Man setzt nun

$$y_1(s_1)+y_2(s_2)+\ldots+y_m(s_m)=\sqrt{m}\,u+b_m, \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

wodurch die neue Veränderliche u auf b_m (Mittelwert) als Koordinatenursprung bezogen ist und der Massstab wie $1:\sqrt{m}$ verkleinert ist. Das besagt dasselbe, wie wenn ich im alten Massstab annehme, dass statt eines Fluges bestehend aus m Elementarflügen einer bestimmten Richtung G_r mit der resultierenden Verteilung $v_r(s_r)$, in diese Richtung nur einer dieser Elementarflüge ausgeführt wird, für den die Streuungskomponenten sind $s_{\lambda\mu}^{(r)}:m$ und dessen Mittelwert ist $a_r:\sqrt{m}$ mit den Komponenten $a_{r1}:\sqrt{m}$ und $a_{r2}:\sqrt{m}$ (s. (6) und (9)). Die durch (4) definierten zwei Gleichungen lassen sich nach s_{m-1} und s_m, wenn die entsprechende Funktionaldeterminante $D\neq 0$ ist, auflösen. Die Werte für s_{m-1} und s_m kann ich in (3) einsetzen und erhalte:

$$J_m(u)=\int\ldots\int v_1(s_1)\ldots v_{m-2}(s_{m-2})\,v_{m-1}(s_{m-1}[u])\,v_m(s_m[u]) \times\left|\frac{1}{D}\right| ds_1\ldots ds_{m-2}\,du_1\,du_2{}^{*} \quad \ldots\ldots(3a)$$

erstreckt über das m-dimensionale der Transformation entsprechende Integrationsgebiet. Dabei ist die Integration nach du_1 und du_2 gerade über das Intervall $du_1 du_2$ erstreckt. Wenn

$$(u+du)\sqrt{m}+b_m \text{ in } u\sqrt{m}+b_m$$

übergeht, kommen wir von der Wahrscheinlichkeit $J_m(u)$ zur Wahrscheinlichkeitsdichte $w_m(u)$, indem wir bilden:

$$\lim_{\substack{du_1\to 0\\ du_2\to 0}}\frac{J_m}{du_1\,du_2}=w_m(u)$$

$$=\int_{(m-2)}\ldots\int v_1(s_1)\ldots v_{m-2}(s_{m-2})\,v_{m-1}(s_{m-1}[u])\,v_m(s_m[u]) \times\left|\frac{1}{D}\right| ds_1\ldots ds_{m-2} \quad \ldots\ldots(5).$$

Dieses $(m-2)$-fache Integral multipliziert mit N geht nun für grosse m in unser gesuchtes Φ_1 über. Es ist also $\lim_{m\to\infty} w_m(u)$ zu untersuchen.

Es sei nun noch zu den $v_r(s_r)$ ein Mittelwert definiert als:

$$a_r=\int y_r v_r(s_r)\,ds_r{}^{\dagger} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

und es sei:

$$\sum_{r=1}^{m} a_r=b_m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

und als "Streuung" die zweireihige symmetrische Matrix

$$(s_{\lambda\mu}^{(r)})=\begin{pmatrix} s_{11}^{(r)} & s_{12}^{(r)} \\ s_{21}^{(r)} & s_{22}^{(r)} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

* s_{m-1} und s_m hängen natürlich auch noch von s_1 bis s_{m-2} ab.

† *Es soll im folgenden heissen:*

$$\int=\int_{-\infty}^{+\infty}, \quad \int_a=\int_a^{+\infty}, \quad \int^b=\int_{-\infty}^{b}.$$

wobei
$$s_{\lambda\mu}^{(r)} = \int v_r (y_{\lambda r} - a_{\lambda r})(y_{\mu r} - a_{\mu r})\, ds_r \quad \ldots\ldots (9),$$

und es sei:
$$|a_r| < a, \quad |s_{\lambda\mu}^{(r)}| < s \quad \ldots\ldots (10),$$

$$\sum_{r=1}^{m} s_{\lambda\mu}^{(r)} = t_{\lambda\mu}^{(m)} = m h_{\lambda\mu}^{(m)} \quad \ldots\ldots (11),$$

also
$$h_{\lambda\mu}^{(m)} = t_{\lambda\mu}^{(m)} : m \quad \text{und} \quad \lim_{m\to\infty} h_{\lambda\mu}^{(m)} = h_{\lambda\mu} \quad \ldots\ldots (12).$$

Die $h_{\lambda\mu}$ seien endlich, sie bilden die positive definite Matrix der reduzierten Streuungen. Die Summe der Streuungskomponenten der $v_r(s_r)$ wachse also wie m ins Unendliche, d.h. die Streuungskomponenten der verschiedenen Richtungen seien fast alle von der gleichen Grössenordnung.

Zu v_r sei eine komplexe Adjunkte

$$f_r(u) = \int e^{i(u-a_r)(y_r(t)-a_r)}\, v_r(t)\, dt^{*} \quad \ldots\ldots (13)$$

definiert, sodass
$$f_r(a_r) = 1 \quad \ldots\ldots (14),$$

$$\left(\frac{\partial f_r}{\partial s_\lambda}\right)_{a_r} = 0 \quad \ldots\ldots (15),$$

$$\left(\frac{\partial^2 f_r}{\partial s_\lambda \partial s_\mu}\right)_{a_r} = -s_{\lambda\mu}^{(r)}; \quad \lambda, \mu = 1, 2 \quad \ldots\ldots (16).$$

Analog sei die komplexe Adjunkte von w_m:

$$p_m(u) = \iint e^{izu} w_m(z)\, dz_1\, dz_2 \quad \ldots\ldots (17).$$

Führe ich nun statt z_1, z_2 die alten Variablen s_{m-1}, s_m ein, nämlich

$$z_\lambda = \frac{1}{\sqrt{m}} \sum_{r=1}^{m} (y_{\lambda r} - a_{\lambda r}); \quad \lambda = 1, 2 \quad \ldots\ldots (18),$$

so wird unter Berücksichtigung von (5),

$$\begin{aligned} p_m(u) &= \int_{(m)} \cdots \int D \frac{1}{D} e^{\frac{i}{\sqrt{m}}\left\{u_1 \sum\limits_{r=1}^{m} (y_{1r} - a_{1r}) + u_2 \sum\limits_{r=1}^{m} (y_{2r} - a_{2r})\right\}} v_1(s_1) \ldots v_m(s_m)\, ds_1 \ldots ds_m \\ &= \int e^{\frac{i}{\sqrt{m}} u (y_1 - a_1)} v_1(s_1)\, ds_1 \ldots \int e^{\frac{i}{\sqrt{m}} u (y_m - a_m)} v_m(s_m)\, ds_m \\ &= f_1\left(a_1 + \frac{u}{\sqrt{m}}\right) \ldots f_m\left(a_m + \frac{u}{\sqrt{m}}\right) \quad \ldots\ldots (19), \end{aligned}$$

d.h. die komplexe Adjunkte von $w_m(u)$ (ein Integralprodukt) ist auf ein gewöhn-

* ab bedeutet skalares Produkt der beiden Vektoren a und b. Also $ab = a_1 b_1 + a_2 b_2$, wobei a_1, a_2, b_1, b_2 die Komponenten von a bzw. b sind. In obigem Produkt entspricht: a dem Vektor $(u - a_r)$ und b dem Vektor $y_r(t) - a_r$).

liches Produkt der komplexen Adjunkten der einzelnen Verteilungen $v_r\left(a_r+\frac{u}{\sqrt{m}}\right)$ zurückgeführt. Nach einem Satz über gewöhnliche Produkte von v. Mises (s. Lit.-Verz. Nr. (8)) gilt nun unter gewissen Voraussetzungen, die hier zutreffen, wenn für die Funktionen f_r (14) bis (16) gilt:

$$\begin{aligned}\lim_{m\to\infty} f_1\left(a_1+\frac{u}{\sqrt{m}}\right) f_2\left(a_2+\frac{u}{\sqrt{m}}\right)\dots f_m\left(a_m+\frac{u}{\sqrt{m}}\right)\\ =\lim_{m\to\infty} p_m(u)\\ =e^{-\frac{1}{2}(h_{11}u_1^2-2h_{12}u_1u_2+h_{22}u_2^2)}, \quad\dots\dots(20),\end{aligned}$$

$h_{\lambda\mu}$ berechnet sich aus den $s_{\lambda\mu}^{(r)}$ nach (11), (12); ausserdem gilt:

$$\lim_{m\to\infty}\iint p_m(u)\,\psi(u)\,du_1\,du_2=\iint e^{-\frac{1}{2}\sum_{\lambda\mu} h_{\lambda\mu}u_\lambda u_\mu}\,\psi(u)\,du_1\,du_2 \quad\dots\dots(21).$$

Mit Hilfe des Fourier'schen Integraltheorems lässt sich nun umgekehrt aus der Adjunkte einer Verteilung diese selbst wiedergewinnen und zwar ist:

$$w_m(u)=\left(\frac{1}{2\pi}\right)^2\iint e^{-izu}\,p_m(z)\,dz_1\,dz_2 \quad\dots\dots(22),$$

wobei p_m die Adjunkte von w_m ist.

Also bei Benutzung der vorigen Formel (21) mit $\psi(u)=e^{-izu}$ wird:

$$\begin{aligned}\lim_{m\to\infty} w_m(u)=\lim_{m\to\infty}\left(\frac{1}{2\pi}\right)^2\iint e^{-izu}\,p_m(z)\,dz_1\,dz_2\\ =\left(\frac{1}{2\pi}\right)^2\iint e^{-izu}\,e^{-\frac{1}{2}\sum_{\lambda\mu} h_{\lambda\mu}z_\lambda z_\mu}\,dz_1\,dz_2 \quad\dots\dots(23).\end{aligned}$$

Das Doppelintegral lässt sich auswerten durch zweimalige Anwendung der Formel

$$\int e^{-pz^2-2qz}\,dz=\sqrt{\frac{\pi}{p}}\,e^{\frac{q^2}{p}} \quad\dots\dots(24),$$

wobei ich erhalte:

$$\lim_{m\to\infty} w_m(u)=w(u)=\frac{1}{2\pi\sqrt{h_{11}h_{22}-h_{12}^2}}\,e^{-\frac{h_{22}u_1^2-2h_{12}u_1u_2+h_{11}u_2^2}{2(h_{11}h_{22}-h_{12}^2)}} \quad\dots\dots(25).$$

Das wäre also die Wahrscheinlichkeitsdichte für den Punkt P, den Endpunkt des Vektors $u\sqrt{m}+b$, den man durch die angegebene Vektoraddition erhält. Oder bei der oben auf Seite 198 gemachten Annahme, die ja im wesentlichen nichts anderes besagt, als dass die Länge einer gradlinigen Flugstrecke umgekehrt proportional ist der Anzahl der überhaupt ausgeführten gradlinigen Flugstrecken, ist

$$Nw(u)=\Phi_1(u) \quad\dots\dots(26).$$

Das ist die Dichte der Mücken—also Anzahl pro Flächeneinheit—nach dem ersten Flug an einem Punkt P dem Endpunkt des von Q abgetragenen Vektors u. (Dabei

ist $\lim\limits_{m\to\infty}\frac{b_m}{\sqrt{m}}=b=0$ angenommen, andernfalls ist die gesammte Verteilung um den Vektor b zu verschieben.)

Als Lösung des Grundproblems erhalten wir also eine Gauss'sche Verteilung

$$\Phi_1(u)=\frac{N}{2\pi\sqrt{h_{11}h_{22}-h_{12}^2}}e^{-\frac{h_{22}u_1^2-2h_{12}u_1u_2+h_{11}u_2^2}{2(h_{11}h_{22}-h_{12}^2)}}\quad\ldots\ldots\ldots\ldots(27).$$

Durch geeignete Wahl des Koordinatensystem, Drehung um den Winkel

$$\phi=\tfrac{1}{2}\operatorname{arc}tg\frac{2h_{12}}{h_{11}-h_{22}},$$

mittels orthogonaler Transformation der Koordinaten, indem man setzt:

$$u_1=x\cos\phi-y\sin\phi,\quad u_2=x\sin\phi+y\cos\phi,$$

lässt sich (27) noch vereinfachen, und man erhält:

$$\Phi_1(u)=\frac{N}{2\pi s_1 s_2}e^{-\frac{x^2}{2s_1^2}-\frac{y^2}{2s_2^2}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28),$$

dabei ist
$$s_1^2=\frac{2(h_{11}h_{22}-h_{12}^2)}{h_{11}+h_{22}+\sqrt{4h_{12}^2+(h_{22}-h_{11})^2}},$$

und
$$s_2^2=\frac{2(h_{11}h_{22}-h_{12}^2)}{h_{11}+h_{22}-\sqrt{4h_{12}^2+(h_{22}-h_{11})^2}}.$$

Sind nun keine Teile der Ebene, über die sich die Verteilung erstreckt, irgendwie ausgezeichnet, was ich hier der Einfachheit halber annehmen möchte, so muss aus Symmetriegründen $s_1^2=s_2^2$ sein, und es wird:

$$\left.\begin{aligned}\Phi_1(r)&=\frac{N}{2\pi s^2}e^{-\frac{r^2}{2s^2}}\qquad(r^2=x^2+y^2)\\&=\frac{N}{\pi\sigma^2}e^{-\frac{r^2}{\sigma^2}}\end{aligned}\right\}\ldots\ldots(29).$$

$2s^2=\sigma^2$ kann man als "Streuung" des Abstandes r vom Ursprung bezeichnen, wenn man als Definition der Streuung ansieht:

$$\sigma^2=\frac{1}{N}\sum_\nu r_\nu^2\Phi(r_\nu)f_\nu,$$

oder bei geometrischer Verteilung:

$$\begin{aligned}\sigma^2&=\int_0\int_0^{2\pi}\frac{1}{2\pi s^2}e^{-\frac{r^2}{2s^2}}r^3\,dr\,d\phi=\frac{1}{2\pi s^2}\iint(x^2+y^2)\,e^{-\frac{x^2+y^2}{2s^2}}dx\,dy\\&=\frac{1}{\sqrt{2\pi}s}\int\left\{e^{-\frac{y^2}{2s^2}}\frac{1}{\sqrt{2\pi}s}\int x^2e^{-\frac{x^2}{2s^2}}dx\right\}dy\\&\qquad+\frac{1}{\sqrt{2\pi}s}\int\left\{e^{-\frac{x^2}{2s^2}}\frac{1}{\sqrt{2\pi}s}\int y^2e^{-\frac{y^2}{2s^2}}dy\right\}dx\\&=s^2+s^2=2s^2=\sigma^2\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(30),\end{aligned}$$

da allgemein gilt (s. Formel (24)):

$$\frac{1}{\sqrt{2\pi}\,s}\int x^2 e^{-\frac{x^2}{2s^2}}dx = s^2, \quad \frac{1}{\sqrt{2\pi}\,s}\int e^{-\frac{x^2}{2s^2}}dx = 1.$$

Analog ist der mittlere oder durchschnittliche Abstand von Q:

$$l = \frac{1}{N}\int_0\int_0^{2\pi} r\Phi_1(r)\, r\,dr\,d\phi = \frac{1}{2\pi s^2}\,2\pi\int_0 r^2 e^{-\frac{r^2}{2s^2}}dr = \sqrt{\frac{\pi}{2}}\,s = \frac{\sigma}{2}\sqrt{\pi}\ldots(31).$$

Das wäre wohl die Grösse, die in den früheren Arbeiten als durchschnittliche Fluglänge l bezeichnet wird, obgleich es genauer heissen müsste: durchschnittlicher Abstand nach einem Fluge von einem Punkt aus, denn die im Durchschnitt tatsächlich durchflogene Strecke ist zweifellos grösser.

3. *Die von einem Punkt ausgehende Verteilung nach dem n-ten Fluge.*

Nach dem ersten Flug befinden sich auf einem Flächenelement $dx'dy'$ mit dem inneren Punkt $\overline{Q}_{x',y'}$ $\frac{N}{\pi\sigma^2}e^{-\frac{r'^2}{\sigma^2}}dx'dy'$ Mücken. Beim nächsten Flug verteilen sich nun diese von $\overline{Q}$ aus wie vorher die Gesamtzahl N von Q aus. Der entsprechende Vorgang findet von sämtlichen Flächenelementen der Ebene statt. Alle diese Verteilungen überlagern sich und ergeben die Gesamtverteilung nach dem zweiten Flug. Entsprechend erhält man alle weiteren Verteilungen. Es sei nun

$$\Phi_{n-1}(r) = \frac{N}{\pi(n-1)\sigma^2}e^{-\frac{r^2}{(n-1)\sigma^2}}\ldots\ldots(32),$$

dann ergibt sich nach der Methode der vollständigen Induktion, da (32) für $n = 2$ richtig ist, nämlich

$$\Phi_1(r) = \frac{N}{\pi\sigma^2}e^{-\frac{r^2}{\sigma^2}}\ldots\ldots(29),$$

$$\Phi_n(r) = \iint\frac{N}{\pi(n-1)\sigma^2}e^{-\frac{r'^2}{(n-1)\sigma^2}}dx'dy'\,\frac{1}{\pi\sigma^2}e^{-\frac{(r-r')^2}{\sigma^2}}$$

$$= \frac{N}{\pi^2(n-1)\sigma^4}e^{-\frac{r^2}{\sigma^2}}\iint e^{-\frac{nr'^2-2(n-1)rr'}{(n-1)\sigma^2}}dx'dy'$$

$$= \frac{N}{\pi^2(n-1)\sigma^4}e^{-\frac{r^2}{\sigma^2}}\int\left\{e^{-\frac{2(n-1)xx'+nx'^2}{(n-1)\sigma^2}}\int e^{-\frac{2(n-1)yy'+ny'^2}{(n-1)\sigma^2}}dy'\right\}dx',$$

nach zweimaliger Anwendung der Formel (24) erhält man:

$$= \frac{N}{\pi^2(n-1)\sigma^4}e^{-\frac{r^2}{\sigma^2}}\sqrt{\frac{\pi(n-1)\sigma^2}{n}}\,e^{\frac{(n-1)\sigma^2y^2}{n}}\sqrt{\frac{\pi(n-1)\sigma^2}{n}}\,e^{\frac{(n-1)\sigma^2x^2}{n}}$$

$$= \frac{N}{n\pi\sigma^2}e^{-\frac{r^2}{n\sigma^2}} = \frac{N}{4nl^2}e^{-\frac{r^2\pi}{4nl^2}} = \Phi_n(r)\ldots\ldots(33).$$

Die Rayleigh-Pearson'sche Lösung war für $n \to \infty$

$$\Phi_n(r) = \frac{N}{n\pi l^2} e^{-\frac{r^2}{nl^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{V}).$$

Sie weicht also für grosse n nur in geringem Masse von der hier erhaltenen ab und würde mit dieser zusammenfallen, wenn mit l^2 die "Streuung" nach dem ersten Flug gemeint wäre.

In einem Kreis um Q mit dem Radius σ befinden sich nach n Flügen:

$$W_n(\sigma) = \int_0^\sigma \int_0^{2\pi} \frac{N}{n\pi\sigma^2} e^{-\frac{r^2}{n\sigma^2}} r\,dr\,d\phi = N\left[-e^{-\frac{r^2}{n\sigma^2}}\right]_0^\sigma$$

$$= N\left(1 - e^{-\frac{1}{n}}\right) = N\left(\frac{1}{n} - \frac{1}{2!\,n^2} + \frac{1}{3!\,n^3} - \ldots\right) \quad \ldots\ldots\ldots(34).$$

Das stimmt in erster Näherung mit dem Resultat von Kluyver (*l.c.* S. 344) bzw. Pearson überein, wenn wie oben mit der mittleren Fluglänge l das σ gemeint ist. Sie erhielten:

$$W_n(\sigma) = N:(n+1) = N\left(\frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^3} - \frac{1}{n^4} + \ldots\right) \quad \ldots\ldots\ldots(34\,a).$$

Die Verteilung nach m Flugperioden, deren jede aus n Flügen besteht, ergibt sich ohne weiteres aus (29), indem man setzt:

$$n\sigma^2 \text{ statt } \sigma^2 \text{ und } m \text{ statt } n.$$

Es wird dann nach (33)

$$_m\Phi_n(r) = \Phi_{mn}(r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(35).$$

Also resultiert die gleiche Verteilung, ob m-mal Flugperioden aus je n Flügen stattfinden oder ob eine Flugperiode aus mn Flügen stattfindet.

Ist nun von einer Mückenart M_1 das σ_1 bekannt, so lässt sich daraus für eine andere $M_2\sigma_2$ berechnen, wenn man auch für diese nach dem sogenannten Prinzip von "Trial and Error" in der hier ausgeführten Weise die Verteilung, durch "alltägliche Flüge" (s. Lit.-Verz. Nr. (17), S. 237) entstehend, annehmen kann, dass also nach der in der Biologie üblichen Ausdrucksweise "Random Migration" vorliegt. Zunächst sei die Dauer eines Fluges von M_2 t-mal so gross wie die von M_1. Dann kann man die Verteilung von M_2 nach einer Flugperiode auffassen als die Verteilung von M_1 nach t Flugperioden. Oder man nimmt an, dass bei t-facher Flugdauer statt eines Elementarfluges einer bestimmten Richtung θ_r t Elementarflüge dieser Art ausgeführt werden, wodurch die Streuungskomponenten $s_{\lambda\mu}^{(r)}$ mit t zu multiplizieren sind und daher auch, wie man aus Formel (8) bis (12), (28) und (30) sieht:

$$\sigma_2^2 = t\sigma_1^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36).$$

Also sowohl nach (29) in Verbindung mit (33) wie nach (35) erhält man:

$$^{(2)}\Phi_n(r) = {}^{(1)}\Phi_{tn}(r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(37).$$

Andererseits sei die lineare Fluggeschwindigkeit von M_2 k-mal so gross wie die von M_1. Dann wird man annehmen können, dass die Wahrscheinlichkeit für

die Länge eines Fluges längs G_r bei M_1 gleich ist der für die k-fache Länge bei M_2 längs derselben Geraden G_r. Man hat also in (6) bis (9) für y_r ky_r zu setzen. Daraus folgt, dass $s_{\lambda\mu}^{(r)}$ in $k^2 s_{\lambda\mu}^{(r)}$ übergeht, und nach Berücksichtigung der gleichen Formeln wie oben wird also

$$\sigma_2^2 = k^2 \sigma_1^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (38).$$

Unterscheidet sich also M_2 von M_1 durch die t-fache Flugdauer und die k-fache Fluggeschwindigkeit, so erhält man die Verteilung für M_2 aus der für M_1 als:

$$^{(2)}\Phi_n(r) = {}^{(1)}\Phi_{tk^2 n}(r) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (39).$$

Umgekehrt lässt sich von den Verteilungen zweier Tierarten auf das Verhältnis ihrer Fluggeschwindigkeiten schliessen, eine Grösse, die bei Untersuchungen von A. J. Lotka über das Fangen und Entkommen der Beute eines Raubtieres von grundlegender Bedeutung sein dürfte (s. Lit.-Verz. Nr. (21)).

4. *Verteilung von einem mit der Dichte N besetzten Gebiet in die unbesetzte Umgebung**.

Es gehe jetzt die Verteilung von einer irgendwie begrenzten und gleichmässig dicht besetzten Fläche Q aus. Die Dichte sei N. Es befinden sich also in einem Flächenelement $dx'dy'$ $N dx'dy'$ Mücken. Die Verteilung von jedem Flächenelement aus sei gleichmässig nach allen Seiten wie in den §§ 2 und 3. Dann wird nach n Flügen die Dichte an einem Punkt $P = (x, y)$ sein:

$$F_n(x, y) = \iint_{(Q)} N\,dx'dy' \frac{1}{N} \Phi_n\left(\sqrt{(x-x')^2 + (y-y')^2}\right)$$

$$= \iint_{(Q)} A\,df \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (40).$$

Das Integral ist über die Fläche Q zu erstrecken.

Sei andererseits für das Äussere eines Gebietes Q die Dichte gleich N, so ergibt sich für die Verteilung in das ursprünglich freie Innere nach dem n-ten Flug:

$$F_n(x, y) = \iint_{(Qa)} A\,df = \iint A\,df - \iint_{(Qi)} A\,df = N - F_n(x, y) \quad \ldots\ldots (41),$$

denn das Integral über das Aussere einer Fläche Q ist gleich dem Integral über die ganze Ebene vermindert um das Integral über das Innere von Q.

Beispiele für verschiedene Gebiete.

(a) Verteilung von einer durch eine gerade Linie L begrenzten Halbebene aus.

Die Dichte rechts von L sei N. Wie gross ist nach n Flügen die Dichte F im Punkt O (Koordinatenanfangspunkt) mit dem Abstand $-x$ von L?

* Teilweise sind die mathematischen Formeln dieses Paragraphen im Prinzip in der Pearson'schen Arbeit (s. Lit.-Verz. Nr. (8)) schon enthalten, da sie aus den dortigen komplizierteren Sätzen jedoch nicht so ohne weiteres ersichtlich sind und ich häufig eine etwas andere Ableitung gewählt habe, habe ich nähere Hinweise unterlassen.

Nach (40), wenn die y-Achse parallel L, ist

$$
\begin{aligned}
F_n(x) = F_n(0,0) &= \int_x\int \Phi_n(\sqrt{x'^2+y'^2})\,dx'dy' \\
&= \frac{N}{n\pi\sigma^2}\int_x\int e^{-\frac{x'^2+y'^2}{n\sigma^2}}\,dx'dy' \\
&= \frac{N}{n\pi\sigma^2}\int_x\left\{e^{-\frac{x'^2}{n\sigma^2}}\int e^{-\frac{y'^2}{n\sigma^2}}dy'\right\}dx' \\
&= \frac{N}{\sqrt{n\pi}\,\sigma}\int_x e^{-\frac{x'^2}{n\sigma^2}}\,dx' \\
&= \frac{N}{\sqrt{\pi n}\,\sigma}\left\{\int e^{-\frac{x'^2}{n\sigma^2}}\,dx' - \int^x e^{-\frac{x'^2}{n\sigma^2}}\,dx'\right\} \\
&= N\left\{1-\frac{1}{\sqrt{\pi n}\,\sigma}\int^x e^{-\frac{x^2}{n\sigma^2}}\,dx\right\} \quad\ldots\ldots\ldots\ldots(42),
\end{aligned}
$$

d.i. also die Dichte an einem Punkt mit dem Abstand $-x$ von L. Die Werte des Integrals sind aus Tabellen, wie sie fast in jedem Werk über Wahrscheinlichkeitsrechnung oder Statistik (z.B. Lit.-Verz. Nr. (9) bis (11)) enthalten sind, zu entnehmen. Für $x=0$, d.h. auf der Grenzlinie L ist die Dichte wie zu erwarten gleich $\frac{1}{2}N$.

(b) Verteilung von einem Rechteck mit den Seiten $2a$ und $2b$.

Die Dichte sei wieder N. Die Koordinatenachsen sein parallel den Rechteckseiten und ihr Anfangspunkt sei der Schnittpunkt der Diagonalen. Dann ist die Dichte nach dem n-ten Flug im Punkt (x, y):

$$
\begin{aligned}
F_n(x,y) &= \int_{-a}^{+a}\int_{-b}^{+b}\Phi_n(\sqrt{(x'-x)^2+(y'-y)^2})\,dx'dy' \\
&= \frac{N}{\pi n\sigma^2}\int_{-a}^{a}\int_{-b}^{b} e^{-\frac{(x'-x)^2+(y'-y)^2}{n\sigma^2}}\,dx'\,dy' \\
&= \frac{N}{\sqrt{\pi n}\,\sigma}\int_{-a}^{+a} e^{-\frac{(x'-x)^2}{n\sigma^2}}\,dx'\;\frac{1}{\sqrt{\pi n}\,\sigma}\int_{-b}^{+b} e^{-\frac{(y'-y)^2}{n\sigma^2}}\,dy' \\
&= N\left\{1-\int^{-a-x}\frac{1}{\sqrt{n\pi}\,\sigma}e^{-\frac{x'^2}{n\sigma^2}}dx' - \int^{-a+x}\frac{1}{\sqrt{n\pi}\,\sigma}e^{-\frac{x'^2}{n\sigma^2}}dx'\right\} \\
&\qquad\times\left\{1-\int^{-b-y}(\ldots y'\ldots)-\int^{-b+y}(\ldots y'\ldots)\right\}\quad\ldots\ldots\ldots(43).
\end{aligned}
$$

Bei der Verteilung von einem schmalen Streifen, d.h. $b=\infty$, wird der zweite Faktor $\{1\}$.

(c) Verteilung von einem Kreis mit dem Radius a um O als Mittelpunkt.

Die Dichte sei wieder N. Dann ist nach n Flügen die Dichte an einem Punkt im Abstand c von O:

$$
\begin{aligned}
F_n(c) &= \int_0^a\int_0^{2\pi}\Phi_n(\sqrt{c^2+r^2-2rc\cos\theta})\,r\,dr\,d\theta \\
&= \frac{N}{\pi n\sigma^2}\int_0^a\int_0^{2\pi} e^{-\frac{c^2+r^2-2rc\cos\theta}{n\sigma^2}}\,r\,dr\,d\theta \quad\ldots\ldots\ldots(44),
\end{aligned}
$$

nach Entwicklung von $e^{\frac{2rc\cos\theta}{n\sigma^2}}$ in die bekannte e-Reihe:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \dots\dots\dots\dots(45),$$

wird also $$F_n(c) = \frac{N}{\pi n\sigma^2} e^{-\frac{c^2}{n\sigma^2}} \int_0^a \int_0^{2\pi} \left\{ e^{-\frac{r^2}{n\sigma^2}} r \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{2rc}{n\sigma^2}\right)^k \cos^k\theta \right\} d\theta\, dr.$$

Da nun $$\int_0^{2\pi} \cos^{2k+1}\theta\, d\theta = 0, \quad \int_0^{2\pi} \cos^{2k}\theta\, d\theta = \frac{2\pi}{2^{2k}} \binom{2k}{k} \dots\dots\dots\dots(46),$$
wird weiter
$$F_n(c) = \frac{N}{\pi n\sigma^2} e^{-\frac{c^2}{n\sigma^2}} \int_0^a \left\{ e^{-\frac{r^2}{n\sigma^2}} r \sum_{k=0}^{\infty} \left(\frac{2rc}{n\sigma^2}\right)^{2k} \frac{2\pi}{(2^k k!)^2} \right\} dr$$
$$= \frac{2N}{n\sigma^2} e^{-\frac{c^2}{n\sigma^2}} \sum_{k=0}^{\infty} \left\{ \left(\frac{c}{n\sigma^2}\right)^{2k} \left(\frac{1}{k!}\right)^2 \int_0^a e^{-\frac{r^2}{n\sigma^2}} r^{2k+1}\, dr \right\}.$$

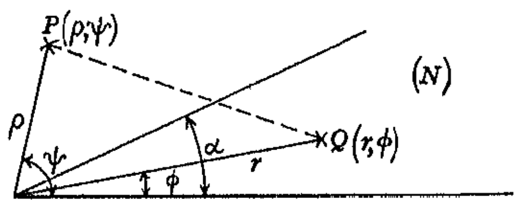
Die Integrale lassen sich sukzessiv durch Produktintegration auswerten, sodass also:

$$F_n(c) = N \left\{ 1 - e^{-\frac{c^2+a^2}{n\sigma^2}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{c^2}{n\sigma^2}\right)^k \left[1 + \frac{a^2}{n\sigma^2} + \frac{1}{2!}\left(\frac{a^2}{n\sigma^2}\right)^2 + \dots + \frac{1}{k!}\left(\frac{a^2}{n\sigma^2}\right)^k \right] \right\} \dots\dots\dots(47).$$

Die Konvergenz dieser Summe ist bei $c^2 \ll n\sigma^2$ verhältnismässig gut. Man kann aber auch zur Auswertung bei Benutzung der vorhergehenden Form Tabellen über die höheren Momente der Gauss'schen Summenfunktion benutzen (s. Lit.-Verz. Nr. (9), S. 22, 23).

Ist bei den behandelten Verteilungen das Innere zunächst frei von Mücken, während ausserhalb der betrachteten Gebiete die Dichte N ist, so ergibt sich die Verteilung in das Gebiet hinein gemäss (41).

(*d*) Verteilung von einem Winkelraum aus.



Innerhalb des von den Schenkeln des Winkels α abgegrenzten Gebietes herrsche wieder eine Dichte N. Dann wird nach n Flügen an einem Punkt P mit den Polarkoordinaten ρ, ψ die Dichte sein:

$${}_\alpha F_n(\rho, \psi) = \int_0 \int_0^\alpha \Phi_n\left(\sqrt{r^2 + \rho^2 - 2r\rho\cos(\psi - \phi)}\right) r\, dr\, d\phi \quad \dots\dots\dots(48)$$

—unter Berücksichtigung von (45) und $\theta = \psi - \phi$

$$= \frac{N}{\pi n\sigma^2} e^{-\frac{\rho^2}{n\sigma^2}} \int_0 \left\{ e^{-\frac{r^2}{n\sigma^2}} r \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{2r\rho}{n\sigma^2}\right)^k \int_{\psi-\alpha}^{\psi} \cos^k\theta\, d\theta \right\} dr$$
$$= \frac{N}{\pi n\sigma^2} e^{-\frac{\rho^2}{n\sigma^2}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{2\rho}{n\sigma^2}\right)^k \int_0 e^{-\frac{r^2}{n\sigma^2}} r^{k+1}\, dr \int_{\psi-\alpha}^{\psi} \cos^k\theta\, d\theta.$$

Wegen der unbedingten Konvergenz dieser Reihe kann man sie aufspalten und als Summe der Reihen mit geradem und ungeradem k schreiben. Berücksichtigt man dann noch, dass:

$$\left.\begin{aligned}\int_0 e^{-\frac{r^2}{n\sigma^2}} r^{2k+1}\,dr &= \frac{k!}{2}(n\sigma^2)^{k+1}\\ \int_0 e^{-\frac{r^2}{n\sigma^2}} r^{2(k+1)}\,dr &= \frac{(2k+1)!}{k!\,2^{2k+1}}\frac{\sqrt{\pi n}\,\sigma}{2}(n\sigma^2)^{k+1}\end{aligned}\right\}\ldots\ldots\ldots\ldots(49),$$

so wird:

$${}_\alpha F_n(\rho,\psi) = Ne^{-\frac{\rho^2}{n\sigma^2}}\left\{\frac{1}{2\pi}\sum_{k=0}^{\infty}\left[\frac{k!}{(2k)!}\left(\frac{2\rho}{\sqrt{n}\,\sigma}\right)^{2k}\int_{\psi-\alpha}^{\psi}\cos^{2k}\theta\,d\theta\right]\right.$$
$$\left.+\frac{1}{2\sqrt{\pi}}\sum_{k=0}^{\infty}\left[\frac{1}{k!}\left(\frac{\rho}{\sqrt{n}\,\sigma}\right)^{2k+1}\int_{\psi-\alpha}^{\psi}\cos^{2k+1}\theta\,d\theta\right]\right\}.$$

Die noch verbleibenden Integrale lassen sich auch auswerten (s. Anmerkung*), und es wird nach Einsetzung und einfacher Umordnung:

$${}_\alpha F_n(\rho,\psi) = N\left\{\frac{\alpha}{2\pi}+\frac{1}{\pi}e^{-\frac{\rho^2}{n\sigma^2}}\sum_{k=1}^{\infty}\left(\frac{\rho^2}{n\sigma^2}\right)^k\frac{k!}{(2k)!}\right.$$
$$\times\sum_{r=0}^{k-1}\binom{2k}{r}\frac{\cos([k-r][2\psi-\alpha])\sin[k-r]\alpha}{k-r}+\frac{\rho}{\sqrt{\pi n}\,\sigma}e^{-\frac{\rho^2}{n\sigma^2}}\sum_{k=0}^{\infty}\left(\frac{\rho^2}{n\sigma^2}\right)^k\frac{1}{2^{2k}k!}$$
$$\left.\times\sum_{r=0}^{k}\binom{2k+1}{r}\frac{\cos(2k+1-2r)\left(\psi-\frac{\alpha}{2}\right)\sin(2k+1-2r)\frac{\alpha}{2}}{2k+1-2r}\right\}\ldots\ldots\ldots\ldots(50).$$

Die Güte der Konvergenz hängt wieder vor allem von $\frac{\rho^2}{n\sigma^2}$ ab.

Für $\alpha=\pi$, $\psi=\frac{1}{2}\pi$ erhält man, wie man sich durch Einsetzen und Reihenvergleichung überzeugen kann, den nach (42) zu erwartenden Wert*:

$$F_n(-\rho) = {}_\pi F_n\left(\rho,\frac{\pi}{2}\right).$$

* Es ist:
$$\cos^n x = (\tfrac{1}{2})^n(e^{ix}+e^{-ix})^n = (\tfrac{1}{2})^n\sum_{r=0}^{n}\binom{n}{r}e^{ix(n-2r)}$$
$$= (\tfrac{1}{2})^n\sum_{r=0}^{n}\binom{n}{r}\{\cos x(n-2r)+i\sin x(n-2r)\}.$$

Da $\cos^n x$ für reelle x reell ist, müssen die imaginären Glieder der Summe 0 ergeben, also:

$$\cos^n x = (\tfrac{1}{2})^n\sum_{r=0}^{n}\binom{n}{r}\cos x(n-2r),$$

$$\int_{\psi-\alpha}^{\psi}\cos^n x\,dx = (\tfrac{1}{2})^n\sum_{r=0}^{n}\binom{n}{r}\frac{\sin x(n-2r)}{n-2r}\Bigg|_{\psi-\alpha}^{\psi}$$

$$= (\tfrac{1}{2})^n\sum_{r=0}^{n}\binom{n}{r}\frac{2\cos(n-2r)\left(\psi-\frac{\alpha}{2}\right)\sin(n-2r)\frac{\alpha}{2}}{n-2r},$$

5. *Ausbreitung in Gebiete, die durch Hindernisse (Gebirge, Wasser od. ähnl.) begrenzt sind.*

Ich beschränke mich auf den Fall, dass die mögliche Ausbreitungsfläche durch eine gradlinige Grenze abgeschlossen ist und mache die Hypothese, dass ein von einem Punkt ausgehender Flug an dieser Geraden gespiegelt wird. Es wären zwar noch andere Möglichkeiten für das Verhalten der Mücken an dem Hindernis denkbar, mir erscheint jedoch bei der Annahme rein zufälliger Flüge die oben gemachte Voraussetzung am glaubwürdigsten.

(*a*) Verteilung der N Mücken von einem Zentrum aus.

Die Grenzgerade H verlaufe parallel der y-Achse im Abstand a von dieser. Von $Q=(0, 0)$ aus mögen wieder N Mücken abfliegen. Dann ist nach dem ersten Flug die Dichte an einem Punkt (x, y) für $x \leqslant a$:

$$\frac{N}{\pi\sigma^2} e^{-\frac{x^2+y^2}{\sigma^2}},$$

dazu kommt die Dichte des Spiegelbildes von (x, y) bezüglich der Geraden H, nämlich:

$$\frac{N}{\pi\sigma^2} e^{-\frac{(x+2[a-x])^2+y^2}{\sigma^2}} = \frac{N}{\pi\sigma^2} e^{-\frac{(2a-x)^2+y^2}{\sigma^2}}.$$

Also wird die gesamte Dichte:

$$_a\overline{\Phi}_1(x, y) = \frac{N}{\pi\sigma^2}\left(e^{-\frac{x^2+y^2}{\sigma^2}} + e^{-\frac{(2a-x)^2+y^2}{\sigma^2}}\right) \quad \ldots\ldots\ldots\ldots(51).$$

Die Gesamtzahl der Mücken ist natürlich wieder:

$$\int^a \int {}_a\overline{\Phi}_1(x, y)\, dx\, dy = N \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(52),$$

wie bei der Verteilung (33), was man leicht nachrechnen kann.

Sei beim $(n-1)$-ten Flug jetzt die Verteilung gegeben durch $_a\overline{\Phi}_{n-1}(x, y)$, dann erhält man $_a\overline{\Phi}_n(x, y)$, indem von allen Flächenelementen $dx'\,dy'$ sich

$$_a\overline{\Phi}_{n-1}(x', y')\, dx'\, dy' \text{ Mücken}$$

gemäss $\frac{1}{N}\,_{a-x'}\overline{\Phi}_1(x, y)$ verteilen, sich überlagern und $_a\overline{\Phi}_n(x, y)$ ergeben.

also $$\int_{\psi-\alpha}^{\psi} \cos^{2k} x\, dx = (\tfrac{1}{2})^{2k}\binom{2k}{k}\alpha + \frac{2}{2^{2k}}\sum_{r=0}^{k-1}\binom{2k}{r}\frac{\cos(k-r)(2\psi-\alpha)\sin(k-r)\alpha}{k-r},$$

$$\int_{\psi-\alpha}^{\psi} \cos^{2k+1} x\, dx = \frac{2}{2^{2k}}\sum_{r=0}^{k}\binom{2k+1}{r}\frac{\cos(2k+1-2r)\left(\psi-\frac{\alpha}{2}\right)\sin(2k+1-2r)\frac{\alpha}{2}}{2k+1-2r}.$$

Für $\alpha=\pi$, $\psi=\frac{\pi}{2}$ erhält man die Formel

$$\frac{2^{2k}(k!)^2}{(2k+1)!} = \frac{1}{2^{2k}}\sum_{r=0}^{k}\binom{2k+1}{r}\frac{(-1)^{k-r}}{2k+1-2r},$$

da bekanntlich durch partielle Integration wird:

$$\int_{-\pi/2}^{+\pi/2}\cos^{2k+1} x\, dx = 2\int_0^{\pi/2}\cos^{2k+1} x\, dx = 2\,\frac{2^{2k}(k!)^2}{(2k+1)!}.$$

Die obige Formel ist beim Vergleich von (50) mit (42) anzuwenden.

Bei der Annahme, dass gilt:

für $x \leqq a$: $${}_a\overline{\Phi}_{n-1}(x,y) = \Phi_{n-1}(x,y) + \Phi_{n-1}(2a-x,y),$$

für $x > a$: $${}_a\overline{\Phi}_{n-1}(x,y) = 0,$$

folgt durch vollständige Induktion, da nach (51)

$${}_a\overline{\Phi}_1(x,y) = \Phi_1(x,y) + \Phi_1(2a-x,y) \quad \ldots\ldots\ldots\ldots(51\,a)$$

ist, dass gilt:

$${}_a\overline{\Phi}_n(x,y) = \int^a \int \left\{\Phi_{n-1}(x',y') + \Phi_{n-1}(2a-x',y')\right\} \frac{1}{N}$$
$$\times \{\Phi_1(x-x',y-y') + \Phi_1(2a-x'-x,y-y')\}\,dx'\,dy' \quad \ldots\ldots(53).$$

Dies Integral lässt sich in die Summe aus vier Integralen aufspalten und zwar:

1. $$\int^a \int \Phi_{n-1}(x',y')\,\Phi_1(x-x',y-y')\,dx'\,dy'.$$

2. $$\int^a \int \Phi_{n-1}(2a-x',y')\,\Phi_1(2a-x'-x,y-y')\,dx'\,dy'$$
$$= \int_a \int \Phi_{n-1}(\overline{x}',y')\,\Phi_1(\overline{x}'-x,y-y')\,d\overline{x}'\,dy',$$

wobei $$2a-x' = \overline{x}'.$$

Da nach (29) $\Phi_1(-x,y) = \Phi_1(+x,y)$, ergibt

1. + 2. $$= \iint \Phi_{n-1}(x',y')\,\Phi_1(x-x',y-y')\,dx'\,dy',$$

das ist nach (33) $= \Phi_n(x,y)$.

3. $$\int^a \int \Phi_{n-1}(x',y')\,\Phi_1(2a-x'-x,y-y')\,dx'\,dy',$$

4. $$\int^a \int \Phi_{n-1}(2a-x',y')\,\Phi_1(x-x',y-y')\,dx'\,dy'$$
$$= \int_a \int \Phi_{n-1}(\overline{x}',y')\,\Phi_1(x-2a+\overline{x}',y-y')\,d\overline{x}'\,dy',$$

wobei $$2a-x' = \overline{x}',$$

also:

3. + 4. $$= \iint \Phi_{n-1}(x',y')\,\Phi_1(2a-x'-x,y-y')\,dx'\,dy',$$

das ist nach (33) $= \Phi_n(2a-x,y)$.

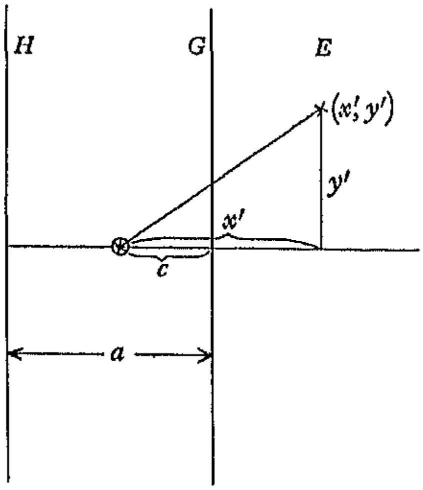
Also ist für $x \leqq a$:

1. + 2. + 3. + 4. $$= {}_a\Phi_n(x,y) = \Phi_n(x,y) + \Phi_n(2a-x,y) \quad \ldots\ldots\ldots\ldots(53a),$$

w.z.b.w.

(*b*) Verteilung von einer mit der Dichte N besetzten Halbebene aus.

Die Verteilung gehe also von einer durch eine Gerade G begrenzten Halbebene E mit der Dichte N aus in ein Gebiet, das durch eine Gerade H, die parallel G sei, abgeschlossen ist.



Dann ist nach n Flügen an einem Punkt (dem Nullpunkt) im Abstand c von G die Dichte:

$$
\begin{aligned}
{}_aF_n(c) &= \int_c \int_{a+x'-c} \bar{\Phi}_n(x', y')\, dx'\, dy' \\
&= \int_c \int \Phi_n(x', y')\, dx'\, dy' + \int_c \int \Phi_n(2a - 2c + x', y')\, dx'\, dy' \quad \ldots\ldots (54),
\end{aligned}
$$

das ergibt nach (42)

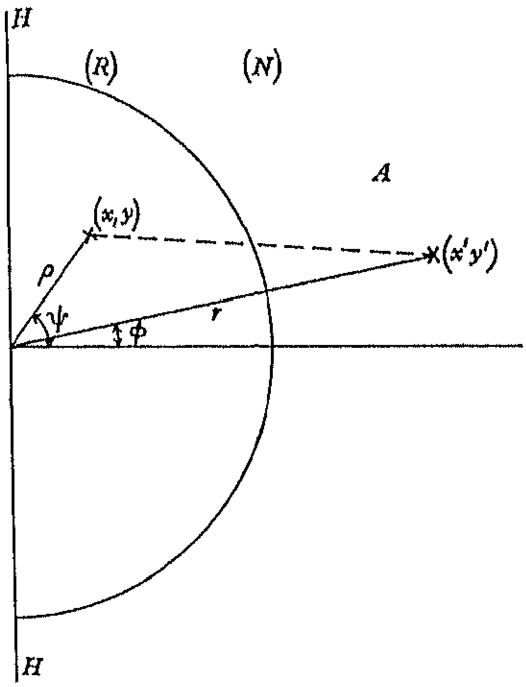
$$
\begin{aligned}
&= N\left\{1 - \frac{1}{\sqrt{\pi n}\,\sigma}\int^c e^{-\frac{x'^2}{n\sigma^2}} dx'\right\} \\
&\quad + N\left\{1 - \frac{1}{\sqrt{\pi n}\,\sigma}\int^c e^{-\frac{(2a-2c+x')^2}{n\sigma^2}} dx'\right\} \\
&= N\left\{2 - \frac{1}{\sqrt{\pi n}\,\sigma}\left[\int^c e^{-\frac{x'^2}{n\sigma^2}} dx' + \int^{2a-c} e^{-\frac{x'^2}{n\sigma^2}} dx'\right]\right\} \\
&= F_n(c) + F_n(2a - c).
\end{aligned}
$$

Die Werte der Integrale lassen sich wieder aus Tabellen entnehmen (s. Lit.-Verz. Nr. (9) bis (11)).

Man kann also (54) erhalten, indem man zu der Halbebene E eine bezüglich H spiegelbildliche fiktive Halbebene E' mit der gleichen Dichte N annimmt.

Analog erhält man die Verteilung der anderen in § 4 besprochenen Gebiete, indem man immer das zu der jeweiligen Grenze H spiegelbildlich gelegene Gebiet mit der gleichen Dichte N hinzunimmt und die aus dem wirklichen und dem fiktiven Gebiet nach § 4 zu erwartenden Verteilungen addiert. Als Illustration dazu möchte ich noch die Verteilung in einen Halbkreis berechnen.

(c) Eine mit der Dichte N besetzte Halbebene sei wieder durch ein Hindernis H abgeschlossen. Gesucht ist die Verteilung in einen zunächst freien Halbkreis mit dem Radius R über H.



Nach dem n-ten Flug ist also an einem Punkt (x, y) (bzw. in Polarkoordinaten: ρ, ψ) die Dichte:

$$
\begin{aligned}
{}_RF_n(x, y) &= \int_{(A)}\int_{x'} \bar{\Phi}_n(x'-x, y'-y)\, dx'\, dy' \\
&= \int_{(A)}\int \Big\{ \Phi_n(x'-x, y'-y) + \Phi_n(x-x', y-y') \Big\}\, dx'\, dy' \quad \ldots\ldots(55).
\end{aligned}
$$

Das Integral ist über das ausserhalb des Halbkreises und rechts von H liegende Gebiet zu erstrecken. Durch Einführung von Polarkoordinaten:

$$x = \rho \cos \psi,\ y = \rho \sin \psi;\quad x' = r \cos \phi,\ y' = r \sin \phi;$$

wird also:

$$
\begin{aligned}
{}_RF_n(\rho, \psi) = \int_R \int_{-\pi/2}^{+\pi/2} \frac{N}{\pi n \sigma^2} \Big\{ & e^{-\frac{r^2+\rho^2-2r\rho\cos(\phi-\psi)}{n\sigma^2}} \\
& + e^{-\frac{r^2+\rho^2+2r\rho\cos(\phi+\psi)}{n\sigma^2}} \Big\}\, r\, dr\, d\phi \ldots(56),
\end{aligned}
$$

nach (45)

$$
\begin{aligned}
= \frac{N}{\pi n \sigma^2} e^{-\frac{\rho^2}{n\sigma^2}} \int_R e^{-\frac{r^2}{n\sigma^2}} r \Big\{ & \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{2r\rho}{n\sigma^2}\right)^k \int_{-\pi/2}^{+\pi/2} \cos^k(\phi-\psi)\, d\phi \\
& + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{2r\rho}{n\sigma^2}\right)^k \int_{-\pi/2}^{+\pi/2} \cos^k(\phi+\psi)\, d\phi \Big\}\, dr.
\end{aligned}
$$

Nun ist aber:

$$
\begin{aligned}
\int_{-\pi/2}^{+\pi/2} \cos^k(\phi-\psi)\, d\phi &= \int_{-\pi/2-\psi}^{+\pi/2-\psi} \cos^k \phi\, d\phi = \int_{-\pi/2+\psi}^{+\pi/2+\psi} \cos^k \phi\, d\phi \\
&= \int_{-\pi/2}^{+\pi/2} \cos^k(\phi+\psi)\, d\phi \ldots\ldots\ldots\ldots(57),
\end{aligned}
$$

folglich heben sich die in der Klammer $\left\{\ldots\ldots\right\}$ stehenden Glieder mit ungeradem k heraus, und es bleibt, da ausserdem:

$$\int_{-\pi/2}^{+\pi/2} \cos^{2k}(\phi \pm \psi)\, d\phi = \frac{\pi}{2^{2k}}\binom{2k}{k},$$

$$\begin{aligned}
{}_R F_n(\rho, \psi) &= \frac{N}{\pi n \sigma^2} e^{-\frac{\rho^2}{n\sigma^2}} \int_R r e^{-\frac{r^2}{n\sigma^2}}\, 2 \sum_{k=0}^{\infty} \frac{1}{(2k)!}\left(\frac{2r\rho}{n\sigma^2}\right)^{2k} \frac{\pi}{2^{2k}}\binom{2k}{k} dr \\
&= \frac{2N}{n\sigma^2} e^{-\frac{\rho^2}{n\sigma^2}} \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{\rho}{n\sigma^2}\right)^{2k} \int_R e^{-\frac{r^2}{n\sigma^2}} r^{2k+1}\, dr \\
&= N\left\{1 - \frac{2}{n\sigma^2} e^{-\frac{\rho^2}{n\sigma^2}} \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{\rho}{n\sigma^2}\right)^{2k} \int_0^R e^{-\frac{r^2}{n\sigma^2}} r^{2k+1}\, dr\right\} \\
&= N - F_n(\rho) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(58).
\end{aligned}$$

Siehe (47), wobei das dortige ϱ dem jetzigen ρ entspricht. ${}_R F_n(\rho, \psi)$ ist also nur von ρ abhängig und ergibt die gleiche Verteilung, wie wenn die Ausbreitung der Mücken von der vollen Umgebung eines Vollkreises vom Radius R in diesen hinein stattfindet.

Bei den behandelten Verteilungen war bisher von einer Sterblichkeit bzw. Vermehrung der Mücken abgesehen. Ist nun die dadurch bedingte Aenderung der Dichte vom Ort unabhängig und ist der Prozentsatz der nach einem Fluge abgeschiedenen bzw. hinzugekommenen Mücken gleich 100Δ, so erhält man die Verteilung nach dem n-ten Flug durch einfache Multiplikation der Dichte mit $(1-\Delta)^n$. Bedeutend komplizierter wird diese Frage, wenn Δ von Ort zu Ort variiert. In seiner ganzen Allgemeinheit lässt sich dies Problem mathematisch wohl kaum lösen.

6. *Die Verteilung, wenn in einer Halbebene nach jedem Flug ein bestimmter Prozentsatz ausscheidet.*

Zunächst möchte ich eine später zu gebrauchende allgemeine Formel ableiten. Bei der Verteilung von einer Halbebene mit der Dichte N in die andere links davon gelegene war nach $n-1$ Flügen die Dichte an einem Punkt (x, y) gemäss (42), wenn der Nullpunkt auf der Grenzgeraden liegt:

$$F_{n-1}(x) = N\left\{1 - \frac{1}{\sqrt{\pi(n-1)}\,\sigma}\int^{-x} e^{-\frac{x''^2}{(n-1)\sigma^2}} dx''\right\}.$$

Nach dem nächsten Flug ist die Dichte:

$$\begin{aligned}
F_n(x) &= \frac{1}{N}\iint F_{n-1}(x')\,\Phi_1(x-x', y-y')\, dx'\, dy' \\
&= N\left(1 - \frac{1}{\sqrt{\pi}\,\sigma}\right)\int\left\{e^{-\frac{(x-x')^2}{\sigma^2}} \frac{1}{\sqrt{\pi(n-1)}\,\sigma}\int^{-x'} e^{-\frac{x''^2}{(n-1)\sigma^2}} dx''\right\} dx' \quad \ldots\ldots(59),
\end{aligned}$$

das aber ist andererseits nach (42):

$$= N\left\{1 - \frac{1}{\sqrt{\pi n}\,\sigma}\int^{-x} e^{-\frac{x'^2}{n\sigma^2}} dx',\right.$$

also muss gelten:

$$\frac{1}{\sqrt{\pi}\,\sigma}\int\left\{e^{-\frac{(x-x')^2}{\sigma^2}}\frac{1}{\sqrt{\pi(n-1)}\,\sigma}\int^{-x'}e^{-\frac{x''^2}{(n-1)\sigma^2}}dx''\right\}dx'=\frac{1}{\sqrt{\pi n}\,\sigma}\int^{-x}e^{-\frac{x'^2}{n\sigma^2}}dx' \quad \ldots\ldots\ldots\ldots\ldots(60),$$

oder wenn ich setze $\frac{x''^2}{(n-1)\sigma^2}=\frac{\bar{x}''^2}{2}$ und $\left(\frac{x-x'}{\sigma}\right)^2=\frac{\bar{x}'^2}{2}$ und dann wieder $\bar{x}''$ mit x'' und $\bar{x}'$ mit x' bezeichne, wird:

$$\frac{1}{\sqrt{2\pi}}\int\left\{e^{-\frac{x'^2}{2}}\int^{\frac{x'}{\sqrt{n-1}}-\frac{x}{\sigma}\sqrt{\frac{2}{n-1}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x''^2}{2}}dx''\right\}dx'$$

$$=\int^{-\frac{x}{\sigma}\sqrt{\frac{2}{n}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx' \quad \ldots\ldots(61),$$

oder, wenn man setzt $\frac{1}{\sqrt{n+1}}=a$ und $-\frac{x}{\sigma}\sqrt{\frac{2}{n-1}}=b$,

$$\int\left\{\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}\int^{ax'+b}\frac{1}{\sqrt{2\pi}}e^{-\frac{x''^2}{2}}dx''\right\}dx'=\int^{b/\sqrt{1+a^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx' \quad \ldots(62).$$

Weiter folgt daraus, dass gelten muss:

$$\int\left\{\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{x'^2}{2\sigma^2}}\int^{ax'+b}\frac{1}{\sqrt{2\pi}\,\rho}e^{-\frac{x''^2}{2\rho^2}}dx''\right\}dx'=\int^{b/\sqrt{\rho^2+\sigma^2a^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx' \quad (63).$$

(a) Es soll jetzt die Verteilung von N Mücken ausgehend von einem Punkt (a, b) nach dem n-ten Flug bestimmt werden, wenn links der y-Achse der Vernichtungsfaktor $\Delta=\delta=$ konst. und rechts $\Delta=0$ ist, also für

$$x<0,\ \Delta=\delta; \quad x>0,\ \Delta=0.$$

Δ wird im allgemeinen natürlich durch Vermehrung und Absterben bestimmt und ist kleiner beziehungsweise grösser Null, je nachdem ob das erste oder zweite überwiegt.

Nach dem ersten Flug erhalten wir dann folgende Verteilung (nach (33))*:

$$\left.\begin{aligned}\bar{\Phi}_1(x-a, y-b)&=\Phi_1(x-a, y-b) && \text{für } x>0\\ &=(1-\delta)\,\Phi_1(x-a, y-b) && \text{für } x<0\end{aligned}\right\} \ldots\ldots\ldots\ldots(64).$$

Diese unstetige Funktion $\bar{\Phi}_1(x-a, y-b)$ kann man nun beliebig gut approximieren durch

$$\Phi_1(x-a, y-b)\left\{1-\delta\int^{-k_1x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx'\right\} \quad \ldots\ldots\ldots\ldots(65),$$

wenn nur k_1 genügend gross gewählt wird.

Ich setze also:

$$\bar{\Phi}_1(x-a, y-b)=\Phi_1(x-a, y-b)\left\{1-\delta\int^{-k_1x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx'\right\} \quad \ldots(65\text{ bis}).$$

* Dabei ist allerdings angenommen, Vermehrung und Absterben finden immer am Ende eines Fluges statt.

Die Verteilung nach dem zweiten Flug wird dann bei Verwendung der noch abzuleitenden Formel (67):

$$\overline{\Phi}_2(x-a, y-b)$$
$$= \frac{1}{N}\iint \overline{\Phi}_1(x'-a, y'-b)\, \Phi_1(x-x', y-y')\, dx'\, dy' \left\{1-\delta \int^{-k_2 x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x'^2}{2}} dx'\right\}$$
$$= \Phi_2(x-a, y-b)\left\{1-\delta \int^{-(a+x)\sqrt{k_1^2/(4+k_1^2\sigma^2)}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x'^2}{2}} dx'\right\}$$
$$\times \left\{1-\delta \int^{-k_2 x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x'^2}{2}} dx'\right\} \quad \ldots\ldots(66),$$

oder bei $k_1 \to \infty$ wird:

$$\overline{\Phi}_2(x-a, y-b) = \phi_2(x-a, y-b)\left\{1-\delta\left[\int^{-(a+x)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{x'^2}{2}} dx'\right.\right.$$
$$\left.\left. + \int^{-k_2 x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x'^2}{2}} dx'\right] + \delta^2 \int^{-(a+x)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{x'^2}{2}} dx' \int^{-k_2 x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x'^2}{2}} dx'\right\}$$
$$\ldots\ldots(66\,a),$$

dabei ist wieder $k_2 \to \infty$ zu nehmen.

Es gilt nämlich allgemein:

$$\frac{1}{N}\iint\left\{\Phi_n(x'-a, y'-b)\, \Phi_1(x-x', y-y') \int^{-(\alpha x'+\beta)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x''^2}{2}} dx''\right\} dx'\, dy'$$
$$= N \iint\left\{\frac{1}{\pi^2 n^2 \sigma^4} e^{-\frac{(x'-a)^2+(y'-b)^2}{n\sigma^2} - \frac{(x-x')^2+(y-y')^2}{\sigma^2}} \int^{-(\alpha x'+\beta)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x''^2}{2}} dx''\right\} dx'\, dy'$$
$$= \frac{N}{\pi^2 n^2 \sigma^4}\int e^{-\frac{(y'-b)^2+n(y-y')^2}{n\sigma^2}} dy' \iint\left\{e^{-\frac{(x'-a)^2+n(x-x')^2}{n\sigma^2}} \int^{-(\alpha x'+\beta)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x''^2}{2}} dx''\right\} dx'$$
$$= \frac{N}{\pi^2 n^2 \sigma^4}\sqrt{\frac{\pi n}{n+1}}\, e^{-\frac{(y-b)^2+(x-a)^2}{(n+1)\sigma^2}} \int e^{-\frac{\left(x'-\frac{a+nx}{n+1}\right)^2}{\frac{n}{n+1}\sigma^2}} \int^{-(\alpha x'+\beta)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x''^2}{2}} dx''\, dx'$$
$$\ldots\ldots(67),$$

wenn $x' - \frac{a+nx}{n+1} = \zeta$,

$$= \Phi_{n+1}(x-a, y-b)\, \frac{1}{\pi\sigma}\sqrt{\frac{\pi(n+1)}{n}} \int\left\{e^{-\frac{\zeta^2}{2\frac{n}{2(n+1)}\sigma^2}}\right.$$
$$\left.\times \int^{-\left(\zeta-\frac{a+nx}{n+1}\right)\alpha-\beta} \frac{1}{\sqrt{2\pi}} e^{-\frac{x''^2}{2}} dx''\right\} d\zeta,$$

daraus folgt unter Anwendung von (63):

$$= \Phi_{n+1}(x-a, y-b) \int^{-K} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx\,,$$

wobei
$$K = \frac{xn\alpha + a\alpha + (n+1)\beta}{\sqrt{\frac{1}{2}(n+1)\{2(n+1)+n\alpha^2\sigma^2\}}} \quad \ldots\ldots\ldots\ldots\ldots(68).$$

Bei Benutzung dieser Formeln folgt aus (66), (66 a), indem man setzt $n=1, \beta=0$, $\alpha=k_1\to\infty$.

Da im allgemeinen $\delta \ll 1$ sein wird, kann man, ohne einen grossen Fehler zu begehen, Glieder mit δ^2 und höherer Ordnung vernachlässigen.

Mit der Methode der vollständigen Induktion lässt sich dann zeigen, dass:

$$\bar{\Phi}_n(x-a, y-b) = \Phi_n(x-a, y-b)\left\{1-\delta\sum_{k=1}^{n}\Theta({}_n\gamma_k(x))\right\} \quad \ldots\ldots\ldots(69),$$

wobei

$$\Theta({}_n\gamma_k(x)) = \int^{{}_n\gamma_k(x)}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(70),$$

und

$$\left.\begin{aligned} {}_n\gamma_k(x) &= -\frac{kx+(n-k)a}{\sigma\sqrt{\frac{1}{2}nk(n-k)}} && \text{für } k\neq n\\ {}_n\gamma_n(x) &= -Mx, && \text{mit } M\to+\infty \end{aligned}\right\} \quad \ldots\ldots\ldots(71).$$

Setzt man $n=2$, so sieht man, dass (69) übergeht in (66 a), abgesehen von dem Glied mit δ^2. Weiter ist:

$$\begin{aligned}\bar{\Phi}_{n+1}(x-a, y-b) &= \frac{1}{N}\iint\bar{\Phi}_n(x'-a, y'-b)\,\Phi_1(x-x', y-y')\,dx'dy'\\ &\qquad\qquad\qquad\qquad\times\{1-\delta\Theta({}_{n+1}\gamma_{n+1}(x))\}\\ &= \frac{1}{N}\iint\Phi_n(x'-a, y'-b)\,\Phi_1(x-x', y-y')\\ &\qquad\times\left\{1-\delta\sum_{k=1}^{n}\Theta({}_n\gamma_k(x'))\right\}dx'\,dy'\,\{1-\delta\Theta({}_{n+1}\gamma_{n+1}(x))\},\end{aligned}$$

das ist unter Berücksichtigung der Formel (67) bei Integration der einzelnen Summanden:

$$= \Phi_{n+1}(x-a, y-b)\left\{1-\delta\sum_{k=1}^{n+1}\Theta({}_{n+1}\gamma_k(x))\right\},$$

wobei die Grössen in der Klammer $\{\ldots\}$ die in (70), (71) angegebene Bedeutung haben. Also gibt (69) die Verteilung nach dem n-ten Flug unter den oben erwähnten Bedingungen an. Allerdings hat diese Formel nur Gültigkeit wenn $\delta\ll 1$, sodass die Glieder mit δ^2, δ^3, ..., etc. wegen immer vorhandner Ungenauigkeit des σ^2 nicht ins Gewicht fallen.

(b) Es sei jetzt die Halbebene rechts der y-Achse mit einer Dichte N besetzt, während links der y-Achse keine Mücken vorhanden seien. Dann wird nach n Flügen bei der vorhergehenden Annahme über die Vernichtungsintensität die Dichte an einem Punkt (x, y) sein:

$$F_n(x) = \iint_0 \bar{\Phi}_n(x-x', y-y')\,dx'\,dy' \quad \ldots\ldots\ldots\ldots\ldots\ldots(72).$$

Ich setze für $\bar{\Phi}_n(x-x', y-y')$

$$\Phi_n(x-x', y-y')\left\{1-\delta\sum_{k=1}^{n-1}\Theta({}_n\gamma_k(x, x'))\right\} \quad \begin{matrix}(1-\delta) & \text{für } x<0\\ 1 & \text{„ } x>0\end{matrix} \quad \ldots\ldots(73)$$

ein, das stimmt nach (69) bis auf das quadratische Glied, das vernachlässigt werden kann.

Θ und ${}_n\gamma_k$ sind durch (70), (71) definiert mit x' statt dem dortigen a.

Es wird also:

$$\bar{F}_n(x) = \left\{\iint_0 \Phi_n(x-x', y-y')\left[1-\delta\sum_{k=1}^{n-1}\Theta({}_n\gamma_k)\right]dx'\,dy'\right\}\begin{matrix}(1-\delta) & \text{für } x<0,\\ 1 & \text{„}\quad x>0,\end{matrix}$$

nach (42)

$$= \left\{F_n(-x) - \delta\sum_{k=1}^{n-1}\int_0\left[\frac{1}{\sqrt{\pi n}\,\sigma}e^{-\frac{(x-x')^2}{n\sigma^2}}\int^{{}_n\gamma_x(x,x')}\frac{1}{\sqrt{2\pi}}e^{-\frac{x''^2}{2}}dx''\right]dx'\right\}\begin{matrix}(1-\delta)\\ 1\end{matrix}$$

$$= \left\{F_n(-x) - \delta\sum_{k=1}^{n-1}\left[P_k(o) + \frac{P_k'(o)}{1!}x + \frac{P_k''(o)}{2!}x^2 + \dots\right]\right\}\begin{matrix}(1-\delta) & \text{für } x<0\\ 1 & \text{„}\quad x>0\end{matrix} \quad \dots\dots(74),$$

dabei ist:

$$P_k(x) = \int_0\left\{\frac{1}{\sqrt{\pi n}\,\sigma}e^{-\frac{(x-x')^2}{n\sigma^2}}\int^{{}_n\gamma_k(x,x')}\frac{1}{\sqrt{2\pi}}e^{-\frac{x''^2}{2}}dx''\right\}dx' \quad \dots\dots\dots\dots\dots\dots(75),$$

$$P_k'(x) = \frac{1}{\sqrt{\pi n}\,\sigma}e^{-\frac{x^2}{n\sigma^2}}\int^{K(x)}\frac{1}{\sqrt{2\pi}}e^{-\frac{x''^2}{2}}dx'' - \frac{1}{\sigma\sqrt{2(n-k)}}\phi\left(\frac{x}{\sigma}\sqrt{\frac{2}{n-k}}\right) \dots(76),$$

wobei $$K(x) = -\frac{x}{\sigma}\sqrt{\frac{2k}{n(n-k)}};\quad \phi(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}} \quad \dots\dots\dots\dots(77).$$

Ich setze dann: $$P'(x) = A(x) - B(x),$$

also $$P^{(s)}(x) = A^{(s-1)}(x) - B^{(s-1)}(x) \quad \dots\dots\dots\dots\dots\dots\dots\dots(78).$$

Die $A^{(s-1)}(x)$ bzw. $B^{(s-1)}(x)$ entstehen durch Differentiation der einzelnen Summanden, aus denen $P'(x)$ besteht. Es wird also:

$$A'(x) = -\frac{2x}{\sigma^3\sqrt{\pi n^3}}e^{-\frac{x^2}{n\sigma^2}}\int^{K(x)}\frac{1}{\sqrt{2\pi}}e^{-\frac{x'^2}{2}}dx' - \frac{1}{\sqrt{\pi n}\,\sigma^2}\sqrt{\frac{2k}{n-k}}\,\phi\left(\frac{x}{\sigma}\sqrt{\frac{2}{n-k}}\right) \quad \dots\dots(79),$$

und allgemein lässt sich dann durch vollständige Induktion zeigen, dass für $s \geqq 2$:

$$A^{(s-1)}(x) = -\frac{1}{n\sigma^2} \times\left\{2x A^{(s-2)}(x) + 2(s-2)A^{(s-3)}(x) + \frac{\sqrt{k}}{\sigma^{s-2}\sqrt{\pi}}\sqrt{\left(\frac{2}{n-k}\right)^{s-1}}\,\phi^{(s-2)}\left(\frac{x}{\sigma}\sqrt{\frac{2}{n-k}}\right)\right\} \quad \dots\dots(80),$$

und ebenso $$B^{(s-1)}(x) = \frac{1}{2\sigma^s}\sqrt{\left(\frac{2}{n-k}\right)^s}\,\phi^{(s-1)}\left(\frac{x}{\sigma}\sqrt{\frac{n}{n-k}}\right),$$

was man durch einfaches Differentiieren verifizieren kann.

Die Koeffizienten der Potenzreihe in (74) erhält man dann, indem man in (75) bis (80) $x=0$ setzt. Die Ableitungen von $P_k(o)$ ergeben sich dabei ohne weiteres,

während $P_k(o)$ aus (75) noch durch eine kleine Rechnung vereinfacht werden muss. Es ist nämlich, wenn ich setze:

$$\frac{x'^2}{n\sigma^2} = \frac{\zeta^2}{2} \quad \text{und} \quad \sqrt{\frac{n-k}{n}} = \alpha \quad \text{.......................(81)},$$

$$P_k(o) = \int_0 \left\{ \frac{1}{\sqrt{2\pi}} e^{-\frac{\zeta^2}{2}} \int^{-\zeta\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \right\} d\zeta = P(\alpha) \quad \text{........(75a)}.$$

Ich bilde dann:

$$\frac{dP(\alpha)}{d\alpha} = \int_0 \frac{1}{\sqrt{2\pi}} e^{-\frac{\zeta^2}{2}} \left\{ -\zeta \frac{1}{\sqrt{2\pi}} e^{-\frac{\alpha^2\zeta^2}{2}} \right\} d\zeta = \int_0 \frac{-1}{2\pi} e^{-\frac{\zeta^2(1+\alpha^2)}{2}} \zeta d\zeta$$

$$= \frac{1}{2\pi(1+\alpha^2)} e^{-\frac{\zeta^2(1+\alpha^2)}{2}} \Big|_0^\infty = -\frac{1}{2\pi(1+\alpha^2)},$$

also
$$P(\alpha) = \frac{1}{2\pi} \int^{*} \frac{-1}{1+\alpha^2} d\alpha = \frac{1}{2\pi} \operatorname{arc\,cotg} \alpha + C.$$

Nun ist aber nach (75a) $P(\alpha)$ für $\alpha = 0$ gleich $\frac{1}{4}$. Also muss $C = 0$ sein, wenn ich den arc cotg auf die Periode 0 bis π beschränke. Setze ich dann den Wert für α (nach (81)) ein, so wird:

$$P_k(o) = \frac{1}{2\pi} \operatorname{arc\,cotg} \sqrt{\frac{n-k}{k}} \qquad \text{(Periode 0 bis } \pi\text{)},$$

weiter nach (76):
$$P'(o) = -\frac{1}{2\sigma\sqrt{\pi}} \left\{ \frac{1}{\sqrt{n-k}} - \frac{1}{\sqrt{n}} \right\}.$$

Allgemein erhält man für die Nullstellen der Ableitungen mit ungeradem Index:

$$P_k^{(2s+1)}(o) = \frac{(-2)^{s-1}\, 1.3.5.7 \ldots (2s-1)}{\sqrt{\pi}\, \sigma^{2s+1}} \left\{ \frac{1}{(n-k)^s \sqrt{n-k}} - \frac{1}{n^s \sqrt{n}} \right\}.$$

Für die Nullstellen der Ableitungen mit geradem Index ist mir keine allgemeine Darstellung geglückt. Es ist jedoch:

$$P_k''(o) = \frac{-1}{\pi n \sigma^2} \sqrt{\frac{k}{n-k}},$$

$$P_k^{\mathrm{iv}}(o) = \frac{2(3n-2k)}{\pi n^2 (n-k) \sigma^4} \sqrt{\frac{k}{n-k}},$$

$$P_k^{\mathrm{vi}}(o) = \frac{-4(15n^2 - 20nk + 8k^2)}{\pi n^3 (n-k)^2 \sigma^6} \sqrt{\frac{k}{n-k}},$$

$$P_k^{\mathrm{viii}}(o) = \frac{24(35n^3 - 70n^2k + 56nk^2 - 16k^3)}{\pi n^4 (n-k)^3 \sigma^8} \sqrt{\frac{k}{n-k}},$$

$$P_k^{\mathrm{x}}(o) = \frac{-48(315n^4 - 840n^3k + 1008n^2k^2 - 576nk^3 + 128k^4)}{\pi n^5 (n-k)^4 \sigma^{10}} \sqrt{\frac{k}{n-k}}.$$

* Unbestimmt integriert.

Die praktische Auswertung dieser Resultate wird vor allem von der Kenntnis der Grösse σ abhängen. Um dies σ zu bestimmen, wird man wohl von N Mücken einer bestimmten Art A experimentell eine Verteilung nach einem oder mehreren Flügen, ausgehend von einem Zentrum Q, beobachten müssen entsprechend Formel (33). Nachdem dann einmal von einer Mückenart A mit einer Flugzeit t_1 und einer linearen Fluggeschwindigkeit v_1 die Streuung des Abstandes von einem Punkt bekannt ist, *kann man mit einiger Berechtigung* die *Streuung einer anderen* Mückenart B mit einer Flugzeit t_2 und einer linearen Fluggeschwindigkeit v_2 berechnen, wie auf Seite 204—205 näher ausgeführt wurde.

Im nächsten Paragraphen soll nun an Hand einiger Beispiele die ungefähre Grösse des σ der *Anopheles quadrimaculatus* bestimmt werden.

7. *Bestimmung der "Streuung" für* Anopheles quadrimaculatus *nach Beobachtungen von M. H. Barber und T. B. Hayne.*

Es muss hier allerdings von vornherein betont werden, dass es sich bei den folgenden Berechnungen nur um rohe Näherungswerte handeln kann, da erstens die Zahl der zur Verfügung stehenden Beobachtungen zu klein ist und zweitens die auszuwertenden Versuche nicht eigentlich zu dem hier erstrebten Zweck angestellt wurden. Prinzipiell hätte man an verschiedenen Punkten und Tagen (wenn pro Tag ein Flug stattfindet) die Dichte der Mücken zu beobachten, wenn am Anfang des ersten Tages an einer bestimmten Stelle eine möglichst grosse Zahl gefärbter Mücken in Freiheit gesetzt wurden.

Aus den so gefundenen Daten wäre dann mit Hilfe der Ausgleichsrechnung aus Formel (33) das σ zu berechnen.

In der Praxis wird diese Bestimmung der Dichte an den verschiedenen Punkten nun ziemlich schwierig sein, da man im allgemeinen wohl kaum eine bestimmte Fläche ganz genau absuchen kann und da wegen Häufungen der Mückenanzahl in Stallungen, Wohnungen usw. zur Berechnung der Dichte an einem Punkt eine etwas andere als die abgesuchte Fläche anzusetzen ist.

Ich führe deshalb als neue unbekannte Grösse die Fläche F ein. Unter der Voraussetzung, dass F verhältnismässig klein gegen σ^2 ist, ist dann nach (33) die in der Entfernung r nach dem n-ten Flug auf F befindlich Anzahl Mücken:

$$Z_n(r) = (1-\delta)^n F\phi_n(r) = \frac{(1-\delta)^n FN}{\pi n\sigma^2} e^{-\frac{r^2}{n\sigma^2}} \dots\dots\dots\dots (82).$$

Es sei also der Vernichtungsfaktor δ örtlich und zeitlich konstant. Er gibt den nach einem Fluge ausgeschiedenen Bruchteil von der am Anfang des Fluges vorhanden gewesenen Gesamtzahl der Mücken an.

Bei den Versuchen hat man also an möglichst vielen Stellen, möglichst gleich grosse Flächen F gleich gut abzusuchen (in dem F ist mit berücksichtigt, dass von den wirklich vorhandenen Mücken nur ein Teil gefunden wird; insofern ist also F praktisch nie die wirklich abgesuchte Fläche). Die Unbekannten F, σ^2 und eventuell

δ sind dann so zu bestimmen, dass sich die Zahlen der an den einzelnen Stellen tatsächlich gefundenen Mücken möglichst gut den $Z_n(r)$ der Formel (82) anpassen. Es ist nun üblich, die Unbekannten so zu bestimmen, dass nicht die beobachteten Zahlen "$B_n(r)$" durch $Z_n(r)$, sondern dass $\log B_n(r)$ durch $\log Z_n(r)$ möglichst gut approximiert wird, da dadurch ein bedeutend gleichmässigerer Einfluss aller beobachteten Daten erreicht wird. Nach (82) ist nun:

$$\log Z_n(r) = \log \frac{N}{n\pi} + n \log(1-\delta) + \log \frac{F}{\sigma^2} - \frac{r^2 \log e}{n} \frac{1}{\sigma^2} \quad \ldots\ldots\ldots(83).$$

Ich setze dann:

$$\left.\begin{aligned} \log(1-\delta) &= x \\ \log \frac{F}{\sigma^2} &= y \\ \frac{1}{\sigma^2} &= z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(84),$$

und suche nach der von Gauss herrührenden Methode der kleinsten Quadrate (s. Lit.-Verz. Nr. (10) bis (14)), die x, y, z so zu bestimmen, dass

$$\Sigma(\log B_n(r) - \log Z_n(r))^2$$

ein Minimum wird.

Von Barber und Hayne sind nun zwei Versuche ausgeführt worden, die ungefähr den hier verlangten Bedingungen entsprechen (s. Lit.-Verz. Nr. (15)).

TABELLE I.

$n \backslash r$	0	14·4	15
1	50 (1·699)	1 (0)	1 (0)
2	24 (1·380)	1 (0)	1 (0)
3	10 (1)	1 (0)	1 (0)
4	24 (1·380)	1 (0)	1 (0)
5	4 (0·602)	—	—

Die Zahlen in den Klammern sind die Briggs'schen Logarithmen der darüber stehenden Zahlen. r ist in 100 Fuss (= 30·48 m.) gemessen.

Tabelle I gibt von 2500 gefärbten ausgesetzten Mücken die Verteilung der in verschiedenen Entfernungen und an verschiedenen Tagen Wiedergefundenen an. Die Zahlen am Kopf der Tabelle sind die Entfernungen in 100 Fuss, während links die Anzahl der seit der Aussetzung verstrichenen Tage steht.

Setzt man nun für die x, y, z vom (84) in Formel (83) die Näherungswerte:

$$x_0 = 0, \quad y_0 = -2, \quad z_0 = 0 \quad \ldots\ldots (85),$$

so erhält man 13 sogenannte Fehlergleichungen:

$$\left.\begin{aligned} v_\nu &= n\xi + 1\eta - \frac{r^2 \log e}{n} \zeta - \left(\log B_n(r) - \log \frac{N}{n\pi} + 2\right) \\ &= a_\nu \xi + b_\nu \eta + \quad c_\nu \zeta - \quad l_\nu \end{aligned}\right\} \quad \ldots\ldots (86),$$

in die die aus Tabelle I zu entnehmenden Werte einzusetzen sind, also:

$$\begin{aligned} v_1 &= 1\xi + 1\eta - 0 - 0{\cdot}798, \\ v_2 &= 2\xi + 1\eta - 0 - 0{\cdot}780, \\ &\ldots\ldots\ldots \\ v_6 &= 1\xi + 1\eta - 90 + 0{\cdot}901, \\ v_7 &= 2\xi + 1\eta - 45 + 0{\cdot}600, \\ &\ldots\ldots\ldots, \text{ usw.} \end{aligned}$$

Die ξ, η, ζ, das sind die an x, y, z anzubringenden Verbesserungen, sind nun so zu bestimmen, dass $[vv]$* ein Minimum wird.

Dazu bildet man die Summen der Produkte der in den Fehlergleichungen stehenden Koeffizienten a_ν, b_ν, c_ν und l_ν, aus denen man dann die sogenannten Normalgleichungen erhält:

$$\left.\begin{aligned} [aa]\,\xi + [ab]\,\eta + [ac]\,\zeta &= [al] \\ [ba]\,\xi + [bb]\,\eta + [bc]\,\zeta &= [bl] \\ [ca]\,\xi + [cb]\,\eta + [cc]\,\zeta &= [cl] \end{aligned}\right\} \quad \ldots\ldots (87).$$

Die Lösung dieses Gleichungssystems gibt die zu den Näherungswerten hinzuzufügenden Verbesserungen an.

Die Normalgleichungen unseres Beispiels lauten:

$$\begin{aligned} 115\ \xi + 35\ \eta - 750{\cdot}4\,\zeta &= 1{\cdot}272, \\ 35\ \xi + 13\ \eta - 390{\cdot}8\,\zeta &= -0{\cdot}813, \\ -750{\cdot}4\,\xi - 390{\cdot}8\,\eta + 25070\ \zeta &= 265{\cdot}9. \end{aligned}$$

Die Lösungen sind:

$$\xi = -0{\cdot}123; \quad \eta = 0{\cdot}89; \quad \zeta = 0{\cdot}021.$$

Also wird nach (85):

$$x = -0{\cdot}123; \quad y = -1{\cdot}11; \quad z = 0{\cdot}021.$$

Für die mittleren Fehler ergeben sich die Werte:

$$m_x = \pm\,0{\cdot}088; \quad m_y = \pm\,0{\cdot}32; \quad m_z = \pm\,0{\cdot}0035.$$

* Es bedeutet: $[aa] = \sum_\nu a_\nu a_\nu$, $[ab] = \sum_\nu a_\nu b_\nu$, usw.

Einzelheiten bezüglich der Ableitung der Theorie der Methode der kleinsten Quadrate sind u.a. in den auf Seite 226 angegebenen Werken zu finden.

Nach (84) erhält man dann die gesuchten Grössen:

$$1-\delta = 0{\cdot}75 \pm 0{\cdot}15\,{}^{*}; \quad \frac{F}{\sigma^2} = 0{\cdot}078 \pm 0{\cdot}057;$$

$$\sigma = (6{\cdot}9 \pm 0{\cdot}6)\,100 \text{ Fuss} = (210 \pm 20) \text{ m.}$$

Wie nach der Ausführung des Versuches zu erwarten, ist die Ungenauigkeit von F am grössten.

Der Wert für $1-\delta$ stimmt ausserordentlich gut mit auf ganz andere Weise angestellten Berechnungen von Herrn Prof. Martini überein (s. Lit.-Verz. Nr. (16)).

Aus σ folgt nach (31):

$$l = \frac{\sigma}{2}\sqrt{\pi} = 186 \text{ m.}$$

Prof. Pearson (s. Lit.-Verz. Nr. (3), S. 36) schätzte nun die Grösse von l zwischen 100 und 500 Yard (91·4 bis 457·2 m.) in guter Übereinstimmung mit dem hier gefundenen Wert.

TABELLE II.

$n \backslash r$	0		14·4		15	
	Beob.	Theor.				
1	50	46·5	1	0·6	1	0·4
2	24	17·5	1	2·0	1	1·6
3	10	8·8	1	2·1	1	1·8
4	(24	4·9)	1	1·7	1	1·5
5	4	2·9	—		—	

Tabelle II gibt eine Gegenüberstellung der beobachteten (links in den Spalten) mit den aus (82) nach Einsetzung der oben gefundenen Zahlen berechneten Werten.

Ein Mass für die Güte der Anpassung ist die von Pearson stammende χ^2 Methode (s. Lit.-Verz. Nr. (18), S. 157—75, und Nr. (9), S. xxxi). Man hat zu diesem Zweck zu bilden:

$$\chi^2 = \Sigma \frac{(\text{beob. Wert} - \text{theor. Wert})^2}{\text{theor. Wert}}.$$

* Man erhält die mittleren Fehler aus denen für x, y, z mit Hilfe des Gauss'schen Fehlerfortpflanzungsgesetzes:

$$m_f = \pm\sqrt{\left(m_x \frac{\partial f}{\partial x}\right)^2 + \left(m_y \frac{\partial f}{\partial y}\right)^2 + \dots}$$

m_f ist der mittlere Fehler einer Funktion $f(x, y, \dots)$, wenn $m_x, m_y, \dots$ die voneinander unabhängigen mittleren Fehler von $x, y, \dots$ sind. Also z.B. wenn

$$f(x, y, \dots) = 1 - \delta = 10^x$$

ist, wird:

$$m_f = \pm 0{\cdot}088 \frac{\log_e 10}{10^{0{\cdot}124}} = \pm 0{\cdot}15.$$

(Die Zahlenwerte sind aus obigem Versuch genommen.)

Für jede betrachtete Anzahl von Klassen entspricht dann den verschiedenen Werten von χ^2 ein P, d.i. die Wahrscheinlichkeit dafür, dass eine durch Zufall entstehende Abweichung der beobachteten von der theoretischen Verteilung ebenso gross oder grösser ist als die der untersuchten Verteilung. Die zu den verschiedenen χ^2 gehörigen Werte P sind tabelliert (s. Lit.-Verz. Nr. (9), S. 26, und Nr. (19)).

In unserem Beispiel erhält man bei ausser acht lassen des ganz aus der Reihe fallenden Wertes für $n=4$, $r=0$ für χ^2 den Wert 6·54, dem entspricht:

$$P = 0{\cdot}8.$$

Die Anpassung ist also als überaus gut zu bezeichnen. Man nennt nämlich im allgemeinen eine Anpassung noch gut, solange $p > 0{\cdot}1$ (näheres hierüber siehe Lit.-Verz. Nr. (20), S. 104—8).

Der zweite Versuch von Barber und Hayne.

TABELLE III.

n \ r	0	1	2	5	14·4
1	277 (2·442)	1 (0)	1 (0)	1 (0)	1 (0)
2	29 (1·462)	4 (0·602)	2 (0·301)	—	—
3	4 (0·602)	—	—	—	—
4	2 (0·301	1 (0)	—	—	—
6	1 (0)	—	—	—	—

Tabelle III gibt diesmal die Verteilung von 2400 Mücken. Sonst ist die Bedeutung der Zahlen die gleiche wie die der Tabelle I des ersten Versuches (S. 220). Wie man ohne weiteres sieht, sind die wiedergefundenen Mücken diesmal bedeutend regelloser verteilt, und es ist anzunehmen, dass durch das Absuchen oder sonstwie systematische Fehler in bezug auf die Anzahl der an den einzelnen Stellen wirklich vorhanden gewesenen Mücken vorgekommen sind. Behandelt man nun die Daten dieses zweiten Versuches genau wie die des ersten, so erhält man:

$$1-\delta = 1{\cdot}024\,;\quad \frac{F}{\sigma^2} = 0{\cdot}009\,;\quad \sigma = 9{\cdot}8\ 100\ \text{Fuss}.$$

Der Wert für $1-\delta$ ist jedoch unmöglich, denn auf jeden Fall muss $1-\delta<1$ sein. Ich habe deshalb bei der Auswertung des zweiten Versuches die Grösse $1-\delta$ als bekannt gleich 0·7 angesetzt, welchen Wert man auf Grund des vorigen Versuches und der erwähnten Berechnungen von Prof. Ma tini wohl als einigermassen richtig ansehen kann.

Setzt man also in (82) $1-\delta=0{\cdot}70,$
und führt im übrigen die analoge Rechnung wie beim ersten Versuch mit den entsprechenden Zahlen der Tabelle III durch, so erhält man als Normalgleichungen:

$$12\ \eta-\ 104{\cdot}2\zeta=\ 2{\cdot}812,$$
$$104{\cdot}2\eta-8222\ \ \zeta=74{\cdot}8,$$

mit den Lösungen: $\eta=0{\cdot}35;\qquad \zeta=0{\cdot}0135.$

Also wegen (85): $y=-1{\cdot}65;\quad z=0{\cdot}0135.$

Die ebenfalls aus den Koeffizienten der Normalgleichungen zu berechnenden mittleren Fehler sind:

$$m_y=\pm 0{\cdot}24;\quad m_z=\pm 0{\cdot}0092.$$

Wegen (84) wird dann endlich:

$$\frac{F}{\sigma^2}=0{\cdot}0224\pm 0{\cdot}0124;\quad \sigma=(8{\cdot}6\pm 2{\cdot}9)\,100\,\text{Fuss}=(260\pm 90)\ \text{m}.$$

Wie man sieht ist mit den Resultaten des ersten Versuches eine weitgehende Uebereinstimmung innerhalb des Bereiches der mittleren Fehler vorhanden.

TABELLE IV.

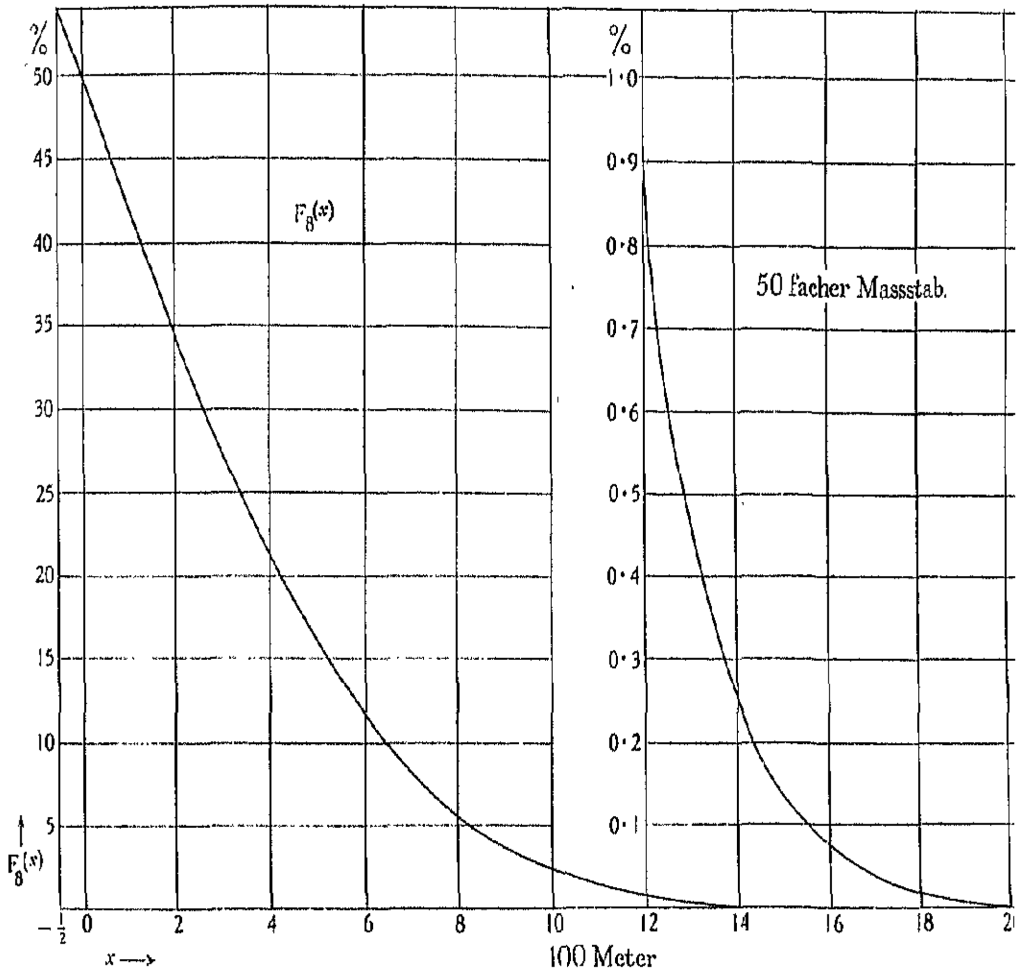
n \ r	0		1		2		5		14·4	
1	277	12·0	1	11·8	1	11·3	1	8·5	1	0·75
2	20	4·2	4	4·2	2	4·1	—		—	
3	4	2·0	—		—		—		—	
4	2	1·0	1	1·9	—		—		—	
6	1	0·3	—		—		—		—	

Diese Tabelle gibt wieder eine Gegenüberstellung der theoretischen (rechts in den Spalten) mit den beobachteten (links in den Spalten) Zahlen. Man sieht sofort, dass im ganzen genommen eine sehr schlechte Anpassung der theoretischen an die beobachteten Daten vorliegt. Ein Vergleich mit Hilfe der erwähnten χ^2 Methode ergibt für P einen weit kleineren Wert als 0·0001. Die 5 stärksten Abweichungen sind nun prozentual im Verhältnis zu den übrigen derartig gross, dass für sie ein systematischer Fehler anzunehmen ist. Man ist deshalb in gewissem Sinne berechtigt, diese bei Beurteilung der Anpassungsgüte auszuschalten. Für die dann übrigbleibenden 7 Beobachtungen erhält man:

$$\chi^2=6{\cdot}2\quad \text{das gibt:}\ P=0{\cdot}4,$$

was immerhin noch sehr gut genannt werden muss.

Man kann also mit einigem Recht auf Grund dieser beiden Versuche für σ einen Wert von etwa 250 m. ansetzen und damit die in dieser Arbeit behandelten Verteilungen für die *Anopheles quadrimaculatus* zahlenmässig berechnen. In folgender Skizze habe ich z.B. die Verteilung von einer Halbebene mit der Dichte 100 in die andere nach 8 Flügen dargestellt.



Die Abzissen sind die Abstände von der Grenzgeraden, während die Ordinaten die in den verschiedenen Entfernungen vorhandenen Dichten, d.h. Anzahl pro Flächeneinheit, angeben. Die Längeneinheit sei 100 m. (Flächeneinheit also 10,000 m.²). Vermehrung und Absterben sollen sich aufheben, also $\delta = 0$.

Dann wird die Dichte im Abstand $x \times 100$ m. von der Grenzlinie nach 8 Flügen zufolge (42):

$$F_8(x) = 1 - \frac{1}{\sqrt{\pi n}\,\sigma} \int^{x} e^{-\frac{x'^2}{n\sigma^2}} dx' \qquad (\text{mit } n = 8,\ \sigma = 2{\cdot}5)$$

$$= 1 - \frac{1}{\sqrt{2\pi}} \int^{x/5} e^{-\frac{x'^2}{2}} dx'.$$

Die Werte dieses Integrals habe ich aus den Pearson'schen Tabellen entnommen (s. Lit.-Verz. Nr. (9), S. 2 bis 9).

$\frac{1}{\sqrt{2\pi}} \int^{x} e^{-\frac{x'^2}{2}} dx'$ ist dort mit $\frac{1}{2}(1-\alpha)$ bezeichnet.

Der zweite Kurvenzug ist die Verlängerung des ersten bei 50-fach vergrössertem Massstab.

Herrn Prof. Dr P. Riebesell sowie Herrn Prof. Dr E. Martini möchte ich auch noch an dieser Stelle meinen besonderen Dank aussprechen für das rege Interesse, dass sie an der Fertigstellung dieser Arbeit stets bewiesen haben.

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THOUGHTS SUGGESTED BY THE PAPERS OF MESSRS WELCH AND KOŁODZIEJCZYK (*Biometrika*, Vol. XXVII. pp. 145—190).

By KARL PEARSON.

THESE authors take populations $\Pi_1, \Pi_2 \ldots \Pi_t \ldots \Pi_k$ and suppose that in any one of these k populations, the distribution of the y's for a given x is normal and homoscedastic. Accordingly the frequency surfaces must be of the form

$$z_t = \phi_t(x)\, e^{-\frac{1}{2\bar{\sigma}^2_{a,t}}(y-\bar{\alpha}_t-\bar{\beta}_t x)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

where $\bar{\sigma}_{a,t}$ is the array standard deviation, and $\bar{\alpha}_t$ and $\bar{\beta}_t$ the constants in the population Π_t, and nothing is stated about the form of $\phi_t(x)$, which *à priori* may be considered continuous or discontinuous. For the truth of the following results it does not matter whether we take a summation or integrate with regard to x between arbitrary limits. But the reader must bear in mind that we are summing for the parent population and *not* for any sample taken from it. We shall use $m_{1,t}$ and $m_{2,t}$ for the x and y means of the population, and $\bar{\sigma}_{1,t}$ and $\bar{\sigma}_{2,t}$ for their standard deviations, $\bar{\rho}_t$ will be their correlation coefficient. For a sample from this population, we shall use $\bar{x}_t$, $\bar{y}_t$ for the means $\sigma_{1,t}$, $\sigma_{2,t}$ for the standard deviations and r_t for the correlation coefficient, and $\sigma_{a,t}$ for an array standard deviation of the y's for a given x_t.

If $\tilde{y}_{x,t}$ be the mean of the x array of y's the form of (i) shows us at once that

$$\tilde{y}_{x,t} = \bar{\alpha}_t + \bar{\beta}_t x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

and there is accordingly a straight regression line for y on x.

Summing for all values of x, it follows whatever be the form of $\phi_t(x)$ that

$$m_{2,t} = \bar{\alpha}_t + \bar{\beta}_t m_{1,t} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii}^{bis}\text{)}.$$

We will now show that for any surface given by (i)

$$\bar{\beta}_t = \bar{\rho}_t \bar{\sigma}_{2,t}/\bar{\sigma}_{1,t}.$$

We have, if N_t be the size of the population Π_t,

$$S_x x\phi_t(x)\int_{-\infty}^{+\infty} e^{-\frac{1}{2\bar{\sigma}^2_{a,t}}(y-\bar{\alpha}-\bar{\beta}_t x)^2}\, y\, dy = N_t(\bar{\rho}_t\bar{\sigma}_{1,t}\bar{\sigma}_{2,t} + m_{2,t}m_{1,t})$$

$$= S_x x\phi_t(x)\int_{-\infty}^{+\infty}(y-\bar{\alpha}_t-\bar{\beta}_t x+\bar{\alpha}_t+\bar{\beta}_t x)\, e^{-\frac{1}{2\bar{\sigma}^2_{a,t}}(y-\bar{\alpha}_t-\bar{\beta}_t x)^2}\, dy$$

$$= S_x x\phi_t(x)\,[0+\bar{\alpha}_t+\bar{\beta}_t x]\sqrt{2\pi}\,\bar{\sigma}_{a,t}$$

$$= [\bar{\alpha}_t m_{1,t} + \bar{\beta}_t(\bar{\sigma}^2_{1,t} + m^2_{1,t})]\, S_x\phi_t(x)\sqrt{2\pi}\,\bar{\sigma}_{a,t}.$$

But $S_x \phi_t(x) \sqrt{2\pi}\,\bar{\sigma}_{a,t} = N_t$ and therefore by (ii)

$$N_t(\bar{\rho}_t \bar{\sigma}_{1,t} \bar{\sigma}_{2,t} + m_{1,t} m_{2,t}) = N_t(\bar{\beta}_t \bar{\sigma}^2_{1,t} + m_{1,t} m_{2,t}),$$

or

$$\bar{\beta}_t = \bar{\rho}_t \bar{\sigma}_{2,t} / \sigma_{1,t} \quad \text{.................................(iii)},$$

whatever be the function $\phi_t(x)$.

We may now proceed further and ask what is the form of $\sigma^2_{a,t}$ for a surface like (i)?

We will find $\sigma^2_{2,t}$.

$$\begin{aligned} N_t(\sigma^2_{2,t} + m^2_{2,t}) &= S_x \phi_t(x) \sqrt{2\pi}\,\bar{\sigma}_{a,t}(\sigma^2_{2,t} + m^2_{2,t}) \\ &= S_x \phi_t(x) \int_{-\infty}^{+\infty} (y - \bar{a}_t - \bar{\beta}_t x + \bar{a}_t + \bar{\beta}_t x)^2 e^{-\frac{1}{2\bar{\sigma}_{a,t}}(y - a_t - \beta_t x)^2} dy \\ &= S_x \phi_2(x) [\bar{\sigma}^2_{a,t} \sqrt{2\pi}\,\bar{\sigma}_{a,t} + 0 + (\bar{a}_t^2 + 2\bar{a}_t \beta_t x + \bar{\beta}_t x^2) \sqrt{2\pi}\,\sigma_{a,t}]. \end{aligned}$$

Thus $\quad N_t(\sigma^2_{2,t} + m^2_{2,t}) = N_t[\bar{\sigma}^2_{a,t} + \bar{a}_t^2 + 2\bar{a}_t \bar{\beta}_t m_{1,t} + \bar{\beta}_t^2 (\bar{\sigma}^2_{1,t} + m^2_{1,t})]$.

Hence by (ii) and (iii)

$$\sigma^2_{2,t} + m^2_{2,t} = \bar{\sigma}^2_{a,t} + (\bar{a}_t + \bar{\beta}_t m_{1,t})^2 + \bar{\beta}_t^2 \bar{\sigma}^2_{1,t},$$

$$\sigma^2_{2,t}(1 - \bar{\rho}_t^2) = \bar{\sigma}^2_{a,t} \quad \text{...........................(iv)},$$

which is exactly the form $\bar{\sigma}^2_{a,t}$ would take for a normal surface, and this whatever be the form of $\phi_t(x)$.

Now if an additional hypothesis be made that $\bar{\sigma}_{a,t}$ is to be the same for all the k populations $\Pi_1, \Pi_2 \ldots \Pi_t \ldots \Pi_k$, then for those populations we must have

$$\bar{\sigma}^2_{2,t} \propto \frac{1}{1 - \bar{\rho}_t^2} \quad \text{....................................(v)},$$

which signifies that for these k populations the variance of y in each of them varies inversely as unity minus the square of its correlation coefficient. It would seem difficult to discover any series of populations for which such a relation would hold. Personally I am not able to think of any series of populations for which the variance and the correlation would obey such an artificial, not to say fantastic, relationship except:

(a) a series of populations with the same variance for y and the same correlation coefficient, i.e.

$$\bar{\sigma}_{2,t} = \bar{\sigma}'_{2,t'} \text{ and } \bar{\rho}_t = \bar{\rho}_{t'} \text{ for all values of } t \text{ and } t' \quad \text{............(vi)},$$

or, (b) a series of populations taken from different parts of one and the same population. This would involve not only (vi), but $\bar{a}_t$ and $\bar{\beta}_t$ being the same, to say nothing of $m_{1,t}$ and $m_{2,t}$, as well as

$$\bar{\sigma}_{1,t} = \bar{\sigma}_{1,t'} \text{ for all values of } t \text{ and } t' \quad \text{.....................(vii)}.$$

Thus case (b) involves far more stringent hypotheses than (a).

But if we accept (a) and then superpose upon it the additional hypothesis that the regression coefficient of these populations, namely $\bar{\beta}_t$, is to be constant, we

have fixed $\bar{\rho}_t\bar{\sigma}_{2,t}/\bar{\sigma}_{1,t}$ and accordingly $\bar{\sigma}_{1,t}$ must be considered constant in this case. (*a*) therefore connotes that the three constants $\bar{\rho}_t$, $\bar{\sigma}_{1,t}$, $\bar{\sigma}_{2,t}$ are *fixed for all k surfaces*, but the means may be different, which is not possible in (*b*).

But we can now go further. Clearly $\bar{\sigma}_{1,t}^2(1-\bar{\rho}_t^2)$ will be constant for all k populations and accordingly the arrays of x on y will be homoscedastic. This does not refer to the samples taken from those populations which may not be random at all, but to the populations $\Pi_1, \Pi_2 \dots \Pi_t \dots \Pi_k$ themselves.

Now the most general equation to a surface having homoscedasticity both ways* is

$$z = z_0 e^{-\left(\frac{\gamma}{\lambda_1\lambda_2}e^{\lambda_1 x+\lambda_2 y} - \frac{\gamma}{\lambda_1^2 m_1}e^{\lambda_1 x} - \frac{\gamma}{\lambda_2^2 m_2}e^{\lambda_2 y} - c_1'' x - c_2'' z\right)} \quad \dots\dots\dots\text{(viii)},$$

where γ, m_1, m_2, λ_1, λ_2, c_1'' and c_2'' are arbitrary constants. This has for its regression curves

$$\tilde{y}_x = c_2 - \frac{1}{\lambda_2}\log\left(1 - \frac{\lambda_2}{\lambda_1}m_2 e^{\lambda_1 x}\right) \quad \dots\dots\dots\dots\dots\dots\dots\dots\text{(ix)}$$

and

$$\tilde{x}_y = c_1 - \frac{1}{\lambda_1}\log\left(1 - \frac{\lambda_1}{\lambda_2}m_1 e^{\lambda_2 y}\right) \quad \dots\dots\dots\dots\dots\dots\dots\dots\text{(x)}.$$

Now by hypothesis (i) we are to have (ix) linear.

Let us expand the logarithm and the exponential. We have

$$\begin{aligned}
\tilde{y}_x &= c_2 + \frac{1}{\lambda_2}\left(\frac{\lambda_2}{\lambda_1}m_2 e^{\lambda_1 x} + \frac{\lambda_2^2 m_2^2}{2\lambda_1^2}e^{2\lambda_1 x} + \frac{\lambda_2^3 m_2^3}{3\lambda_1^3}e^{3\lambda_1 x} + \dots\right) \\
&= c_2 + \frac{m_2}{\lambda_1}\left(1 + \lambda_1 x + \frac{1}{2!}\lambda_1^2 x^2 + \dots\right) \\
&\quad + \frac{m_2^2\lambda_2}{2\lambda_1^2}\left(1 + 2\lambda_1 x + \frac{1}{2!}4\lambda_1^2 x^2 + \dots\right) \\
&\quad + \frac{m_2^3\lambda_2^2}{3\lambda_1^2}\left(1 + 3\lambda_1 x + \frac{1}{2!}9\lambda_1^2 x^2 + \dots\right) \\
&\quad + \text{etc.} \\
&= c_2 + m_2\left(\frac{1}{\lambda_1}\right) + \frac{m_2^2\lambda_2}{2\lambda_1^2} + \frac{m_2^3\lambda_2^2}{3\lambda_1^3} + \dots \\
&\quad + x\left(m_2 + m_2^2\frac{\lambda_2}{\lambda_1} + m_2^3\frac{\lambda_2^2}{\lambda_1^2} + \dots\right) \\
&\quad + \frac{x^2}{2!}\left(m_2\lambda_1 + 2m_2^2\lambda_2 + 3m_2^3\frac{\lambda_2^2}{\lambda_1} + \dots\right) \\
&\quad + \text{etc.}
\end{aligned}$$

To make the right-hand side linear we are bound to make λ_2/λ_1 finite, and both λ_1 and λ_2 zero. The infinite constant now may be added to the arbitrary constant c_2; the coefficients of all powers of x vanish except that of x which is finite and we have

$$\tilde{y}_x = c_2' + m_2' x \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\text{(xi)}.$$

* S. Narumi, *Biometrika*, Vol. XV. pp. 82—85.

But the very same values of λ_1 and λ_2 turn (x) into

$$\bar{x}_y = c_1' + m_1' y \quad \text{..............................(xii).}$$

Accordingly we cannot have a frequency surface homoscedastic both ways and with linear regression one way, without linear regression the other way.

But a frequency surface linear both ways and homoscedastic both ways can only be a bivariate normal surface. If this result be not familiar, it can be deduced from (viii) by making the ratio λ_2/λ_1 finite and λ_1 and λ_2 both zero.

Thus it appears to me, without being dogmatic, that unless a series of surfaces can be produced in which the relation (v) holds other than those supplied by (*a*) or (*b*), we do not narrow our hypothesis by supposing that the populations Π_1, $\Pi_2, \ldots \Pi_t \ldots \Pi_k$ are all normal surfaces differing the one from the other only in the position of their means. These means will not of necessity be collinear; this depends on the constancy of $\bar{a}_t$, supposing that we have accepted the hypotheses that $\bar{\sigma}_{a,t}$ and $\bar{\beta}_t$ are the same for all the populations.

The thoughts here recorded do not question the validity of the processes by which the authors cited would deduce the constancy of $\bar{\sigma}_{a,t}$, $\bar{a}_t$ or $\bar{\beta}_t$. But they do tend to suggest that the surface (i) from which they start is, when the hypotheses $\bar{\sigma}_{a,t} = \bar{\sigma}_{a,t'}$ and $\bar{\beta}_t = \bar{\beta}_{t'}$ are applied, not so general as they, perhaps, imagine and that accordingly there may be other, possibly as simple, methods of attacking their problem as that provided by them.

Let us consider first samples which may be supposed taken from a bivariate parent normal surface defined by means m_1, m_2, σ_1, σ_2, and correlation ρ, and let any sample be defined by $\bar{x}$, $\bar{y}$, Σ_1, Σ_2 and r. The frequency surface in five-way space is known to be*

$$z = z_0 e^{-\frac{1}{2}\left\{\frac{(\bar{x}-m_1)^2}{s_1^2} - \frac{2\rho(\bar{x}-m_1)(\bar{y}-m_2)}{s_1 s_2} + \frac{(\bar{y}-m_2)^2}{s_2^2}\right\}}$$

$$\times e^{-\frac{1}{2}\left\{\frac{\Sigma_1^2}{s_1^2} - \frac{2\rho r \Sigma_1 \Sigma_2}{s_1 s_2} + \frac{\Sigma_2^2}{s_2^2}\right\}} \left(\frac{\Sigma_1}{s_1}\right)^{M-2} \left(\frac{\Sigma_2}{s_2}\right)^{M-2} (1-r^2)^{\frac{M-4}{2}} [d\bar{x}\, d\bar{y}\, d\Sigma_1\, d\Sigma_2\, dr]$$

$$= z_0 F_1(\bar{x}, \bar{y}) \times F_2(\Sigma_1, \Sigma_2, r)[d\bar{x}_1\, d\bar{y}_1\, d\Sigma_1\, d\Sigma_2\, dr] \text{ say, for brevity} \quad \text{......(xiii).}$$

There are five variables here, i.e. those of the sample $\bar{x}$, $\bar{y}$, Σ_1, Σ_2, r. I propose to replace them by the five other variables $\bar{x}$, $a = \bar{y} - \beta\bar{x}$, $\beta = r\Sigma_2/\Sigma_1$, Σ_1, and $\gamma = \Sigma_2\sqrt{1-r^2}$†. In doing this I shall throw any numerical factor or any function of the parental constants m_1, m_2, s_1, s_2 and ρ only, which arises as a multiplier of F_1 or F_2, into z_0.

I shall begin first by getting rid of $\bar{y} = a + \beta\bar{x}$ in $F_1(\bar{x}, \bar{y})$. After re-arranging in powers of $\bar{x}$, we have

$$F_1(\bar{x}, a, \beta) = e^{-\frac{1}{2}\left[Q\left\{\frac{\bar{x}-m_1}{s_1} - \left(\rho - \beta\frac{s_1}{s_2}\right)\frac{a - m_2 + \beta m_1}{s_2 Q}\right\}^2 + \frac{(a-m_2+\beta m_1)^2(1-\rho^2)}{s_2^2 Q}\right]},$$

where

$$Q = 1 - \frac{2\rho\beta s_1}{s_2} + \frac{\beta^2 s_1^2}{s_2^2} = 1 - \frac{2\rho\beta\sigma_1}{\sigma_2} + \beta^2\frac{\sigma_1^2}{\sigma_2^2} \quad \text{..............(xiv).}$$

* We write $s_1^2 = \sigma_1^2(1-\rho^2)/M$, and $s_2^2 = \sigma_2^2(1-\rho^2)/M$, where M is the size of the sample.

† The reader must be careful to bear in mind that a, β, γ here are all sample and not parent-population values.

Now $\bar{x}$ does not occur in $F_2(\Sigma_1, \Sigma_2, r)$ and thus we see if α and β are fixed $\bar{x}$ still follows a normal curve round the value

$$\text{mean } \bar{x} = m_1 - \frac{s_1}{s_2}\left(\rho - \beta\frac{s_1}{s_2}\right)\frac{\alpha - m_2 + \beta m_1}{Q} \quad\text{.................(xv)},$$

with standard deviation
$$\sigma_{\bar{x}, \alpha, \beta} = s_1/\sqrt{Q} \quad\text{................................(xvi)}.$$

We see that this standard deviation is independent of α, which only influences $\bar{x}$. *We now transform our five variables $\bar{x}$, $\bar{y}$, Σ_1, Σ_2, r to the variables $\bar{x}$, α, Σ_1, β, and γ.* Evaluating the determinant we find

$$d\bar{x}\,d\bar{y}\,d\Sigma_1\,d\Sigma_2\,dr = \frac{\Sigma_1}{\Sigma_2}\sqrt{1-r^2}\,d\bar{x}\,d\alpha\,d\Sigma_1\,d\beta\,d\gamma.$$

We can now integrate out for $\bar{x}$ and have a four-way frequency surface:

$$z' = z_0'\sqrt{Q}\,e^{-\frac{1}{2}\frac{(\alpha - m_2 + \beta m_1)^2(1-\rho^2)}{s_2^2 Q}}\frac{\Sigma_1}{\Sigma_2}\sqrt{1-r^2}F_2(\Sigma_1, \Sigma_2, r) \qquad [d\alpha\,d\Sigma_1\,d\beta\,d\gamma].$$

The next stage is to express the function $\dfrac{\Sigma_1\sqrt{1-r^2}}{\Sigma_2}F_2(\Sigma_1, \Sigma_2, r)$ in terms of Σ_1, β and γ. What we have to express is

$$e^{-\frac{1}{2}\left(\frac{\Sigma_1^2}{s_1^2} - \frac{2\rho r\Sigma_1\Sigma_2}{s_1 s_2} + \frac{\Sigma_2^2}{s_2^2}\right)}\left(\frac{\Sigma_1}{s_1}\right)^{M-1}\left(\frac{\Sigma_2}{s_2}\right)^{M-3}(1-r^2)^{\frac{M-3}{2}},$$

and this leads to

$$e^{-\frac{1}{2}\frac{\Sigma_1^2}{s_1^2}\left(1 - \frac{2\rho\beta s_1}{s_2} + \beta^2\frac{s_1^2}{s_2^2}\right) - \frac{1}{2}\frac{\gamma^2}{s_2^2}}\left(\frac{\Sigma_1}{s_2}\right)^{M-1}\left(\frac{\gamma}{s_2}\right)^{M-3},$$

or

$$e^{-\frac{1}{2}\frac{\Sigma_1^2}{s_1^2}Q}\left(\frac{\Sigma_1}{s_1}\right)^{M-1}e^{-\frac{1}{2}\frac{\gamma^2}{s_2^2}}\left(\frac{\gamma}{s_2}\right)^{M-3}.$$

We can now write our fourfold surface of frequency in the form

$$z'\,d\alpha\,d\Sigma_1\,d\beta\,d\gamma = z_0'\frac{1}{\sqrt{Q}}e^{-\frac{1}{2}\frac{(\alpha - m_2 + \beta m_1)^2(1-\rho^2)}{s_2^2 Q}}d\alpha\,d\beta$$
$$\times\, e^{-\frac{1}{2}\frac{\Sigma_1^2}{s_1^2}Q}\left(\frac{\Sigma_1}{s_1}\right)^{M-1}d\Sigma_1\,e^{-\frac{1}{2}\frac{\gamma^2}{s_2^2}}\left(\frac{\gamma}{s_2}\right)^{M-3}d\gamma \quad\text{......(xvii)}.$$

This admits at once of being integrated out with regard to Σ_1, leading us to the complete Γ-function $\frac{1}{2}M$, and other factors which can be thrown into the z_0', as well as the important factor $\dfrac{1}{Q^{M/2}}$. Thus we reduce our expression to a three-way surface of frequency

$$z''\,d\alpha\,d\beta\,d\gamma = z_0''\frac{1}{Q^{\frac{1}{2}(M+1)}}e^{-\frac{1}{2}\frac{(\alpha - m_2 + \beta m_1)^2(1-\rho^2)}{s_2^2 Q}}d\alpha\,d\beta \times e^{-\frac{1}{2}\frac{\gamma^2}{s_2^2}}\left(\frac{\gamma}{s_2}\right)^{M-3}d\gamma \quad\text{...(xviii)}.$$

This result is of considerable importance; we see that the three-way frequency surface breaks up into two independent frequency distributions, the one the bivariate frequency surface of correlation for α and β, and the other the frequency

curve for the distribution of γ, the standard deviation of the arrays of y on x, i.e. $\Sigma_2\sqrt{1-r^2}$. This latter, as pointed out in my memoir of 1926*, is precisely of the same form as the distribution of the standard deviations of samples from a normal curve with the number of the individuals sampled reduced by a unit.

It is accordingly clear that in the problems considered by Messrs Welch and Kolodziejczyk the question of whether their σ_t (= our $\bar{\sigma}_{a,t}$) in a series of samples is the same can be treated without regard to the α and β of the regression line, while the α and β cannot be considered separately as they are correlated. We shall consider later how to deal with this distribution of $\sigma_{a,t}$. We now turn to the correlation surface

$$z''' = z_0''' \frac{1}{Q^{\frac{1}{2}(M+1)}} e^{-\frac{1}{2}\frac{(\alpha - m_2 + \beta m_1)^2(1-\rho^2)}{s_2^2 Q}} [d\alpha d\beta] \quad \text{..........(xix)}.$$

This can be integrated out with regard to α, and gives us

$$z^{\mathrm{iv}} = \frac{z_0^{\mathrm{iv}}}{Q^{\frac{1}{2}M}} = \frac{z_0^{\mathrm{iv}}}{\left(1 - 2\rho\frac{\beta s_1}{s_2} + \beta^2\frac{s_1^2}{s_2^2}\right)^{\frac{1}{2}M}}$$

$$= \frac{z_0^{\mathrm{v}}}{\left\{\frac{s_2^2}{s_1^2}(1-\rho^2) + \left(\beta - \rho\frac{s_2}{s_1}\right)^2\right\}^{\frac{1}{2}M}} \quad \text{....................(xx)}.$$

This is the distribution curve for the regression coefficients β of samples of size M drawn from a bivariate normal surface. It is a curve of my Type VII, symmetrical about the regression coefficient of the parent population, $\rho\sigma_2/\sigma_1$, and as M increases it approaches a normal distribution†. Remembering that $s_2/s_1 = \sigma_2/\sigma_1$ we remark that

$$\bar{\beta} = \rho\sigma_2/\sigma_1, \quad \sigma_\beta = \frac{1}{\sqrt{M-3}}\frac{\sigma_2}{\sigma_1}\sqrt{1-\rho^2} \quad \text{................(xxi)}.$$

The odd moments all vanish and $\mu_4/\mu_2^2 = 3(M-3)/(M-5)$.

For low values of M the moments of the β curve become infinite. For $M=3$ μ_2 is infinite, and for $M=4$ the value of μ_4 is infinite. Generally M must be $> n+2$ for the nth moment to be finite; i.e. if we desire the nth moment to be finite, the number in the sample on which the regression line is based must be $> n+2$.

Returning to (xix) we see that it may be written

$$z''' = z_0''' e^{-\frac{1}{2}\frac{(\alpha - m_2 + \beta m_1)^2}{\sigma_2^2 Q/M}} \frac{1}{Q^{\frac{1}{2}(M+1)}} \quad \text{....................(xxii)}.$$

Since Q is a function only of β we see that for constant regression coefficients α will be distributed normally about the

$$\text{Mean } \alpha_\beta = m_2 - \beta m_1,$$

* *R.S. Proc.* Vol. 112 A, p. 4.

† I discovered this distribution in 1926 (see *R.S. Proc.* Vol. 112 A, pp. 5—7), but Mr Welch has pointed out to me the proof then given was not valid.

with
$$\sigma_{\alpha,\beta} = \frac{\sigma_2\sqrt{Q}}{\sqrt{M}} = \frac{\sigma_2}{\sqrt{M}}\left(1 - 2\rho\frac{\beta\sigma_1}{\sigma^2} + \beta^2\frac{\sigma_1^2}{\sigma_2^2}\right)^{\frac{1}{2}}$$
$$= \frac{1}{\sqrt{M}}(\sigma_2^2 - 2\rho\beta\sigma_1\sigma_2 + \beta^2\sigma_1^2)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(xxiii)}.$$

Thus we learn that the regression of α on β is linear, but the system is heteroscedastic, the variance being given by a hyperbola.

I have tried unsuccessfully to integrate (xxii) for β, so as to obtain the distribution curve for α. This I fear is unlikely to be achieved, but the moments of the α curve can be found, and if the reader be so inclined a curve of one of my types fitted from the first four moments. This is the method adopted in a similar problem by Mr Welch (p. 151), but in the case of samples based on few individuals, I feel doubtful how far this approximation to the true distribution curve of a variate may not be misleading. In the case of many individuals it would for practical purposes be adequate.

We will turn now to the moments of α. First we will find the value of z_0''' supposing the total frequency to be N. We have
$$N = z_0'''\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}\frac{(\alpha - m_2 + m_1\rho(\sigma_2/\sigma_1) + m_1 R_2')^2}{\sigma_2^2 Q/M}}\frac{1}{Q^{\frac{1}{2}(M+1)}}\,d\alpha\, dR'_2,$$
where
$$Q = \frac{\sigma_1^2}{\sigma_2^2}\left(\frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2) + R_2'^2\right)$$
and
$$R_2' = \beta - \rho\frac{\sigma_2}{\sigma_1}.$$

Integrating out with regard to α,
$$N = z_0'''\int_{-\infty}^{+\infty}\frac{\sqrt{2\pi}\,\sigma_2\sqrt{Q}}{\sqrt{M}}\,\frac{1}{Q^{\frac{1}{2}(M+1)}}\,dR_2'$$
$$= \frac{\sqrt{2\pi}\,\sigma_2 z_0'''}{\sqrt{M}}\int_{-\infty}^{+\infty}\frac{1}{Q^{\frac{1}{2}M}}\,dR_2' = \frac{\sqrt{2\pi}}{\sqrt{M}}\,\frac{\sigma_2^{M+1}}{\sigma_1^M z_0'''}\int_{-\infty}^{+\infty}\frac{dR_2'}{(c^2 + R_2'^2)^{\frac{1}{2}M}}$$
where
$$c = \frac{\sigma_2}{\sigma_1}\sqrt{1-\rho^2}.$$
Hence
$$N = z_0'''\sqrt{2\pi}\,\frac{\sigma_2^2}{\sigma_1}\,\frac{\sqrt{M}\,\Gamma(\frac{1}{2}(M-1))}{(1-\rho^2)^{\frac{1}{2}(M-1)}\,\Gamma(\frac{1}{2}M)},$$
or
$$z_0''' = \frac{N\sigma_1(1-\rho^2)^{\frac{1}{2}(M-1)}\sqrt{M}\,\Gamma(\frac{1}{2}M)}{\sqrt{2}\,\pi\sigma_2^2\,\Gamma(\frac{1}{2}(M-1))}\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(xxiv)}.$$

While considering the value of the z_0's it is worth while writing down what value arises from the γ distribution—i.e.
$$z = z_0^{\mathrm{iv}}\,e^{-\frac{1}{2}\frac{\gamma^2}{s_2^2}}\left(\frac{\gamma}{s_2}\right)^{M-3}[d\gamma],$$
where $s_2^2 = \sigma_2^2(1-\rho^2)/M$—of the three-way frequency surface.

Putting $v=\frac{1}{2}\frac{\gamma^2}{s_2^2}$, integrating from 0 to ∞, and taking the total frequency to be unity we find

$$z_0^{\text{iv}}=\frac{\sqrt{M}}{\sigma_2\sqrt{1-\rho^2}\,2^{\frac{1}{2}(M-4)}\,\Gamma\left(\frac{1}{2}M-1\right)},$$

or the distribution of γ, the array standard deviation, is given by

$$z=\frac{\sqrt{M}}{\sigma_2\sqrt{1-\rho^2}\,2^{\frac{1}{2}(M-4)}\,\Gamma\left(\frac{1}{2}M-1\right)}e^{-\frac{1}{2}\frac{\gamma^2}{s_2^2}}\left(\frac{\gamma}{s_2}\right)^{M-3}\text{ ....(xxv).}$$

We are now in a position to write down the complete three-way surface for the frequency distribution of α, β, γ. It is

$$z=\frac{N\sigma_1(1-\rho^2)^{\frac{1}{2}(M-3)}(M-2)\,M}{2^{\frac{1}{2}(M-1)}\,\pi\sigma_2^2\,\Gamma\left(\frac{1}{2}(M-1)\right)}e^{-\frac{1}{2}\frac{(\alpha-m_2+\beta m_1)^2}{\sigma_2^2 Q/M}}\frac{1}{Q^{\frac{1}{2}(M+1)}}$$
$$\times e^{-\frac{1}{2}\frac{M\gamma^2}{\sigma_2^2(1-\rho^2)}}\left(\frac{\sqrt{M}\gamma}{\sigma_2\sqrt{1-\rho^2}}\right)^{M-3}[d\alpha\, d\beta\, d\gamma]\text{ ......(xxvi).}$$

For the two-way surface of α and β only

$$z=\frac{N\sigma_1(1-\rho^2)^{\frac{1}{2}(M-1)}\,\Gamma\left(\frac{1}{2}M\right)\sqrt{M}}{\sqrt{2}\,\pi\sigma_2^2\,\Gamma\left(\frac{1}{2}(M-1)\right)}e^{-\frac{1}{2}\frac{(\alpha-m_2+\beta m_1)^2}{\sigma_2^2 Q/M}}\frac{1}{Q^{\frac{1}{2}(M+1)}}[d\alpha\, d\beta]\text{ ...(xxvii).}$$

The importance of these forms is that for any hypothesis as to α, β, γ we may require to find the best values to take for m_1, m_2, σ_1, σ_2, and ρ in order to give the most probable values to the observed quantities*.

We now return to the moments of α, applying formula (xxvii). It is necessary to find $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z\alpha\, d\alpha\, d\beta$ to obtain the mean of α, or $\bar{\alpha}$. Now

$$N\bar{\alpha}=z_0'''\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\{(\alpha-m_2+\beta m_1)+(m_2-\beta m_1)\}\,e^{-\frac{1}{2}\frac{(\alpha-m_2+\beta m_1)^2}{\sigma_2^2 Q/M}}\frac{1}{Q^{\frac{1}{2}(M+1)}}\,d\alpha\, d\beta,$$

and if we write $X=\alpha-m_2+\beta m_1$,

$$N\bar{\alpha}=z_0'''\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\{X+(m_2-\beta m_1)\}\,e^{-\frac{1}{2}\frac{X^2}{\sigma_2^2 Q/M}}\frac{1}{Q^{\frac{1}{2}(M+1)}}\,dX\, d\beta,$$

the integration of the first term in the curled brackets with regard to X is zero, and of the second term is $\sqrt{2\pi}\sqrt{Q/M}\,\sigma_2\times(m_2-\beta m_1)$. Accordingly

$$N\bar{\alpha}=z_0'''\sqrt{2\pi}\int_{-\infty}^{+\infty}\sqrt{\frac{Q}{M}}\,\sigma_2(m_2-\beta m_1)\frac{1}{Q^{\frac{1}{2}(M+1)}}\,d\beta$$
$$=\frac{z_0'''\sqrt{2\pi}}{\sqrt{M}}\sigma_2\int_{-\infty}^{+\infty}\left\{\left(m_2-m_1\rho\frac{\sigma_2}{\sigma_1}\right)-m_1R_2'\right\}\frac{dR_2'}{\left(\frac{\sigma_1}{\sigma_2}\right)^M\left(\frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2)+R_2'^2\right)^{\frac{1}{2}M}},$$

where $R_2'=\beta-\rho\frac{\sigma_2}{\sigma_1}$ as before.

* The method was first used by Gauss to demonstrate the arithmetic mean of observations and the best value of the standard deviation in the case of the normal curve. It was largely applied by Professor Filon and myself in dealing with the variability of frequency constants in samples (*Phil. Trans.* (1898), Vol. 198 A, pp. 231—33).

Since the R_2' curve is symmetrical, the integral involving the R_2' term in the curled brackets vanishes, and we have

$$N\bar{a} = \frac{z_0''' \sqrt{2\pi}}{\sqrt{M}} \sigma_2 \left(m_2 - m_1 \rho \frac{\sigma_2}{\sigma_1}\right) \left(\frac{\sigma_2}{\sigma_1}\right)^M \int_{-\infty}^{+\infty} \frac{dR_2'}{\left(\frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2) + R_2'^2\right)^{\frac{1}{2}M}}.$$

Take $R_2' = \frac{\sigma_2}{\sigma_1}\sqrt{1-\rho^2}\sqrt{\frac{v}{1-v}}$, and we find

$$\int_{-\infty}^{+\infty} \frac{dR_2'}{\left(\frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2) + R_2'^2\right)^{M/2}} = \left(\frac{\sigma_1}{\sigma_2}\right)^{M-1} \frac{1}{(1-\rho^2)^{\frac{1}{2}(M-1)}} \int_0^1 v^{-\frac{1}{2}}(1-v)^{\frac{1}{2}(M-3)} dv$$

$$= \left(\frac{\sigma_1}{\sigma_2}\right)^{M-1} \frac{1}{(1-\rho^2)^{\frac{1}{2}(M-1)}} \frac{\Gamma(\frac{1}{2})\,\Gamma\frac{1}{2}(M-1)}{\Gamma(\frac{1}{2}M)},$$

and accordingly

$$N\bar{a} = z_0''' \frac{\sqrt{2}\pi}{\sqrt{M}} \frac{\sigma_2^2}{\sigma_1} \frac{1}{(1-\rho^2)^{\frac{1}{2}(M-1)}} \frac{\Gamma\frac{1}{2}(M-1)}{\Gamma(\frac{1}{2}M)} \left(m_2 - m_1\rho\frac{\sigma_2}{\sigma_1}\right).$$

Substituting for z_0''' from (xxiv), we have

$$\bar{a} = m_2 - m_1 \rho \frac{\sigma_2}{\sigma_1} \quad \text{..........................(xxviii)}.$$

This checks the value of z_0''', giving to the mean a the population value. But it is not obvious *à priori* that the mean value of the triple product $\bar{x}r\frac{\Sigma_2}{\Sigma_1}$ will take the population value.

We will now proceed to compute σ_a, the standard deviation of a,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} za^2 da = z_0''' \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \{a - m_2 + \beta m_1 + m_2 - \beta m_1\}^2 e^{-\frac{1}{2}\frac{(a-m_2+\beta m_1)^2}{\sigma_2^2 Q/M}} \times \frac{1}{Q^{\frac{1}{2}(M+1)}} da\, d\beta.$$

Writing $a - m_2 + \beta m_1 = X$ as before, the curled bracket term gives us

$$X^2 + 2(m_2 - \beta m_1) X + (m_2 - \beta m_1)^2;$$

this has to be multiplied by $e^{-\frac{1}{2}\frac{X^2}{\sigma_2^2 Q/M}}$, and first integrated with regard to X. This gives us

$$\sqrt{2\pi}\,\sigma_2 \sqrt{\frac{Q}{M}}\,\sigma_2^2 Q/M + 0 + (m_2 - \beta m_1)^2 \sqrt{2\pi}\,\sigma_2 \sqrt{\frac{Q}{M}},$$

and thus $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} za^2 da\, d\beta = \frac{\sqrt{2\pi}\,\sigma_2 z_0'''}{\sqrt{M}} \int_{-\infty}^{+\infty} \left\{\frac{\sigma_2^2 Q}{M} + (m_2 - \beta m_1)^2\right\} \frac{d\beta}{Q^{\frac{1}{2}M}},$

and this after some rearranging and introduction of $R_2' = \beta - \rho\frac{\sigma_2}{\sigma_1}$

$$= \frac{\sqrt{2\pi}\,\sigma_2 z_0'''}{\sqrt{M}} \int_{-\infty}^{+\infty} \left\{\frac{\sigma_2^2(1-\rho^2)}{M} + \left(m_2 - m_1\rho\frac{\sigma_2}{\sigma_1}\right)^2 - 2m_1 R_2' \left(m_2 - m_1 \rho \frac{\sigma_2}{\sigma_1}\right) + R_2'^2\left(\frac{\sigma_1^2}{M} + m_1^2\right)\right\} \left(\frac{\sigma_2}{\sigma_1}\right)^M \frac{dR_2'}{(c^2 + R_2'^2)^{\frac{1}{2}M}}.$$

The term in the curled brackets in R_2' will disappear on integration, since the R_2' frequency is symmetrical. Hence

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z\alpha^2\,d\alpha\,d\beta = \frac{\sqrt{2\pi}\,\sigma_2 z_0'''}{\sqrt{M}}\int_{-\infty}^{+\infty}\left\{\left(m_2 - m_1\rho\frac{\sigma_2}{\sigma_1}\right) - m_1^2\frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2) + (c^2+R_2'^2)\left(\frac{\sigma_1^2}{M}+m_1^2\right)\right\}\left(\frac{\sigma_2}{\sigma_1}\right)^M\frac{dR_2'}{(c^2+R_2'^2)^{\frac{1}{2}M}}$$

$$= \frac{\sqrt{2\pi}\,\sigma_2 z_0'''}{\sqrt{M}}\left(\frac{\sigma_2}{\sigma_1}\right)^M\left[\left\{\left(m_2 - m_1\rho\frac{\sigma_2}{\sigma_1}\right) - m_1^2\frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2)\right\}\int_{-\infty}^{+\infty}\frac{dR_2'}{(c^2+R_2'^2)^{\frac{1}{2}M}} + \left(\frac{\sigma_1^2}{M}+m_1^2\right)\int_{-\infty}^{+\infty}\frac{dR_2'}{(c^2+R_2'^2)^{\frac{1}{2}(M+2)}}\right] \quad\text{......(xxix).}$$

But
$$\int_{-\infty}^{+\infty}\frac{dR_2'}{(c^2+R_2'^2)^{\frac{1}{2}M}} = \frac{\sqrt{\pi}}{c^{M-1}}\frac{\Gamma(\frac{1}{2}(M-1))}{\Gamma(\frac{1}{2}M)},$$

and
$$\int_{-\infty}^{+\infty}\frac{dR_2'}{(c^2+R_2'^2)^{\frac{1}{2}(M-2)}} = \frac{c^2\sqrt{\pi}}{c^{M-1}}\frac{\Gamma(\frac{1}{2}(M-1))}{\Gamma(\frac{1}{2}M)}\frac{M-2}{M-3}.$$

Substituting in (xxix) these values, and also from (xxiv) the value of z_0''', we have

$$\int_{-\infty}^{+\infty} z\alpha^2\,d\alpha\,d\beta = N(\sigma_\alpha^2+\bar{\alpha}^2) = N\left[\left(m_2-m_1\rho\frac{\sigma_2}{\sigma_1}\right)^2 + \frac{\sigma_2^2(1-\rho^2)}{M-3}\left(1-\frac{2}{M}+\frac{m_1^2}{\sigma_1^2}\right)\right].$$

Or, since $\bar{\alpha} = m_2 - m_1\rho\frac{\sigma_2}{\sigma_1}$, we have

$$\sigma_\alpha^2 = \frac{\sigma_2^2(1-\rho^2)}{M-3}\left(1+\frac{m_1^2}{\sigma_1^2}-\frac{2}{M}\right) \quad\text{.....................(xxx).}$$

The approximate values given by me for *large* samples in 1913* were, for α and β,

$$\sigma_\alpha^2 = \frac{\sigma_2^2(1-\rho^2)}{M}\left(1+\frac{m_1^2}{\sigma_1^2}\right) \quad\text{and}\quad \sigma_\beta^2 = \frac{\sigma_2^2}{\sigma_1^2}\frac{(1-\rho^2)}{M},$$

as against the exact values

$$\sigma_\alpha^2 = \frac{\sigma_2^2(1-\rho^2)}{M-3}\left(1+\frac{m_1^2}{\sigma_1^2}-\frac{2}{M}\right) \quad\text{and}\quad \sigma_\beta^2 = \frac{\sigma_2^2}{\sigma_1^2}\frac{1-\rho^2}{M-3};$$

thus the former values are seen to be closely approximate for large samples.

To indicate the dependence of α and β, we will find their correlation coefficient, at the same time reminding the reader that while the regression curve of α on β is linear, that of β on α may be very far from being so. It is indeed so complicated as to be unmanageable.

Proceeding as before, we find

$$\frac{S(\alpha\beta)}{N} - \bar{\alpha}\bar{\beta} = -\frac{m_1}{\sigma_1}\frac{\sigma_2^2}{\sigma_1}\frac{1-\rho^2}{M-3} \quad\text{.....................(xxxi),}$$

leading to

$$r_{\alpha\beta} = -\frac{\dfrac{m_1}{\sigma_1}}{\sqrt{1+\dfrac{m_1^2}{\sigma_1^2}-\dfrac{2}{M}}} \quad\text{.......................(xxxii).}$$

* *Biometrika*, Vol. IX. p. 9. See also *Phil. Trans.* (1898), Vol. 191 A, p. 245.

The corresponding values found for large samples in 1913* were

$$-\frac{m_1}{\sigma_1}\frac{\sigma_2^2}{\sigma_1}\frac{1-\rho^2}{M} \text{ and } -\frac{\frac{m_1}{\sigma_1}}{\sqrt{1+\frac{m_1^2}{\sigma_1^2}}}$$

respectively. *These approximations are now justified by the determination of the* accurate values for samples of any size.

One more theoretical point may be noted. If $\bar{y}_x$ be the mean of the x array of y's, then

$$\bar{y}_x = a + \beta x,$$

and the equation $\tilde{y}_x - \bar{y} = \alpha - \bar{\alpha} + (\beta - \bar{\beta})\, x$ is exact.

Hence we have

$$\sigma^2_{\bar{y}_x} = \sigma_\alpha^2 + x^2\sigma_\beta^2 + 2r_{\alpha\beta}\sigma_\alpha\sigma_\beta x$$

$$= \frac{1}{M-3}\left[\sigma_2^2(1-\rho^2)\left(1+\frac{m_1^2}{\sigma_1^2}-\frac{2}{M}\right)+\frac{x^2}{\sigma_1^2}\sigma_2^2(1-\rho^2)-2xm_1\frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2)\right],$$

or

$$\sigma^2_{\bar{y}_x} = \frac{\sigma_2^2(1-\rho^2)}{M-3}\left\{1+\frac{m_1^2}{\sigma_1^2}-\frac{2x}{\sigma_1}\frac{m_1}{\sigma_1}+\frac{x^2}{\sigma_1^2}-\frac{2}{M}\right\}.$$

Finally

$$\sigma^2_{\bar{y}_x} = \frac{\sigma_2^2(1-\rho^2)}{M-3}\left(1+\frac{(x-m_1)^2}{\sigma_1^2}-\frac{2}{M}\right) \quad \text{..............(xxxiii).}$$

This gives the exact standard error of the mean of an array as found from a regression line. *The value for large samples found in 1913 was*

$$\sigma^2_{\bar{y}_x} = \frac{\sigma_2^2(1-\rho^2)}{M}\left(1+\frac{(x-m_1)^2}{\sigma_1^2}\right),$$

which is accordingly justified.

We note one further point. Formula (xxxii) may be written

$$r_{\alpha\beta} = -\frac{1}{\sqrt{1+\left(1-\frac{2}{M}\right)\frac{\sigma_1^2}{m_1^2}}} \quad \text{.................(xxxii}^{bis}\text{)},$$

and accordingly we see that if the coefficient of variation of the x variate be very small the relation of α to β tends to become perfect (negatively); but if this coefficient of variation be large, the correlation tends to be small. Since coefficients of variation are in practice usually small, we see that the relation of α and β is close, and the regression curve of β on α will tend to be linear like that of α on β.

Starting with the variables α, β, γ, we have seen that γ is distributed according to a frequency curve which is absolutely independent of α and β, but that the latter are given by a bivariate frequency surface, exhibiting a high degree of

* *Biometrika*, Vol. IX. p. 10.

correlation. We propose to replace α, β, γ by three new variates which are mutually independent. These are as before

$$\left.\begin{aligned} &X = \frac{\alpha - m_2 + \beta m_1}{\sigma_2 \sqrt{Q}/\sqrt{M}}, \text{ where } Q = \frac{\sigma_1^2}{\sigma_2^2}\{c^2 + (\beta - \bar{\beta})^2\}, \text{ and } c^2 = \frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2) \\ &Y = \frac{c^2}{c^2 + (\beta - \bar{\beta})^2}, \text{ and therefore } Y = (1-\rho^2)/Q, \\ &\text{and finally} \\ &Z = \frac{1}{2}\frac{\gamma^2}{s_2^2} = \frac{1}{2}\frac{M\gamma^2}{\sigma_2^2(1-\rho^2)} \end{aligned}\right\} \quad \text{......(xxxiv).}$$

As to the limits they are independent. Those of Z depend only on γ and accordingly are from 0 to $+\infty$. Those of Y depend only on β, but it will have equal values when $\beta - \bar{\beta}$ takes plus or minus equal numerical values. Hence integrals for Y must be doubled, its limits are $Y = 1$, for $\beta = \bar{\beta}$, and $Y = 0$, for $\beta - \bar{\beta} = \pm\infty$.

Turning to X we may write it, having regard to (xxiii), in the form

$$X = \frac{\bar{y} - m_2 - \beta(\bar{x} - m_1)}{\sigma_{\alpha \cdot \beta}} = \frac{\bar{y} - m_2 - \beta(\bar{x} - m_1)}{\sigma_{(\bar{y} - \beta\bar{x}) \cdot \beta}}.$$

Now $\bar{y} - m_2 - \beta(\bar{x} - m_1) = -(\bar{y}_x - \bar{y}) + \beta(x - \bar{x}) + \bar{y}_x - m_2 - \beta(x - m_1)$, where $\bar{y}_x$ is the mean of the sample array we are considering, and accordingly

$$\bar{y}_x - \bar{y} - \beta(x - \bar{x}) = 0.$$

Again $\quad \sigma_{(\bar{y} - \beta\bar{x}) \cdot \beta} = \sigma_{\bar{y} - \bar{y}_x + \bar{y}_x - \beta(\bar{x} - x + x)} = \sigma_{\bar{y}_x - \beta x} = \sigma_{\bar{y}_x - m_2 - \beta(x - m_1)}$,

since m_2 and m_1 do not vary for the sample from the particular population, which has given the regression coefficient β and the mean $\bar{x}$ and $\bar{y}$. Accordingly we have

$$X = \frac{\bar{y}_x - m_2 - \beta(x - m_1)}{\sigma_{\bar{y}_x - m_2 - \beta(x - m_1)}}.$$

Let us consider the geometrical meaning of X. $S(m_1, m_2)$ is the mean of any one of the k populations, and $T(\bar{x}, \bar{y})$ of the sample from this population with regression coefficient β (see figure, p. 239). AB is the regression line of this sample; CD is a parallel line through S of which the equation will be

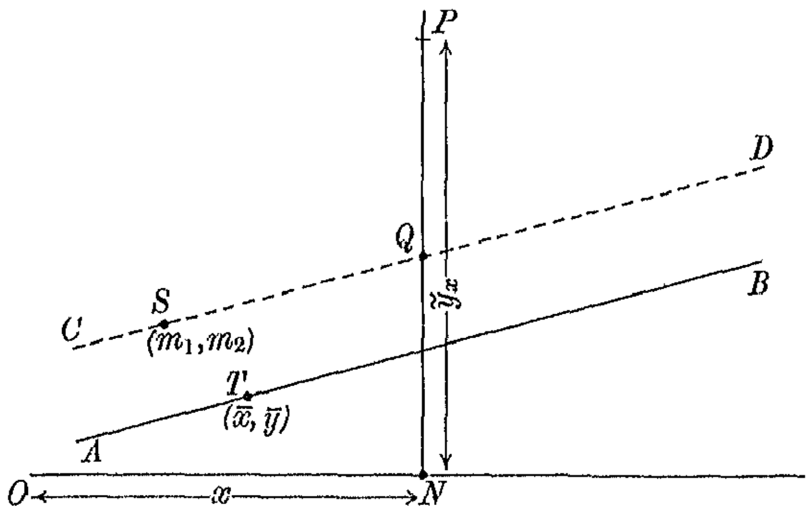
$$y - m_2 - \beta(x - m_1) = 0.$$

$ON = x$ gives the position of the array of y's for this sample of which the mean is $\bar{y}_x$ at P. Now

$$PQ = PN - QN = \bar{y}_x - \{m_2 + \beta(x - m_1)\} = \bar{y}_x - m_2 - \beta(x - m_1).$$

Thus we have $\quad X = \dfrac{PQ}{\text{standard deviation of } PQ}$ ....(xxxiv*bis*).

But by hypothesis the arrays are to form normal distributions and accordingly fixing any particular sample by $\beta, \bar{x}, \bar{y}$ the limits of X are from $-\infty$ to $+\infty$.



We now proceed to transform our equation to the three-way surface from α, β, γ to X, Y, Z.

In order to transform the integral we must replace $d\alpha\, d\beta$ by

$$dX\,dY \begin{vmatrix} \dfrac{d\alpha}{dX}, & \dfrac{d\alpha}{dY} \\ \dfrac{d\beta}{dX}, & \dfrac{d\beta}{dY} \end{vmatrix} = dX\,dY \frac{d\alpha}{dX}\frac{d\beta}{dY},$$

since β is a function of Y only.

Now $\alpha = \sigma_2 \sqrt{Q/M}\, X + m_2 - \beta m_1$, where both Q and β can be expressed as a function of Y only. Accordingly

$$\frac{d\alpha}{dX} = \sigma_2 \sqrt{Q/M} \quad \text{.............................(xxxv)}.$$

Further since

$$Y = \frac{c^2}{c^2 + (\beta - \bar{\beta})^2},$$

$$\beta - \bar{\beta} = \pm c \sqrt{1/Y - 1},$$

$$\frac{d\beta}{dY} = \mp \tfrac{1}{2} c Y^{-\frac{3}{2}} (1 - Y)^{-\frac{1}{2}} \quad \text{.......................(xxxvi)},$$

where we may drop the double sign on the understanding that we have to double the result if we integrate out for Y. We will now write down the transformation of (xxvi) remembering that $1/Q = Y/(1 - \rho^2)$, and that

$$d\alpha\, d\beta = \frac{1}{2}\frac{1}{\sqrt{M}} \sigma_2 c \sqrt{1 - \rho^2}\, Y^{-2} (1 - Y)^{-\frac{1}{2}}\, dX\, dY,$$

$$z = \frac{N\sigma_1 (1 - \rho^2)^{\frac{1}{2}(M-2)} (M - 2) M}{2^{\frac{1}{2}(M-1)} \pi \sigma_2^3 \Gamma(\frac{1}{2}(M-1))} \frac{e^{-\frac{1}{2}X^2}}{\sqrt{M}}\, dX\, \tfrac{1}{2}\sigma_2 c \sqrt{1 - \rho^2} \left(\frac{Y}{1 - \rho^2}\right)^{\frac{1}{2}(M+1)}$$
$$\times Y^{-2} (1 - Y)^{-\frac{1}{2}}\, dY \frac{1}{\sqrt{2}} \frac{\sigma_2 \sqrt{1 - \rho^2}}{\sqrt{M}}\, 2^{\frac{1}{2}(M-3)} e^{-Z} Z^{\frac{1}{2}(M-4)}\, dZ,$$

or

$$\begin{aligned} z\,dX\,dY\,dZ &= \frac{N(M-2)}{2\sqrt{2\pi}\,\Gamma(\frac{1}{2}(M-1))} e^{-\frac{1}{2}X^2}\,\tfrac{1}{2}Y^{\frac{1}{2}(M-3)}(1-Y)^{-\frac{1}{2}}\,e^{-Z}Z^{\frac{1}{2}(M-4)}\,dX\,dY\,dZ \\ &= N\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}X^2}\,dX \times \frac{\frac{1}{2}Y^{\frac{1}{2}(M-3)}(1-Y)^{-\frac{1}{2}}\,dY}{B(\frac{1}{2}(M-1),\frac{1}{2})} \times \frac{e^{-Z}Z^{\frac{1}{2}(M-4)}}{\Gamma(\frac{1}{2}(M-2))}\,dZ \end{aligned} \quad \text{......(xxxvii).}$$

Thus our three-way surface in α, β, γ has been reduced to the product of three frequency curves independent of each other; namely a normal curve in X, a Type I curve in Y and a Type III curve in Z.

Let us now consider how we may proceed. If we want to know whether on the k observed series we may suppose the array standard deviations are constant we calculate k Z's, and ask if these Z's are a random selection from the Z curve. To do this we calculate their probability integrals by means of the *Tables of the Incomplete* Γ-*Function*, and then find the value of P_{λ_n} by aid of the same tables*.

If we wish to know whether the k series have the same regression coefficient, we calculate the values of Y (or, if we prefer, β) for these k series, and by aid of the *Tables of the Incomplete* B-*Function* find the values of the k corresponding probability integrals, then by the *Tables of the Incomplete* Γ-*Function* again find P_{λ_n}. This may be done without first supposing the k-array standard deviations to be the same. If we suppose both the array standard deviations and the regression coefficients to be the same, then we have $2k$ probability integrals, and we can determine P_{λ_n} for the $2k$ as one set, instead of two sets of k.

Lastly if we want the equations to the regression lines to be the same we have to deal with X, which first introduces α_t, and ask whether the k values of X form a random sample from the X normal curve. We calculate the probability integrals of our k values of X from the normal curve probability integral table† and we thus have three sets of k probability integrals, or one set of $3k$, to test by the P_{λ_n} method. The advantage of this method is that in each of the tests of k X's, Y's and Z's we are working with the actual curve of distribution, and not with an empirical curve fitted by two moments.

Now I have said above that we must compute the observed values of the X's, Y's and Z's, but this is not possible unless we have a knowledge of the constants of the k parent populations. These must be obtained in the customary manner by choosing them so as to make the probability of the observed results a maximum. I have expressed the opinion that on the assumptions of common array standard deviations and normal array distributions we lose little, if anything, in generality by assuming the k populations to be all normal distributions. The determination of the constancy of the array standard deviations indicates that we must maximise the product of the k Type III values to find s_2 or $\sigma_2\sqrt{1-\rho^2}/\sqrt{M}$. This constant is not to vary for the k populations. If we now desire further that the regression coefficients should be constant for the k populations, we may do it in two ways: (a) By supposing each population has a different $\bar{\sigma}_{1,t}$, in which case the c_t^2 are all different. But if we have fixed the array standard deviation and the regression

* Or by aid of Miss F. N. David's Table, *Biometrika*, Vol. xxvi. pp. 7—11.

† *Tables for Statisticians and Biometricians*, Part I. pp. 2—8.

coefficient $\bar{\beta}_i$ of all surfaces as the same, this is only rendered possible by assuming that for our k populations, $\bar{\sigma}^2_{2,i} \propto 1/(1-\rho_i^2)$. This I have termed the relation (v), and find difficult of acceptance: see p. 228. Turning to the other alternative: (b) By supposing $\bar{\sigma}_{2,i}$ and $\bar{\rho}_i$ to have the same value for all the populations; then the fixing of $\bar{\beta}_i$ demands that $\bar{\sigma}_{1,i}$ shall be the same for all the populations, and we have c_i fixed. In this latter case, since σ_1, σ_2 and ρ are fixed for all populations it follows that our k populations may be treated as described by a single surface moved parallel to itself, but having a different mean for each population. If we accept (a) then $\sigma^2_{2,i}(1-\rho_i^2) = \text{constant} = \sigma^2_{2,i} - \bar{\beta}_i^2\sigma^2_{1,i}$, and accordingly there is a hyperbolic relation between the two standard deviations—or a linear one between their variances—as singular and remarkable a relation among k populations as (v).

When we turn to the X distribution and consider the most likely values for $m_{1,i}$ and $m_{2,i}$ defining the x and y means of the ith population, we find, if there be only one array value sampled from that population, that the most likely values of $m_{1,i}$ and $m_{2,i}$ are not separately determinable, but that the best position for them is on the observed regression line of the ith population sample. This leads to $X_i = 0$, or the probability integral $= \cdot 5$ for all k samples. Accordingly we reach $P_{\lambda_k} = \Gamma(\cdot 693, 1472\sqrt{k}, k-1)$. For example: $P_{\lambda_4} = \cdot 3020$, $P_{\lambda_{25}} = \cdot 0489$, $P_{\lambda_{49}} = \cdot 0093$. Thus we require the number k of populations to be large before it indicates an improbable series of means when the mean of each population is placed in its most likely position, namely somewhere on the observed regression line of the sample. With a small number of populations, such as four, no discrimination against the hypothesis that they have independent means is likely to be provided by the X distribution. If we suppose that all the k populations may be represented by normal surfaces having the same σ_1, σ_2 and ρ, but differing means, we get no adequate test from the distribution of X, but if we suppose them to have the same mean, i.e. all the k samples to be drawn from the same normal surface, or possibly each sample from a different region of the same surface*, then we shall obtain a test from the X curve. We can again obtain a test from the X curve, if we break each i sample up into two or more subsamples, for in this case we have equations enough to determine $m_{1,i}$ and $m_{2,i}$. If we have s subsamples in the ith sample of the ith population, and we have already fixed c^2, $\bar{\beta}$ and the array variability, we shall have in order to determine $\bar{a}_i$ and $m_{1,i}$ to minimise the expression

$$\mathop{S}_{t=0}^{s} \frac{\{\alpha_{i,t} - \bar{a}_i + m_{1,i}(\beta_{i,t} - \bar{\beta})\}^2}{\sigma_1^2\{c^2 + (\beta_{i,t} - \bar{\beta})^2\}}$$

with regard to $\bar{a}_i$ and $m_{1,i}$, which leads to the equations

$$\left.\begin{aligned}
\bar{a}_i \mathop{S}_{t=1}^{s} \frac{1}{c^2 + (\beta_{i,t} - \bar{\beta})^2} - m_{1,i} \mathop{S}_{t=1}^{s} \frac{(\beta_{i,t} - \bar{\beta})}{c^2 + (\beta_{i,t} - \bar{\beta})^2} &= \mathop{S}_{t=1}^{s} \frac{\alpha_{i,t}}{c^2 + (\beta_{i,t} - \bar{\beta})^2} \\
\bar{a}_i \mathop{S}_{t=1}^{s} \frac{\beta_{i,t} - \bar{\beta}}{c^2 + (\beta_{i,t} - \bar{\beta})^2} - m_{1,i} \mathop{S}_{t=1}^{s} \frac{(\beta_{i,t} - \bar{\beta})^2}{c^2 + (\beta_{i,t} - \bar{\beta})^2} &= \mathop{S}_{t=1}^{s} \frac{\alpha_{i,t}(\beta_{i,t} - \bar{\beta})}{c^2 + (\beta_{i,t} - \bar{\beta})^2}
\end{aligned}\right\} \quad \ldots\text{(xxxviii)}.$$

* This is the condition (b) of p. 228.

Since $\bar{\alpha}_i = m_{2,i} - \bar{\beta} \times m_{1,i}$, we clearly have equations to determine $m_{1,i}$ and $m_{2,i}$ the means of the ith parent population. If there be only one array for the sample the sums reduce to a single term, the factor $\dfrac{1}{c^2 + (\beta_{i,1} - \bar{\beta})^2}$ divides out of the first equation and the factor $\dfrac{\beta_{i,1} - \bar{\beta}}{c^2 + (\beta_{i,1} - \bar{\beta})^2}$ from the second, and both reduce to

$$\bar{\alpha}_i - m_{1,i}(\beta_{i,1} - \bar{\beta}) = \alpha_{i,1},$$

or

$$m_{2,i} - m_{1,i}\bar{\beta} - m_{1,i}(\beta_{i,1} - \bar{\beta}) = \bar{y}_i - \bar{x}_i\beta_{i,1},$$

which is $m_{2,i} - \bar{y}_i = \beta_{i,1}(m_{1,i} - \bar{x}_i)$, and is equivalent to saying that the most probable values of $m_{1,i}$ and $m_{2,i}$ lie on the regression line of the sample, but it does not further define their position.

There are innumerable hypotheses we can make with regard to the constants $m_{1,i}$, $m_{2,i}$, $\sigma_{1,i}$, $\sigma_{2,i}$, ρ_i of the k surfaces. The method of testing them is the same in all cases, namely to deduce the most likely values of these constants by the usual method of maximising the total probability of the observed event, subject to the conditions imposed on the $5k$ constants, and then apply the P_{λ_n} test of reasonableness to the k values of X, Y and Z.

I will now proceed to illustrate these methods on the data provided by Mr Wilsdon of the Building Research Station, which were used by Mr Welch in his paper, and which Mr Wilsdon has most kindly permitted me also to use. The raw data will be given here as we need to table more constants than are published by Mr Welch. We agree absolutely with Mr Welch's values of β_i and have adopted his values of the array variances; all the other constants except the latter have been worked out by Miss David and myself. The populations each represent eight tests of the crushing strengths of concrete and mortar in lbs. per sq. inch; the individual results represent different cements used in making the mortar and the concrete, the same cement being used for both. The four groups ($i = 1, 2, 3, 4$) represent tests made when the age of mortar and concrete was one day, three days, seven days and twenty-eight days respectively. The regression line is for concrete (y), on mortar (x).

Crushing Strength ($i=1$)		Crushing Strength ($i=2$)		Crushing Strength ($i=3$)		Crushing Strength ($i=4$)	
Normal Concrete	Dry Mortar	Normal Concrete	Dry Mortar	Normal Concrete	Dry Mortar	Normal Concrete	Dry Mortar
868	2632	2608	5847	4252	7497	5568	8947
1738	4932	4455	8165	5678	9358	7298	10658
256	1368	1560	3242	3189	4533	5568	6318
1914	4772	4199	7715	5362	8927	7550	11000
724	2334	2363	4477	3448	5452	5013	7165
2190	5680	4667	7505	5755	7970	6903	8972
834	2304	2200	4783	3335	6314	5860	8458
1696	4021	3476	6809	4492	8256	6138	9548

From this table the following constants were deduced:

Constant	Sample ($i=1$)	Sample ($i=2$)	Sample ($i=3$)	Sample ($i=4$)
$\bar{y}_i$	1280	3101	4438·875	6244·375
$\bar{x}_i$	3505·375	6067·875	7288·375	8883·25
$\sigma_{2,i}$	944·5122	1092·0895	904·9920	860·6380
$\sigma_{1,i}$	1448·3186	1050·1133	1503·9983	1491·5566
r_i	·986,649	·965,638	·892,629	·837,205
β_i	·43907	·63677	·55719	·48307
α_i	−259·0895	−672·8414	+377·873	+1953·1193
$\gamma_i^2 = \sigma^2_{2,i}(1-r_i^2)$	3323	9541	7920	5926

As we have seen there are five independent constants for each population on the assumption that it may be treated as a normal population, namely, $m_{1,i}$, $m_{2,i}$, $\sigma_{1,i}$, $\sigma_{2,i}$, ρ_i. We may replace these by five equally independent constants, namely,

$$\bar{\beta}_i = \rho_i \sigma_{2,i}/\sigma_{1,i}, \quad c_i^2 = \sigma^2_{2,i}(1-\rho_i^2)/\sigma^2_{1,i}, \quad \bar{\gamma}_i^2 = \sigma^2_{2,i}(1-\rho_i^2), \quad m_{1,i},$$

and

$$\bar{a}_i = m_{2,i} - \bar{\beta}_i m_{1,i}.$$

Hence the distribution curve of γ_i's in the ith population is from (xxv)

$$z_i = \frac{\sqrt{M}}{\bar{\gamma}_i 2^{\frac{1}{2}(M-1)}\Gamma(\frac{1}{2}M-1)} e^{-\frac{1}{2}\frac{M\gamma_i^2}{\bar{\gamma}_i^2}} \left(\frac{\gamma_i\sqrt{M}}{\bar{\gamma}_i}\right)^{M-3} [d\gamma_i],$$

where the constants $\bar{\gamma}_i$ will *not* occur in any other of the three independent distributions. Thus in determining the most likely value of $\bar{\gamma}_i$ we need only consider this the Z distribution. Now the hypothesis we are to test is whether the array standard deviation is the same for all k populations, or $\bar{\gamma}_i = \bar{\gamma}$ a quantity independent of i. Accordingly we must make

$$P = \frac{(\sqrt{M})^k}{\bar{\gamma}^k} \frac{1}{2^{\frac{1}{2}k(M-4)}(\Gamma(\frac{1}{2}M-1))^k} e^{-\frac{M}{2}\overset{k}{\underset{1}{S}}\left(\frac{\gamma_i}{\bar{\gamma}}\right)^2} \underset{i=1 \text{ to } k}{\text{Product}} \left(\frac{\sqrt{M}\gamma_i}{\bar{\gamma}}\right)^{M-3} d\gamma_1 \ldots d\gamma_k$$

a maximum with regard to $\bar{\gamma}$.

Taking logarithmic differentials we have

$$-\frac{k}{\bar{\gamma}} + M \overset{k}{\underset{1}{S}} \frac{(\gamma_i^2)}{\bar{\gamma}^3} - \frac{k(M-3)}{\bar{\gamma}} = 0,$$

or

$$\bar{\gamma}^2 = \frac{M}{M-2} \overset{k}{\underset{1}{S}} \frac{(\gamma_i^2)}{k} \quad \text{..........................(xxxix)}.$$

Hence the most likely value of the array standard deviation is given by

$$\sigma^2_{2,i}(1-\rho_1^2) = \frac{M}{M-2} \quad (\text{mean value of } \gamma_i^2),$$

and the quantity we have called $s_{2,i}$ by

$$s^2_{2,i} = \frac{1}{M-2} \quad (\text{mean value of } \gamma_i^2) \quad \text{.....................(xl)}.$$

16—2

Here we are met by the old difficulty to which we have referred, namely, we have to suppose that

$$\sigma^2_{2,t} \propto \frac{1}{1-\rho_t^2} \quad \text{.................................(v).}$$

We prefer to take as our hypothesis that the k populations are all normal, and having the same σ_2 and ρ. This justifies the form we have assumed for the distribution of γ_t above, and we shall now show that the hypothesis is not contradicted by the Mortar-Cement data. We may also remark that there seems no greater reason in the present case for supposing that the crushing strength of normal concrete (y) on dry mortar (x) is given by a linear regression, than *vice-versa* that of x on y. Thus we may propound an additional argument in favour of the populations being normal, if any were required beyond the artificiality of assuming the truth of (v).

In Mr Welch's illustration $k=4$, and we have from the four tests the four array variances of:

$$\gamma_1^2 = 3323, \quad \gamma_2^2 = 9541, \quad \gamma_3^2 = 7920, \quad \gamma_4^2 = 5926,$$

giving a mean of $\gamma^2 = 6677{\cdot}5$ and by (xl) s_2^2 equal to $\frac{1}{6}$ of this value since each test consisted of $M=8$ individual cases. Accordingly $s_2^2 = 1112{\cdot}91\dot{6}$.

We have now before us the following problem: The most suitable value of s_2 being the root of $1112{\cdot}91\dot{6}$, for it gives the highest probability of the observed result, may the values observed of γ be taken as a reasonably random sample from the curve of distribution

$$z = \frac{1}{2^{\frac{1}{2}(M-4)}\,\Gamma(\frac{1}{2}M-1)}\, e^{-\frac{1}{2}\frac{\gamma^2}{s_2^2}} \left(\frac{\gamma}{s_2}\right)^{M-3} \left[d\left(\frac{\gamma}{s_2}\right)\right] ?$$

I know no better way of solving this problem than by the P_{λ_n} method, and we will proceed to calculate the probability integrals of the four γ's. Write

$$v_t = \frac{1}{2}\left(\frac{\gamma_t}{s_2}\right)^2,$$

and we find that the probability integral p_t becomes

$$p_t = \frac{1}{\Gamma(\frac{1}{2}M-1)} \int^{\frac{1}{2}\left(\frac{\gamma_t}{s_2}\right)^2} e^{-v_t} v_t^{\frac{1}{2}(M-4)}\, dv_t.$$

Hence in the notation of the *Tables of the Incomplete* Γ-*Function**, $p = \frac{1}{2}(M-4)$ and this equals 2, so that

$$p_t = I\left(\frac{1}{2}\left(\frac{\gamma_t^2}{s_2^2}\right)\frac{1}{\sqrt{p+1}}, p\right) = I\left(\frac{1}{2}\frac{\gamma_t^2}{s_2^2}\frac{1}{\sqrt{3}}, 2\right) \quad \text{..............(xli).}$$

Thus all we have to do is to calculate the four values of $\frac{1}{2\sqrt{3}}\frac{\gamma_t^2}{s_2^2}$ and interpolate into the above Tables. The values of

$$u_t = v_t \frac{1}{\sqrt{3}} = \frac{\gamma_t^2}{2225{\cdot}8333} \times {\cdot}577{,}3503$$

* Published by the *Biometrika* Office, University College, London.

are as follows:

$u_1 = \cdot 861{,}9401$, $u_2 = 2\cdot 474{,}8031$, $u_3 = 2\cdot 054{,}3382$, and $u_4 = 1\cdot 537{,}1222$.

From these by merely linear interpolation into the Γ-function table which will suffice practically we have, if $p' = 1 - p$,

$p_1 = \cdot 189{,}6896$, $\log p_1 = \bar{1}\cdot 278{,}0435$, $p_1' = \cdot 810{,}3104$, $\log p_1' = \bar{1}\cdot 908{,}6514$,
$p_2 = \cdot 800{,}7515$, $\log p_2 = \bar{1}\cdot 903{,}4977$, $p_2' = \cdot 199{,}2485$, $\log p_2' = \bar{1}\cdot 299{,}3952$,
$p_3 = \cdot 689{,}4926$, $\log p_3 = \bar{1}\cdot 838{,}5297$, $p_3' = \cdot 310{,}5074$, $\log p_3' = \bar{1}\cdot 492{,}0720$,
$p_4 = \cdot 496{,}8960$, $\log p_4 = \bar{1}\cdot 696{,}2655$, $p_4' = \cdot 503{,}1040$, $\log p_4' = \bar{1}\cdot 701{,}6578$.

Hence

$$\lambda_n = \log p_1 p_2 p_3 p_4 = -1\cdot 283{,}6636, \quad \lambda_n' = \log p_1' p_2' p_3' p_4' = -1\cdot 598{,}2236,$$

and $\dfrac{-\lambda_n}{\sqrt{n}\log_{10} e}$, since $n = 4$, gives us

$$-\frac{\lambda_n}{\cdot 868{,}5890} = 1\cdot 477{,}8725 \text{ and } -\frac{\lambda_n'}{\cdot 868{,}5890} = 1\cdot 840{,}0228,$$

leading to $P_{\lambda_n} = I(1\cdot 477{,}8725, 3)$, and $P_{\lambda_n'} = I(1\cdot 840{,}0228, 3)$,

or $P_{\lambda_n} = \cdot 34288$, $P_{\lambda_n'} = \cdot 50149$,

$$Q_{\lambda_n} = 1 - P_{\lambda_n} = \cdot 65712, \quad Q_{\lambda_n'} = P_{\lambda_n'} = \cdot 49851.$$

Thus whether we reckon our probability integrals from the left or right end of our frequency curve the same conclusion must be drawn, that the four array standard deviations form a most reasonable sample from a set of normal populations having a common array standard deviation

$$\sigma_2 \sqrt{1-\rho^2} = \sqrt{\frac{M}{M-2}}\,(\text{mean value of } \gamma_i^2)^{\frac{1}{2}} = \sqrt{8903\cdot 3333}$$
$$= 94\cdot 3575, \text{ with standard error } 23\cdot 1365,$$

as against the observed values, 57·6455, 97·6780, 88·9944 and 76·9805. The only deviation claiming special consideration is the first, which is about 1·59 times the standard error, *and a deviation as small as or smaller than this would occur in* about 19 in 100 trials*, or about ·76 times in four trials; it occurs once. Mr Welch gives a far higher improbability, = ·0126, for the four values being a random sample from his curve of distribution of the γ's. I think the difference arises from our methods. I have given, on the assumption that the surfaces are normal, what I think is the real curve of distribution of the γ's from which a little sample of four has been taken. Mr Welch obtains a compound criterion and selects a curve of my Type I from the first two moments to describe the variation of his criterion. While I believe for considerable material this method might give adequate results, I have grave doubts whether, if there really exist only four observed values, it may not possibly be misleading. At any rate, if my arithmetic be correct, I cannot raise objection to these observed values being a random sample from four *normal* frequency surfaces giving exact curves of distribution, and I will start on the assumption that they do.

* See p_1 above.

We now pass to the question whether if $\sigma_{2,t}\sqrt{1-\rho_t^2}$ be treated as a constant for the arrays of all the k populations, the values of $R_2 = \rho_t\sigma_{2,t}/\sigma_{1,t}$ can be considered as a random sample from a series $\Pi_1, \Pi_2, \ldots \Pi_k$ of normal populations having the same ρ_t, $\sigma_{2,t}$, but not necessarily the same $\sigma_{1,t}$, $\bar{x}_t$ and $\bar{y}_t$, i.e. the α_t's and $\bar{x}_t$'s are not necessarily the same. In order to get rid of α_t we must start from the distribution of R_2 or β_t in each of the individual populations. This will be of the form (xxi), or

$$z = \frac{z_0}{\left\{\dfrac{\sigma^2_{2,t}}{\sigma^2_{1,t}}(1-\rho_t^2) + \left(\beta_t - \rho_t\dfrac{\sigma_{2,t}}{\sigma_{1,t}}\right)^2\right\}^{\frac{1}{2}M}} \quad \text{..................(xlii).}$$

Here the quantity $\sigma^2_{2,t}(1-\rho_t^2)$ has already been supposed constant $= \bar{\gamma}$ for all populations. Our hypothesis to be tested is now that $\rho_t\dfrac{\sigma_{2,t}}{\sigma_{1,t}}$ is constant $= \bar{\beta}$ for all populations. Putting $c_t^2 = \dfrac{\sigma^2_{2,t}(1-\rho_2^2)}{\sigma^2_{1,t}} = \dfrac{\bar{\gamma}^2}{\sigma^2_{1,t}}$, we can suppose either (a) that c_t^2 varies from population to population, or (b) that $c_t^2 = c^2 =$ a constant for all populations. Let us see whether one or other of these hypotheses is the more probable. We may write (xlii) in the form

$$z_t = \frac{Nc_t^{M-1}\Gamma(\frac{1}{2}M)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}(M-1))}\frac{1}{\{c_t^2 + (\beta_t - \bar{\beta})^2\}^{\frac{1}{2}M}} \quad \text{..............(xlii}^{bis}\text{).}$$

(a) Since c_t only occurs in z_t the equation to determine its most likely value is

$$\frac{M-1}{2c_t^2} = \tfrac{1}{2}M\frac{1}{c_t^2 + (\beta_t - \bar{\beta})^2}, \quad \text{or} \quad c_t^2 = (M-1)(\beta_t - \bar{\beta})^2 \quad \text{......(xliii),}$$

and for $\bar{\beta}$ occurring in all the k populations

$$\mathop{S}_{t=0}^{t=k}\frac{\beta_t - \bar{\beta}}{c_t^2 + (\beta_t - \bar{\beta})^2} = 0 \quad \text{........................(xliv),}$$

which by (xliii) gives, for the determination of $\bar{\beta}$,

$$\mathop{S}_{t=0}^{t=k}\frac{1}{\beta_t - \bar{\beta}} = 0 \quad \text{...............................(xlv).}$$

Assume $\bar{\beta}_0$ an approximate value of $\bar{\beta}$ and let $\bar{\beta} = \bar{\beta}_0 + \epsilon$, where ϵ is supposed small, then we find

$$\epsilon = -\mathop{S}_{t=0}^{t=k}\frac{1}{\beta_t - \bar{\beta}_0}\bigg/\mathop{S}_{t=0}^{t=k}\frac{1}{(\beta_t - \bar{\beta}_0)^2} \quad \text{....................(xlvi).}$$

A good taking-off point is to assume $\bar{\beta}_0$ to be the mean value of the k β_t's. This in our example is ·529,025, and substituting the values of β_t in (xlvi) we obtain $\epsilon = -$·006,126. Thus $\bar{\beta} = \bar{\beta}_0 + \epsilon =$ ·522,899, not a very great change on $\bar{\beta}_0$, and for practical purposes needing no further approximation. We have now to consider the values taken by

$$Y_t = \frac{c_t^2}{c_t^2 + (\beta_t - \bar{\beta})^2},$$

but this equals in every case $(M-1)/M$. Accordingly we have, since for $i=1$ and 4, $\beta_i - \bar{\beta}$ is negative, and for $i=2$ and 4, $\beta_i - \bar{\beta}$ is positive,

$$p_1 = p_4 = \tfrac{1}{2} I_{(M-1)/M}(3\cdot 5, \tfrac{1}{2}),$$

and
$$p_2 = p_3 = 1 - \tfrac{1}{2} I_{(M-1)/M}(3\cdot 5, \tfrac{1}{2}),$$

as the probability integrals derived from the Type III curve

$$z = z_0 Y_t^{\frac{1}{2}(M-3)} (1 - Y_t)^{-\frac{1}{2}},$$

$I_x(p, q)$ being the incomplete Beta function. Looking up these values in the Table of that function, we find for $(M-1)/M = \cdot 875$,

$$p_1 = p_4 = \cdot 175{,}3968, \text{ and } p_2 = p_3 = \cdot 824{,}6032.$$

It follows that
$$\lambda_n = \log(p_1 p_2 p_3 p_4) = -1\cdot 679{,}4666$$

and
$$-\frac{\lambda_n}{\sqrt{n}\log_{10} e}, \text{ since } n=4, = 1\cdot 932{,}176.$$

Accordingly
$$P_{\lambda_n} = I(1\cdot 932{,}176, 3) = \cdot 53949$$

and
$$Q_{\lambda_n} = \cdot 46051.$$

There is no improbability therefore in supposing each of our four sets drawn from populations having the same regression coefficient, but different values of the quantity $c_t^2 = \sigma_2^2 (1-\rho^2)/\sigma^2_{1,t}$.

It may be asked why we have not included the c_t^2 and $\bar{\beta}$ terms in X, when maximising the probability of the total observed data. The answer is that the maximising of X with regard to $\bar{a}_t$ and $m_{1,t}$ causes the vanishing of the factor $\{a_t - \bar{a}_t + m_{1,t}(\beta_t - \bar{\beta})\}^2$ in X and so the other factor containing c_t^2 and $\bar{\beta}$ disappears with it (see our p. 241).

(b) Now on the other hypothesis we have for the four c_t^2's

$$c_1^2 = \cdot 056{,}643, \quad c_2^2 = \cdot 081{,}263, \quad c_3^2 = \cdot 005{,}553, \text{ and } c_4^2 = \cdot 014{,}783,$$

and these c_t^2's do not at first sight look like the result of random sampling from a common value. But if they were, this would mean that $\sigma_{1,t}$ would be the same for all four populations. The $\sigma_{1,t}$ values are given in the Table on p. 243, and *à priori* they do not look so improbable as a selection of four values of a standard deviation from four bivariate normal surfaces with the same σ_1, that is to say four random values taken from the frequency curve

$$z = \frac{N\sqrt{M}}{\sigma_1 2^{\frac{1}{2}(M-3)} \Gamma(\frac{1}{2}(M-1))} e^{-\frac{1}{2}M\left(\frac{\sigma_{1,t}}{\sigma_1}\right)^2} \left(\frac{\sqrt{M}\sigma_{1,t}}{\sigma_1}\right)^{M-2} [d\sigma_{1,t}] \quad \ldots\ldots\text{(xlvii)}.$$

This leads us to the most likely value $\bar{\sigma}_1^2$ for σ_1^2, namely,

$$\bar{\sigma}_1^2 = \frac{M}{M-1} \mathop{S}_{t=1}^{t=k} \frac{(\sigma^2_{1,t})}{k} = \frac{M}{M-1} \text{ (mean value of observed } x \text{ variances)}$$
$$= 1656\cdot 6673 \text{ for our data, giving } \bar{\sigma}_1 = 407\cdot 0218.$$

Taking
$$v_t = \tfrac{1}{2} M \left(\frac{\sigma_{1,t}}{\sigma_1}\right)^2,$$
we find
$$p_t = I\left(\frac{v_t}{\sqrt{\frac{1}{2}(M-1)}}, \tfrac{1}{2}(M-3)\right) = I\left(\frac{v_t}{\sqrt{3\cdot 5}}, 2\cdot 5\right).$$
Hence we obtain
$$p_1 = \cdot 473{,}364, \quad p_1' = \cdot 526{,}636,$$
$$p_2 = \cdot 666{,}665, \quad p_2' = \cdot 333{,}335,$$
$$p_3 = \cdot 611{,}974, \quad p_3' = \cdot 388{,}026,$$
$$p_4 = \cdot 515{,}390, \quad p_4' = \cdot 484{,}610,$$
which give us
$$\lambda_n = -1\cdot 002{,}028, \quad \lambda_n' = -1\cdot 481{,}355,$$
and
$$P_{\lambda_n} = \cdot 2013, \quad P_{\lambda_n'} = \cdot 4439,$$
$$Q_{\lambda_n} = \cdot 7987, \quad Q_{\lambda_n'} = \cdot 5561.$$

Thus, according to the P_{λ_n} criterion, there is no improbability in the four values of $\sigma_{1,t}$ being random samples from parent populations having a common $\sigma_1 = 1656\cdot 6673$.

Returning, however, to our different c_t^2's and accordingly our different $\sigma^2_{1,t}$'s we have to find for those c_t^2's and the corresponding $\bar{\beta}$ ($= \cdot 522{,}899$), whether the resulting four Y_t's given by
$$Y_t = \frac{c_t^2}{c_t^2 + (\beta_t - \bar{\beta})^2}$$
provide an improbable set of probability integrals. We have just seen (p. 247) that they give
$$P_{\lambda_n} = P_{\lambda_n'} = \cdot 5395, \quad Q_{\lambda_n} = Q_{\lambda_n'} = \cdot 4605.$$

Comparing the latter with $Q_{\lambda_n'} = \cdot 5561$ as the more stringent series above, we see that to suppose the four surfaces to have different $\sigma_{1,t}$'s is a somewhat more probable hypothesis than supposing them to have a common σ_1, but the probability of the latter is so considerable that it certainly would not prevent us from accepting the hypothesis that all four populations have the same σ_1, σ_2 and ρ.

It may be objected, however, that for the proper determination of σ_1 we ought to have taken into consideration the general effect of fixing c and $\bar{\beta}$, and that we ought in determining σ_1 to proceed by maximising first the combined probability with regard to c and $\bar{\beta}$. This we will now proceed to do. The curve for the Y variate for a single population is, if as before, $c^2 = \sigma_2^2(1-\rho^2)/\sigma_1^2$ and $\bar{\beta} = \rho\sigma_2/\sigma_1$,
$$z = \frac{Nc^{M-1}\Gamma(\frac{1}{2}M)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}(M-1))} \frac{1}{\{c^2 + (\beta_t - \bar{\beta})^2\}^{\frac{1}{2}M}} \quad \text{..............(xlviii)},$$
and accordingly the probability to be maximised is
$$P = \frac{c^{k(M-1)}}{\pi^{\frac{1}{2}k}} \frac{(\Gamma(\frac{1}{2}M))^k}{\{\Gamma(\frac{1}{2}(M-1))\}^k} \overset{k}{\underset{1}{P}} \frac{1}{\{c^2 + (\beta_t - \bar{\beta})^2\}^{\frac{1}{2}M}};$$

hence differentiating with regard to c and $\bar{\beta}$ we find

$$\frac{k(M-1)}{c} - Mc \overset{k}{\underset{1}{S}} \frac{1}{\{c^2 + (\beta_i - \bar{\beta})^2\}},$$

or
$$\frac{M-1}{M}\frac{1}{c^2} = \frac{1}{k}\overset{k}{\underset{1}{S}} \frac{1}{\{c^2 + (\beta_i - \bar{\beta})^2\}} \quad\text{.....................(xlix)},$$

and
$$\overset{k}{\underset{1}{S}}\left(\frac{\beta_i - \bar{\beta}}{c^2 + (\beta_i - \bar{\beta})^2}\right) = 0 \quad\text{.............................(l)}.$$

These are the equations from which c^2 and $\bar{\beta}$ are to be found. Clearly if M be large and β_i not very different from $\bar{\beta}$, that is σ_β small (see p. 246), we can start with $\bar{\beta}$ = mean β_i as a first approximation.

Again, if M be large then (xlviii) approaches closely a normal curve, and the constants c^2 and $\bar{\beta}$ can be found directly without approximation*. When M is very small, as in the present case, we can only proceed by successive approximation. Let us suppose approximations c_0^2 and $\bar{\beta}_0$ have been found to c^2 and $\bar{\beta}$, and that $c_0^2 + \zeta = c^2$ and $\bar{\beta}_0 + \xi = \bar{\beta}$, where ξ and ζ are small quantities. Neglecting their squares, our equations for finding ξ and ζ are from (xlii^{bis})

$$\overset{k}{\underset{1}{S}} \frac{(\beta_i - \bar{\beta}_0)}{c_0^2 + (\beta_i - \bar{\beta}_0)^2} = \xi \overset{k}{\underset{1}{S}} \frac{c_0^2 - (\beta_i - \bar{\beta}_0)^2}{\{c_0^2 + (\beta_i - \bar{\beta}_0)^2\}^2} + \zeta \overset{k}{\underset{1}{S}} \frac{(\beta_i - \bar{\beta}_0)}{\{c_0^2 + (\beta_i - \bar{\beta}_0)^2\}^2} \quad\text{......(li)},$$

$$k\frac{M-1}{M} - \overset{k}{\underset{1}{S}} \frac{c_0^2}{c_0^2 + (\beta_i - \bar{\beta}_0)^2} = \xi \overset{k}{\underset{1}{S}} \frac{2c_0^2(\beta_i - \bar{\beta}_0)}{\{c_0^2 + (\beta_i - \bar{\beta}_0)^2\}^2} + \zeta \overset{k}{\underset{1}{S}} \frac{(\beta_i - \bar{\beta}_0)^2}{\{c_0^2 + (\beta_i - \bar{\beta}_0)^2\}^2} \quad\text{...(lii)}.$$

In our case, $k = 4$, $M = 8$, and the mean value of the four observed β_i's, i.e. $\beta_1 = \cdot 43907$, $\beta_2 = \cdot 63677$, $\beta_3 = \cdot 55719$ and $\beta_4 = \cdot 48307$, is $\cdot 529025$. We will start then with $\bar{\beta}_0 = \cdot 53$, which nearly corresponds to this value, and $c_0^2 = \cdot 04$, and compute our coefficients in (li) and (lii). We have

$$\overset{k}{\underset{1}{S}} \frac{(\beta_i - \bar{\beta}_0)}{c_0^2 + (\beta_i - \bar{\beta}_0)^2} = -\cdot 2512{,}0918, \quad \overset{k}{\underset{1}{S}} \frac{c_0^2}{\{c_0^2 + (\beta_i - \bar{\beta}_0)^2\}^2} = 78\cdot 8686{,}5354,$$

$$\overset{k}{\underset{1}{S}} \frac{(\beta_i - \bar{\beta}_0)^2}{\{c_0^2 + (\beta_i - \bar{\beta}_0)^2\}^2} = 9\cdot 5458{,}5438, \quad \overset{k}{\underset{1}{S}} \frac{(\beta_i - \bar{\beta}_0)}{\{c_0^2 + (\beta_i - \bar{\beta}_0)^2\}^2} = -8\cdot 5823{,}6398,$$

$$\overset{k}{\underset{1}{S}} \frac{c_0^2}{c_0^2 + (\beta_i - \bar{\beta}_0)^2} = 3\cdot 5365{,}8033, \quad k\frac{M-1}{M} = 3\cdot 5.$$

Thus there result the numerical equations for ξ, ζ,

$$-\cdot 2512{,}0918 = 69\cdot 3227{,}9916\,\xi - 8\cdot 5823{,}6398\,\zeta,$$
$$-\cdot 0365{,}8033 = -\cdot 0865{,}8912\,\xi + 9\cdot 5458{,}5438\,\zeta,$$

leading to
$$\xi = -\cdot 0041{,}3500, \quad \zeta = -\cdot 0041{,}2948,$$

and thus to a second approximation

$$\bar{\beta} = \bar{\beta}_0 + \xi = \cdot 53 - \cdot 004{,}135 = \cdot 525{,}865,$$
$$c^2 = c_0^2 + \zeta = \cdot 04 - \cdot 004{,}129 = \cdot 035{,}871.$$

* In the *Annals of Eugenics*, Vol. v. p. 408 *et seq.* I have dealt with a case of M large, but varying in magnitude from one population to a second, replacing (xlviii) by a normal curve.

It would be easy to improve these slightly by a further approximation, but they are adequate for our purpose. They show that the normal populations from which we suppose the samples $\Pi_1, \Pi_2, \dots \Pi_4$ selected have a regression coefficient in common most probably of the value ·525,865, which is strikingly near the value found for $\bar{\beta}$ on the hypothesis that the populations have independent $\sigma^2_{1_t}$'s—i.e. ·522,899. Further these populations have, as we have seen, the same array variance $\sigma_2^2(1-\rho^2) = 8903{\cdot}3333$ (see p. 245), but the value of

$$c^2 = \frac{\sigma_2^2(1-\rho^2)}{\sigma_1^2} = \frac{8903{\cdot}3333}{\sigma_1^2} = {\cdot}035{,}871.$$

Accordingly $\qquad \sigma_1^2 = \dfrac{8903{\cdot}3333}{{\cdot}035{,}871} = 248{,}204{\cdot}21,$

or $\qquad \sigma_1 = 498{\cdot}201$*.

Now this result is of considerable interest. It is of course not the standard deviation of the values of x considered in the experiments; it is the most likely value of the standard deviation of x in the several normal surfaces with fixed c and $\bar{\beta}$ from which the observations are supposed drawn. It seems to me that σ_1 has a most probable value, the moment we drop the unacceptable (v). But we can now proceed further to find the probable values of σ_2 and ρ. We have

$$c^2 = \frac{\sigma_2^2}{\sigma_1^2} - \rho^2\frac{\sigma_2^2}{\sigma_1^2} = \frac{\sigma_2^2}{\sigma_1^2} - \bar{\beta}^2,$$

or $\qquad \sigma_2^2 = \sigma_1^2(c^2 + \bar{\beta}^2) = 248{,}204{\cdot}21\,({\cdot}035{,}871 + {\cdot}276{,}534) = 77540{\cdot}2362,$

and $\qquad \sigma_2 = 278{\cdot}460{,}475.$

Accordingly

$$\rho = \bar{\beta} \times \sigma_1/\sigma_2 = {\cdot}525{,}865 \times 498{\cdot}201/278{\cdot}460{,}475 = {\cdot}94084$$

gives the likely correlation.

We have now ascertained the σ_1, σ_2 and ρ of the most probable surfaces from which the regression lines have been ascertained. The next step is to consider whether β_1, β_2, β_3 and β_4 are probable values to be drawn from the now known curve of distribution of β. This curve is given in (xlviii), where we can insert the probable values of c^2 and β. The values of $\beta_t - \bar{\beta}$ are $\beta_1 - \bar{\beta} = -{\cdot}086{,}795$, $\beta_2 - \bar{\beta} = +{\cdot}110{,}905$, $\beta_3 - \bar{\beta} = +{\cdot}031{,}335$, $\beta_4 - \bar{\beta} = -{\cdot}042{,}795$.

From these and $c^2 = {\cdot}035{,}871$, we have to calculate

$$v_t = \frac{(\beta_t - \bar{\beta})^2}{c^2 + (\beta_t - \bar{\beta})^2} \quad \dots\dots\dots\dots\dots\dots\dots\dots \text{(liii)},$$

giving $\qquad v_1 = {\cdot}173{,}563, \quad v_2 = {\cdot}255{,}339, \quad v_3 = {\cdot}026{,}627, \quad v_4 = {\cdot}048{,}575.$

By the above transformation of β_t to v_t, the proportional area of the distribution curve of β up to the value β_t from left to right, or the probability integral

$$p_t = \tfrac{1}{2}\{1 \pm I_{v_t}(\tfrac{1}{2}, 3{\cdot}5)\},$$

* The value of σ_1 without any constraint on the four populations to have the same $\bar{\beta}$ and c was 407·0218, which marks the considerable influence of the restraints (see p. 247).

where the plus sign corresponds to $\beta_i > \bar{\beta}$ and the minus sign to $\beta_i < \bar{\beta}$. These values of the incomplete Beta function must be found from the *Tables of the Incomplete Beta Function*, using as sufficient for the present purpose linear interpolation. But the Tables give $I_x(3{\cdot}5, \frac{1}{2})$, so that we must write

$$I_{v_i}(\tfrac{1}{2}, 3{\cdot}5) = {\cdot}1 - I_{1-v_i}(3{\cdot}5, \tfrac{1}{2}),$$

or

$$p_i = 1 - \tfrac{1}{2} I_{1-v_i}(3{\cdot}5, \tfrac{1}{2}), \quad \text{if } \beta_i > \bar{\beta},$$
$$= \tfrac{1}{2} I_{1-v_i}(3{\cdot}5, \tfrac{1}{2}), \quad \text{if } \beta_i < \bar{\beta}.$$

We deduce accordingly the following values:

$$p_1 = \tfrac{1}{2} I_{\cdot 820,502}(3{\cdot}5, 0{\cdot}5) = {\cdot}13238, \quad p_1' = {\cdot}86762;$$
$$p_2 = 1 - \tfrac{1}{2} I_{\cdot 744,661}(3{\cdot}5, 0{\cdot}5) = {\cdot}91734, \quad p_2' = {\cdot}08266;$$
$$p_3 = 1 - \tfrac{1}{2} I_{\cdot 973,373}(3{\cdot}5, 0{\cdot}5) = {\cdot}66182, \quad p_3' = {\cdot}33818;$$
$$p_4 = \tfrac{1}{2} I_{\cdot 951,425}(3{\cdot}5, 0{\cdot}5) = {\cdot}28457, \quad p_4' = {\cdot}71543;$$

$$\lambda_n = -\log(p_1 p_2 p_3 p_4) = -1{\cdot}640{,}7183, \quad \lambda_n' = -\log(p_1' p_2' p_3' p_4') = -1{\cdot}760{,}6600.$$

Dividing these values by $\sqrt{n} \log_{10} e = {\cdot}868{,}5890$, we have

$$P_{\lambda_n} = I(1{\cdot}888{,}947, 3), \qquad P_{\lambda_n'} = I(2{\cdot}027{,}035, 3),$$

whence from the *Tables of the Incomplete Γ-Function* we deduce by linear interpolation as adequate for our present purposes

$$P_{\lambda_n} = {\cdot}5219, \qquad P_{\lambda_n'} = {\cdot}5768,$$
$$Q_{\lambda_n} = 1 - P_{\lambda_n} = {\cdot}4781, \qquad Q_{\lambda_n'} = 1 - P_{\lambda_n'} = {\cdot}4232.$$

It is clear from these results that the four values of the regression coefficient provided might easily have been randomly selected from the distribution curve of samples of 8 drawn from normal bivariate surfaces differing only in their means. But we can hardly draw the inference that the distributions of the populations Π_1, Π_2, Π_3, Π_4 *are* accordingly normal, because the array standard deviations and the regression coefficients are capable of being considered random samples from the corresponding appropriate Type III and Type I curves. They, consisting of only *four* values each, may be shown to be equally random samples from normal curves or rectangles. But the method which, I think, would be legitimate when applied to larger samples can be illustrated on this example.

We now turn back to formula (xxvii) for the bivariate surface of α and β, namely,

$$z = \frac{N\sigma_1(1-\rho^2)^{\frac{1}{2}(M-1)}\,\Gamma(\frac{1}{2}M)\sqrt{M}}{\sqrt{2}\,\pi\sigma_2^2\,\Gamma(\frac{1}{2}(M-1))} e^{-\frac{1}{2}\frac{(\alpha - m_2 + \beta m_1)^2}{\sigma_2^2 Q/M}} \frac{1}{Q^{\frac{1}{2}(M+1)}},$$

and

$$Q = 1 - \frac{2\rho\beta\sigma_1}{\sigma_2} + \beta^2 \frac{\sigma_1^2}{\sigma_2},$$

and the subscript i should be attached to all the letters when we wish to emphasise that we are dealing solely with the parent population Π_i.

Now we have seen how to test the hypothesis, that the array standard deviations for the populations are the same, and further the additional hypothesis that their regression coefficients are all the same, i.e. that their regression lines are all parallel. To find the probable means of each population, we should have to maximise the only term in which $m_{1,i}$ and $m_{2,i}$ occur, namely,

$$e^{-\frac{1}{2}\frac{(a_i - m_{2,i} + \beta_i m_{1,i})^2}{\sigma^2_{2,1} Q_i/M}}.$$

with regard to $m_{1,i}$ and $m_{2,i}$. But these give

$$\beta_i(a_i - m_{2,i} + \beta_i m_{1,i}) = 0 \text{ and } a_i - m_{2i} + \beta_i m_{1i} = 0.$$

These, supposing β_i to be finite, reduce to the one equation

$$a_i - m_{2,i} + \beta_i m_{1,i} = 0,$$

and tell us that the position of the mean of the individual population is not fixed, but its most probable position is on the observational regression line of the ith sample. If, on the other hand, we suppose not a series of surfaces with the same σ_1, σ_2, and ρ and variable means, but one surface with one mean m_1, m_2, then we have to minimise

$$\overset{k}{\underset{1}{S}} \frac{(a_i - m_2 + \beta_i m_1)^2}{\sigma_2^2 Q_i} = \overset{k}{\underset{1}{S}} (X_i)^2/M.$$

We will put this into another form. Let $\bar{a} = m_2 + \bar{\beta} m_1$, and

$$\sigma_2^2 Q_i = \sigma_1^2 \left\{\frac{\sigma_2^2}{\sigma_1^2}(1-\rho^2) + (\beta_i - \bar{\beta})^2\right\} = \sigma_1^2 \{c^2 + (\beta_i - \bar{\beta})^2\}.$$

Thus we have to minimise

$$\overset{k}{\underset{1}{S}} \left(\frac{(a_i - \bar{a} + m_1(\beta_i - \bar{\beta}))^2}{c^2 + (\beta_i - \bar{\beta})^2}\right)$$

by differentiation with regard to $\bar{a}$ and m_1. We at once obtain the equations

$$\left.\begin{aligned} \bar{a}\overset{k}{\underset{1}{S}} \frac{1}{c^2 + (\beta_i - \bar{\beta})^2} - m_1 \overset{k}{\underset{1}{S}} \frac{\beta_i - \bar{\beta}}{c^2 + (\beta_i - \bar{\beta})^2} &= \overset{k}{\underset{1}{S}} \frac{a_i}{c^2 + (\beta_i - \bar{\beta})^2} \\ \bar{a}\overset{k}{\underset{1}{S}} \frac{\beta_i - \bar{\beta}}{c^2 + (\beta_i - \bar{\beta})^2} - m_1 \overset{k}{\underset{1}{S}} \frac{(\beta_i - \bar{\beta})^2}{c^2 + (\beta_i - \bar{\beta})^2} &= \overset{k}{\underset{1}{S}} \frac{a_i(\beta_i - \bar{\beta})}{c^2 + (\beta_i - \bar{\beta})^2} \end{aligned}\right\} \quad \text{......(liv)},$$

to find $\bar{a}$ and m_1, or, of course, m_2 and m_1. Actually we ought to substitute in the above equations the value of c and $\bar{\beta}$ found from differentiating these quantities as they appear not only in Y but in X also. But the resulting equations then involve four variables $\bar{a}$, m_1, c and $\bar{\beta}$ to be solved by approximation, and we will merely ask whether the likely values found for c and $\bar{\beta}$, when the populations have independent means, are compatible with a common mean.

Collecting the data we require, we have

$$c^2 = \cdot 035{,}871, \qquad \bar{\beta} = \cdot 525{,}865, \qquad \sigma_1 = 498 \cdot 201.$$

$$\beta_1 = \cdot 43907, \qquad \beta_2 = \cdot 63677, \qquad \beta_3 = \cdot 55719, \qquad \beta_4 = \cdot 48307.$$

$$\beta_1 - \bar{\beta} = - \cdot 086{,}795, \quad \beta_2 - \bar{\beta} = + \cdot 110{,}905, \quad \beta_3 - \bar{\beta} = + \cdot 031{,}325, \quad \beta_4 - \bar{\beta} = - \cdot 042{,}795.$$

$$(\beta_1 - \bar{\beta})^2 = \cdot 0075{,}3337, \quad (\beta_2 - \bar{\beta})^2 = \cdot 0122{,}9992, \quad (\beta_3 - \bar{\beta})^2 = \cdot 0009{,}8126, \quad (\beta_4 - \bar{\beta})^2 = \cdot 0018{,}3141.$$

Hence

	$(i=1)$	$(i=2)$	$(i=3)$	$(i=4)$
$c^2+(\beta_i-\breve{\beta})^2$	$\cdot 0434{,}0437$,	$\cdot 0481{,}7092$,	$\cdot 0368{,}5226$,	$\cdot 0377{,}0241$,
$\dfrac{1}{c^2+(\beta_i-\breve{\beta})^2}$	$23\cdot 039{,}154$,	$20\cdot 759{,}413$,	$27\cdot 135{,}381$,	$26\cdot 523{,}502$,
$\dfrac{\beta_i-\breve{\beta}}{c^2+(\beta_i-\breve{\beta})^2}$	$-1\cdot 9996{,}8337$,	$+2\cdot 3023{,}2270$,	$+\cdot 8500{,}1581$,	$-1\cdot 1350{,}7327$,
α_i	$-259\cdot 0895$,	$-672\cdot 8414$,	$+377\cdot 8726$,	$+1953\cdot 1193$,
$\dfrac{\alpha_i}{c^2+(\beta_i-\breve{\beta})^2}$	$-5969\cdot 2029$,	$-13967\cdot 7925$,	$+10253\cdot 7170$,	$+51803\cdot 5637$,
$\dfrac{(\beta_i-\breve{\beta})^2}{c^2+(\beta_i-\breve{\beta})^2}$	$+\cdot 1735{,}6247$,	$+\cdot 2553{,}3912$,	$+\cdot 0266{,}2686$,	$+\cdot 0485{,}7541$,
$\dfrac{\alpha_i(\beta_i-\breve{\beta})}{c^2+(\beta_i-\breve{\beta})^2}$	$+518\cdot 00696$,	$-1549\cdot 09803$,	$+321\cdot 19768$,	$-2216\cdot 93351$.

Thus we obtain the summations, namely,

$$\overset{k}{\underset{1}{S}}\frac{1}{c^2+(\beta_i-\breve{\beta})^2}=97\cdot 457{,}450,\quad \overset{k}{\underset{1}{S}}\frac{\beta_i-\breve{\beta}}{c^2+(\beta_i-\breve{\beta})^2}=\cdot 0175{,}8187,\quad \overset{k}{\underset{1}{S}}\frac{(\beta_i-\breve{\beta})^2}{c^2+(\beta_i-\breve{\beta})^2}=\cdot 5041{,}0386,$$

$$\overset{k}{\underset{1}{S}}\frac{\alpha_i}{c^2+(\beta_i-\breve{\beta})^2}=+42120\cdot 2853,\quad \overset{k}{\underset{1}{S}}\frac{\alpha_i(\beta_i-\breve{\beta})}{c^2+(\beta_i-\breve{\beta})^2}=-2926\cdot 73690,$$

and our equations for $\breve{a}$ and m_1 take the forms

$$\left.\begin{array}{l} 97\cdot 457{,}450\,\breve{a}-\cdot 0175{,}8187\,m_1=\quad 42120\cdot 2853 \\ \cdot 0175{,}8187\,\breve{a}-\cdot 5041{,}0386\,m_1=-2926\cdot 73690 \end{array}\right\}\quad \ldots\ldots\ldots\ldots(\text{lv}).$$

Solving these equations, we find*

$$\breve{a}=433\cdot 2416,\quad m_1=5820\cdot 9316,\quad m_2=\breve{a}+m_1\breve{\beta}=3494\cdot 2658.$$

If we examine the Table on p. 243 these values do not look unreasonable. But the problem is: Will they give reasonable values for X_i? Accordingly we proceed to compute X_i by insertion of $\breve{a}$, m_2, c, $\breve{\beta}$, σ_1 in (xxxiv). We find

$$X_1=-32\cdot 6340,\quad X_2=-11\cdot 9121,\quad X_3=+3\cdot 7550,\quad X_4=+37\cdot 1555.$$

It is idle to work out P_{λ_n} for four such values supposed to be selected at random from a normal curve of standard deviation unity! The probability is enormously against them.

We conclude confidently that the four sets although they might be samples from a single *shifted* normal population could not possibly be drawn from the same normal population fixed in position. Thus (*b*) of p. 228 is absolutely excluded in this case.

* I have endeavoured in the course of this paper to give some evidence of the nature and amount of arithmetical work required, as this seems to me an essential need for judgment of the value of any statistical process.

We may now proceed to further hypotheses. We may ask, admitting that the four samples could have been drawn from the same normal surface with the values of σ_1, c and $\bar{\beta}$ determined on pp. 245 and 249, shifted parallel to itself, whether the means of these surfaces $m_{1,i}$, $m_{2,i}$ could be reasonably supposed to lie on one and the same straight line

$$y = L + Kx.$$

We have first to determine the most probable value of L and K by minimising

$$\overset{k}{\underset{1}{S}}(X_i^2) = \overset{k}{\underset{1}{S}} \frac{M(a_i - m_{2,i} + \beta_i m_{1,i})^2}{\sigma_1^2\{c^2 + (\beta_i - \bar{\beta})^2\}} \quad \text{(lvi)},$$

where c, $\bar{\beta}$ and σ_1 have the values already determined.

This may be written

$$\overset{k}{\underset{1}{S}}(X_i^2) = \overset{k}{\underset{1}{S}} \frac{M(\bar{y}_i - \beta_i \bar{x}_i - L + (\beta_i - K)m_{1,i})^2}{\sigma_1^2\{c^2 + (\beta_i - \bar{\beta})^2\}} \quad \text{(lvi}^{bis}\text{)}.$$

The problem now is: What value shall we give to $m_{1,i}$?

On pp. 147—8 of his memoir, Mr Welch appears to give $m_{1,i}$ the x mean of the ith population the value of the sample mean $\bar{x}_i$. I am a little doubtful if this is admissible. He writes:

"It will be necessary to express algebraically the condition that the population means of the groups [my $m_{1,i}$ and $m_{2,i}$] are collinear. If $\bar{x}_i$ and $\bar{y}_i$ are the means for the n_i observations in the ith group and the symbol [ ] is used to denote expected or population value, then

$$[\bar{y}_i] = \beta_i \bar{x}_i + a_i \qquad (i = 1, 2, \ldots k)$$

and the condition that the point $(\bar{x}_i, [\bar{y}_i])$ lies on the line

$$y = lx + m$$

is that

$$\beta_i \bar{x}_i + a_i = l\bar{x}_i + m \qquad (i = 1, 2, \ldots k)."$$

In my notation $[\bar{y}_i] = m_{2,i}$ and $m_{1,i}$ is not *à priori* $= \bar{x}_i$, my $\bar{x}_i$. We have the condition that

$$m_{2,i} = L + Km_{1,i}.$$

Further, by our (iii),

$$\bar{y}_i = a_i + \beta_i \bar{x}_i,$$

and, if we assume all possible samples taken from the parent population, this leads to

$$m_{2,i} = a_i + \beta_i m_{1,i},$$

and not to

$$m_{2,i} = a_i + \beta_i \bar{x}_i,$$

unless we assume that $m_{1,i} = \bar{x}_i$.

Now the distribution of $\bar{x}_i$ and $\bar{y}_i$ is given by

$$z = z_0 e^{-\frac{1}{2}M\left\{\frac{(\bar{x}_i - m_{1,i})^2}{\sigma^2_{1,i}} - \frac{2\rho(\bar{x}_i - m_{1,i})(\bar{y}_i - m_{2,i})}{\sigma_{1,i}\sigma_{2,i}} + \frac{(\bar{y}_i - m_{2,i})^2}{\sigma^2_{2,i}}\right\}} \quad \text{(lvii)},$$

where z_0 does not contain $m_{1,i}$ or $m_{2,i}$. Hence if we maximise the probability of $\bar{x}_i$ and $\bar{y}_i$ occurring—the parent-population means $m_{1,i}$ and $m_{2,i}$ being unknown and not linked together—we find the most likely values of $m_{1,i}$ and $m_{2,i}$ are those of the sample, i.e. $\bar{x}_i$ and $\bar{y}_i$; but if we introduce a bond between $m_{1,i}$ and $m_{2,i}$ this is

not the case. For example, if we make $m_{2,i} = L + Km_{1,i}$ substituting this in (lvii) above, and then minimise

$$\frac{(\bar{x}_i - m_{1,i})^2}{\bar{\sigma}^2_{1,i}} - \frac{2\rho\,(\bar{x}_i - m_{1,i})(\bar{y}_i - L - Km_{1,i})}{\bar{\sigma}_{1,i}\,\bar{\sigma}_{2,i}} + \frac{(\bar{y}_i - L - Km_{1,i})^2}{\bar{\sigma}^2_{2,i}}$$

with regard to $m_{1,i}$, this leads to

$$m_{1,i} - \bar{x}_i = \frac{(K - \bar{\beta}_i)(\bar{y}_i - L - K\bar{x}_i)}{\dfrac{\bar{\sigma}^2_{2,i}}{\bar{\sigma}^2_{1,i}}(1 - \rho^2) + (K - \bar{\beta}_i)^2} \quad \ldots\ldots\ldots\ldots\ldots(\text{lvii}^{bis}).$$

Accordingly $m_{1,i}$ will not be equal to $\bar{x}_i$ unless either the slope of the proposed line of population means is equal to the slope of the ith population regression line (not the regression line of the sample), or the line of population means actually passes through the mean of the ith sample. As these are very special cases, we may assume that in general $m_{1,i}$'s most likely value when it is linked to $m_{2,i}$ is not $\bar{x}_i$.

It is quite true that: (*a*) the k populations may not be normal populations, although if we feel unable to accept (v) they narrow down to something very close to it, owing to the form (i) of the population surface selected; and (*b*) the *sample may not be a random sample from the ith population—the x-values may* be directly selected; still $\bar{x}_i$, $\bar{y}_i$ must correspond to one of the possible samples from the ith population*, and we think (lvii^{bis}) will give a more likely value for $m_{1,i}$ than taking $m_{1,i}$ equal to $\bar{x}_i$.

Now we have seen (p. 252) that

$$\overset{k}{\underset{1}{S}}(X_i)^2 = \overset{k}{\underset{1}{S}} \frac{M\,(\alpha_i - m_{2,i} + \beta_i m_{1,i})^2}{\bar{\sigma}^2_{1,i}\,\{c_i^2 + (\beta_i - \bar{\beta}_i)^2\}}$$
$$= \overset{k}{\underset{1}{S}} \frac{M\,\{\bar{y}_i - L - K\bar{x}_i + (\beta_i - K)(m_{1,i} - \bar{x}_i)\}^2}{\bar{\sigma}^2_{1,i}\,\{c_i^2 + (\beta_i - \bar{\beta}_i)^2\}}.$$

Or, substituting from (lvii) for the most probable value of $m_{1,i} - \bar{x}_i$, we have, since

$$c_i^2 = \frac{\bar{\sigma}^2_{2,i}}{\bar{\sigma}^2_{1,i}}(1 - \rho_i^2) = \frac{\bar{\gamma}_i^2}{\bar{\sigma}^2_{1,i}},$$

$$\overset{k}{\underset{1}{S}}(X_i^2) = \overset{k}{\underset{1}{S}} \frac{Mc_i^2}{\bar{\gamma}_i^2}\,\frac{(\bar{y}_i - L - K\bar{x}_i)^2\,\{c_i^2 + (K - \beta_i)(\beta_i - \bar{\beta}_i)\}^2}{\{c_i^2 + (\beta_i - \bar{\beta}_i)^2\}\,\{c_i^2 + (\bar{\beta}_i - K)^2\}^2},$$

or, as we may write it,

$$= \overset{k}{\underset{1}{S}} \frac{Mc_i^2}{\bar{\gamma}_i^2}\,\frac{\{\bar{y}_i - \beta_i - L - (K - \bar{\beta}_i)\,\bar{x}_i\}^2\,\{c_i^2 + (K - \bar{\beta}_i)(\beta_i - \bar{\beta}_i)\}^2}{\{c_i^2 + (\beta_i - \bar{\beta}_i)^2\}\,\{c_i^2 + (K - \bar{\beta}_i)^2\}^2} \quad \ldots\ldots(\text{lviii}).$$

The most general problem would be one in which we should maximise both the X_i and Y_i functions for c_i, $\bar{\beta}_i$, L and M. But this leads to extremely complicated equations, and accordingly we will suppose $\bar{\gamma}_i$, c_i and $\bar{\beta}_i$ are the constants determined on pp. 245 and 249, which give reasonable normal surfaces for the

* I think in the actual example chosen, the values of x are more or less a random selection, and accordingly $\bar{x}_i$, $\bar{y}_i$ have random values. But if they are not, and it has been suggested that they might be taken at one end of the surface, is there any logic in assuming that a single probability integral (Welch) of a combination of such integrals, which are interpreted as random measures, will really provide a measure of the possibility of an hypothesis, and not of the selected values of x?

k populations. What we have then to do is to minimise $S(X_i^2)$ with regard to L and $\kappa = K - \bar{\beta}$. Accordingly we start from

$$\overset{k}{\underset{1}{S}}(X_i^2) = \frac{Mc^2}{\bar{\gamma}^2}\overset{k}{\underset{1}{S}}\frac{(\bar{y}_i - \bar{\beta}x_i - L - \kappa\bar{x}_i)^2\{c^2 + \kappa(\bar{\beta} - \beta_i)\}^2}{(c^2+\kappa^2)^2\{c^2 + (\bar{\beta} - \beta_i)^2\}}.$$

The differentiation with regard to L gives us, after dividing out the terms independent of i,

$$\overset{k}{\underset{1}{S}}\frac{(\bar{y}_i - \bar{\beta}\bar{x}_i - L - \kappa\bar{x}_i)\{c^2 + \kappa(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots(\text{lix}),$$

and for κ, after again dividing out terms independently of i,

$$\begin{aligned}&-\overset{k}{\underset{1}{S}}\frac{\bar{x}_i(\bar{y}_i - \bar{\beta}x_i - L - \kappa x_i)\{c^2 + \kappa(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2}\\ &+\overset{k}{\underset{1}{S}}\frac{(\bar{y}_i - \bar{\beta}x_i - L - \kappa x_i)^2\{c^2 + \kappa(\beta_i - \bar{\beta})\}(\beta_i - \bar{\beta})}{c^2 + (\beta_i - \bar{\beta})^2}\\ &-\frac{2\kappa}{c^2+\kappa^2}\overset{k}{\underset{1}{S}}\frac{(\bar{y}_i - \bar{\beta}\bar{x}_i - L - \kappa\bar{x}_i)^2\{c^2 + \kappa(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} = 0 \quad \ldots\ldots\ldots\ldots(\text{lx}).\end{aligned}$$

We will now assume that approximations to L and κ or K are known, and will take $L = L_0 + \xi$, $\kappa = \kappa_0 + \zeta$; thus neglecting squares and products, we have from (lix),

$$\begin{aligned}&\overset{k}{\underset{1}{S}}\frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} = \xi\overset{k}{\underset{1}{S}}\frac{\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta}_i)^2}\\ &\quad+\zeta\left[\overset{k}{\underset{1}{S}}\frac{x_i\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta}_i)^2} - \overset{k}{\underset{1}{S}}\frac{2(\bar{y}_i - L_0 - K_0\bar{x}_i)\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}(\beta_i - \bar{\beta})}{c^2 + (\beta_i - \bar{\beta})^2}\right]\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ldots\ldots(\text{lxi}).\end{aligned}$$

We have now to arrange (lx) in like manner—

$$\begin{aligned}&\overset{k}{\underset{1}{S}}\frac{\bar{x}_i(y_i - L_0 - K_0\bar{x}_i)\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} - \overset{k}{\underset{1}{S}}\frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)^2\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}(\beta_i - \bar{\beta})}{c^2 + (\beta_i - \bar{\beta})^2}\\ &\quad+\frac{2\kappa_0}{c^2+\kappa_0^2}\overset{k}{\underset{1}{S}}\frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)^2\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2}\\ &=\xi\left[\overset{k}{\underset{1}{S}}\frac{\bar{x}_i\{c^2 + \kappa(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} - 2\overset{k}{\underset{1}{S}}\frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)^2\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}(\beta_i - \bar{\beta})}{c^2 + (\beta_i - \bar{\beta})^2}\right.\\ &\quad\left.+\frac{4\kappa_0}{c^2+\kappa_0^2}\overset{k}{\underset{1}{S}}\frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2}\right]\\ &+\zeta\left[\overset{k}{\underset{1}{S}}\frac{\bar{x}_i^2\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2} - 4\overset{k}{\underset{1}{S}}\frac{\bar{x}_i(\bar{y}_i - L_0 - K_0\bar{x}_i)\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}(\beta_i - \bar{\beta})}{c^2 + (\beta_i - \bar{\beta})^2}\right.\\ &\quad+\overset{k}{\underset{1}{S}}\frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)^2(\beta_i - \bar{\beta})^2}{c^2 + (\beta_i - \bar{\beta})^2} + \frac{4\kappa_0}{c^2+\kappa_0^2}\overset{k}{\underset{1}{S}}\frac{x_i(\bar{y}_i - L_0 - K_0\bar{x}_i)\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2}\\ &\quad-\frac{4\kappa_0}{c^2+\kappa_0^2}\overset{k}{\underset{1}{S}}\frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)^2\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}(\beta_i - \bar{\beta})}{c^2 + (\beta_i - \bar{\beta})^2}\\ &\quad\left.-\frac{2(c^2-\kappa_0^2)}{(c^2+\kappa_0^2)^2}\overset{k}{\underset{1}{S}}\frac{(\bar{y}_i - L_0 - K_0\bar{x}_i)^2\{c^2 + \kappa_0(\beta_i - \bar{\beta})\}^2}{c^2 + (\beta_i - \bar{\beta})^2}\right] \qquad\ldots\ldots(\text{lxii}).\end{aligned}$$

The problem is now how to find approximate values of L_0 and $\kappa_0 = K_0 - \bar{\beta}$, which may be inserted in the above equations. Let us see whether Mr Welch's assumption $m_{1,t} = \bar{x}_t$ will serve as a starting point. We have from (lvi)

$$\overset{k}{\underset{1}{S}}(X_t^2) = \overset{k}{\underset{1}{S}} \frac{M(\alpha_t - m_{2,t} + \beta_t m_{1,t})^2}{\bar{\sigma}^2_{1,t}(c_t^2 + (\beta_t - \bar{\beta}_t)^2)}.$$

Putting $m_{2,t} = L + Km_{1,t} = L + K\bar{x}_t$, $\bar{\sigma}^2_{1,t} = \bar{\gamma}^2/c^2$, $\bar{\beta}_t = \bar{\beta}$ and $c_t = c$, we have

$$\overset{k}{\underset{1}{S}}(X_t^2) = \frac{Mc^2}{\bar{\gamma}^2} \overset{k}{\underset{1}{S}} \frac{(\bar{y}_t - L - K\bar{x}_t)^2}{c^2 + (\beta_t - \bar{\beta})^2}.$$

Differentiating this result with regard to L and K we have the following relations to determine those quantities after dividing out quantities independent of i:

$$\overset{k}{\underset{1}{S}} \frac{(\bar{y}_t - L - K\bar{x}_t)}{c^2 + (\beta_t - \bar{\beta})^2} = 0, \quad \overset{k}{\underset{1}{S}} \frac{(\bar{y}_t - L - K\bar{x}_t)\bar{x}_t}{c^2 + (\beta_t - \bar{\beta})^2} = 0.$$

These lead to

$$\left.\begin{aligned} \overset{k}{\underset{1}{S}} \frac{\bar{y}_t}{c^2 + (\beta_t - \bar{\beta})^2} &= L\overset{k}{\underset{1}{S}} \frac{1}{c^2 + (\beta_t - \bar{\beta})^2} + K\overset{k}{\underset{1}{S}} \frac{\bar{x}_t}{c^2 + (\beta_t - \bar{\beta})^2} \\ \overset{k}{\underset{1}{S}} \frac{\bar{x}_t\bar{y}_t}{c^2 + (\beta_t - \bar{\beta})^2} &= L\overset{k}{\underset{1}{S}} \frac{\bar{x}_t}{c^2 + (\beta_t - \bar{\beta})^2} + K\overset{k}{\underset{1}{S}} \frac{\bar{x}_t^2}{c^2 + (\beta_t - \bar{\beta})^2} \end{aligned}\right\} \text{......(lxiii)},$$

delightfully simple equations after those we have found for a more probable value of $\bar{x}_{1,t}$.

Substituting the values given for $\bar{x}_t$, $\bar{y}_t$ and β_t on p. 248 and for c^2 and $\bar{\beta}$ on p. 252 we find

$$\overset{k}{\underset{1}{S}} \frac{\gamma_t}{c^2 + (\beta_t - \bar{\beta})^2} = 381{,}806\cdot66114, \quad \overset{k}{\underset{1}{S}} \frac{1}{c^2 + (\beta_t - \bar{\beta})^2} = 97\cdot457{,}450,$$

$$\overset{k}{\underset{1}{S}} \frac{\bar{x}_t}{c^2 + (\beta_t - \bar{\beta})^2} = 640{,}114\cdot12925, \quad \overset{k}{\underset{1}{S}} \frac{\bar{x}_t^2}{c^2 + (\beta_t - \bar{\beta})^2} = 45{,}8190{,}8819\cdot95,$$

and

$$\overset{k}{\underset{1}{S}} \frac{\bar{x}_t\bar{y}_t}{c^2 + (\beta_t - \bar{\beta})^2} = 28{,}5448{,}6571\cdot56.$$

Thus we obtain the equations

$$\left.\begin{aligned} 381{,}806\cdot66114 &= 97\cdot457{,}450L + 640{,}114\cdot12925K \\ 28{,}5448{,}6571\cdot56 &= 640{,}114\cdot12925L + 45{,}8190{,}8819\cdot95K \end{aligned}\right\} \text{......(lxiv)},$$

which provide the solutions

$$L = -2114\cdot2489 \quad \text{and} \quad K = \cdot9183{,}6118,$$

and the most likely lines for the means

$$y = \cdot9183{,}6118x - 2114\cdot2489.$$

We have now to consider what is the probability of the resulting set of X_t's. We have

$$X_t = \frac{\sqrt{M}c}{\bar{\gamma}} \frac{\bar{y}_t - L - K\bar{x}_t}{\sqrt{c^2 + (\beta_t - \bar{\beta})^2}}.$$

Here $M=8$, $c^2=\cdot035,871$, $\bar{\beta}=\cdot525,865$ and $\bar{\gamma}=94\cdot3575$: see pp. 250 and 245. Thus

$$X_t=\frac{\cdot005,67728\,(\bar{y}_t+2114\cdot2489-\cdot9183,6118\bar{x}_t)}{\sqrt{\cdot035,871+(\beta_t-\cdot525,685)^2}},$$

whence we obtain

$$X_1=+4\cdot7702,\quad X_2=-6\cdot9129,\quad X_3=-4\cdot1473,\quad X_4=5\cdot8650.$$

These values as a random sample from a normal curve are most highly improbable —we do not need to calculate their P_{λ_n}—and accordingly if we adopt the hypothesis $m_{1,t}=\bar{x}_t$, it is clear that the means of the populations cannot lie on a straight line. Our problem, however, is whether the values thus found, i.e.

$$L_0=-2114\cdot2489,\quad K_0=\cdot9183,6118,$$

and accordingly $\qquad \kappa_0=K_0-\bar{\beta}=\cdot3924,9618,$

will be suitable approximate values for the solution. The actual arithmetical work is not so laborious as it appears, the summations often repeating themselves. We find (lxi) gives

$$-41\cdot8567,47095=\cdot2035,5520\xi+1458\cdot100,055\zeta,$$

and (lxii) $\quad -226,852\cdot3567=-24767\cdot384,440\xi+850,2270\cdot6475,7242\zeta.$

As solution of these equations we find

$$\xi=-\cdot6633,5362,\quad \zeta=-\cdot0286,13755 \quad\text{.................(lxv).}$$

These denote a very small change in L, and a fairly small change in K or κ, so we shall not go to a second approximation. They give us

$$L=-2114\cdot9122,5362,\quad K=\cdot8897,4742 \text{ and } \kappa=\cdot3638,8242,$$

or the most likely line for the means of the four populations is

$$y=\cdot8897,4742x-2114\cdot9122,5362 \quad\text{.................(lxvi).}$$

We have now to discover whether this line gives a reasonable confirmation of the hypothesis that the four populations means really lie on it. For this purpose we must compute the four X_t's from the formula

$$X_t=\frac{\sqrt{M}c}{\bar{\gamma}}\,\frac{1}{c^2+\kappa^2}\,\frac{(\bar{y}_t-L-M\bar{x}_t)\{c^2+\kappa(\beta_t-\bar{\beta})\}}{\sqrt{c^2+(\beta_t-\bar{\beta})^2}}$$

$$=\cdot0337,36821\times\frac{(\bar{y}_t+2114\cdot91225-\cdot8897,4742\bar{x}_t)\{\cdot035,871+\cdot3638,8242\,(\beta_t-\bar{\beta})\}}{\sqrt{\cdot035,871+(\beta_t-\cdot525,865)^2}},$$

and then see if they are a random sample from a normal curve of standard deviation unity. The values of X_t are

$$X_1=+\cdot191,657,\quad X_2=-1\cdot089,254,\quad X_3=+\cdot573,000,\quad X_4=+1\cdot606,296,$$

giving

$p_1=\cdot575,9942$, $\log p_1=\bar{1}\cdot760,4181$, $p_1'=\cdot424,0058$, $\log p_1'=\bar{1}\cdot627,3719$,
$p_2=\cdot138,0218$, $\log p_2=\bar{1}\cdot139,9476$, $p_2'=\cdot861,9782$, $\log p_2'=\bar{1}\cdot935,4963$,
$p_3=\cdot716,6757$, $\log p_3=\bar{1}\cdot855,3227$, $p_3'=\cdot283,3243$, $\log p_3'=\bar{1}\cdot452,2838$,
$p_4=\cdot945,8935$, $\log p_4=\bar{1}\cdot975,8422$, $p_4'=\cdot054,1065$, $\log p_4'=\bar{2}\cdot733,2494$.

$$\lambda_n=-1\cdot268,4694,\quad \lambda_n'=-2\cdot251,5986,$$

and hence

$$P_{\lambda_n} = I(1{\cdot}460{,}3793,\ 3) = {\cdot}3351, \quad P_{\lambda_n'} = I(2{\cdot}592{,}2486,\ 3) = {\cdot}7598,$$

or

$$Q_{\lambda_n} = {\cdot}6649, \quad Q_{\lambda_n'} = {\cdot}2402.$$

$Q_{\lambda_n'}$ is the more stringent, but we see that it is quite compatible with the means of the populations being co-linear.

Summary. (*a*) The hypothesis that the four populations are normal and have a common array standard deviation can be accepted. This is opposed to Mr Welch's conclusion that the latter must be rejected, and is curious considering that our assumption of normality is more stringent than his assumption (i). I attribute the difference, supposing the arithmetic of both of us to be correct, to his assuming for the distribution of his compound criterion a curve which is probably not the true curve. In my case the hypothesis of normality leads to the exact curve.

(*b*) I find no improbability in the four normal populations having the same regression coefficient. This is in accordance with Mr Welch's conclusion. This result can be expressed by saying that the four populations have the same values for $\sigma_{2,t}\sqrt{1-\rho^2}$, $\beta = \rho\,\sigma_{2,t}/\sigma_{1,t}$ and $\sigma_{1,t}$.

(*c*) It is highly improbable that the value of $\bar{a}_t$ is the same for all populations; *in particular that they are all selections from a single normal surface.*

(*d*) Mr Welch finds that the means of the four populations are unlikely to be collinear. I find that if I use the values he adopts for the $m_{1,t}$'s—namely, to take them equal to the sample means—that I agree with him. But I doubt the accuracy of this equality, and by taking what I think is a more likely value of $m_{1,t}$, I conclude that the hypothesis of collinearity need by no means be rejected.

(*e*) For such small samples as are here under discussion the conclusion that certain hypotheses are not markedly improbable does not seem to be a justification without further experience that they are true. My method has been to combine the probability integrals deduced from the k populations and determine whether they are a fairly probable random sample of k points from the appropriate curve of distribution. Mr Welch's method is to form a combined criterion, and find the probability integral of this combined criterion on the assumption that it follows an empirical curve. In either case with merely four points, we should get equally probable results had we taken other arbitrary curves for my appropriate curves, or for Mr Welch's empirical curve. It requires not four, but frequently one or two hundred points to make a selection markedly more probable when taken from one curve rather than from another. Hence it seems to me unjustifiable to adopt from four samples of eight individuals each any conclusion as to the algebraical forms of the populations sampled.

MISCELLANEA.

(i) Constancy in the Number of Ligulate Flowers of Chrysanthemum Leucanthemum, variety Pinnatifidum, during the Flowering Season.

By WILLIAM DOWELL BATEN

(University of Michigan).

Many plants produce flowers at the beginning of the flowering season which differ significantly from those produced at the end of the season. Yule* found that the average number of sepals in flowers of *Anemone nemorosa* differed significantly for two periods of the same season. Burkill† shows by his counts that the number of stamens in *Stellaria media* varied greatly during the blossoming period. Species of Salpiglossis produce rather large blossoms at the beginning of the season and very tiny cleistogamous blossoms at the close of the season‡. The average numbers of pistils and stamens of *Ranunculus ficaria* differ widely during the time of blooming§. Shull's‖ samples of *Aster prenanthoides* Muhl. show that the number of bracts, the number of disk flowers and the number of ligulate flowers differ significantly for different times in the blossoming period. Tower¶ found there was a significant difference between the average number of ray flowers per head of *Chrysanthemum leucanthemum* for different periods of the flowering season.

The object of this study was to determine whether or not *Chrysanthemum leucanthemum*, variety *pinnatifidum*, exhibits any significant differences in the average number of ray flowers per head at different times during the flowering season.

Material.

On May 17, 1932, a sample of one thousand flower heads of *Chrysanthemum leucanthemum*, variety *pinnatifidum*, was gathered at random from a plot of about one-half acre, located about four miles east of Dexter, Michigan, along the Huron River Drive. On this date there were

* Yule, G. U. "Variation of the number of sepals in *Anemone nemorosa*." *Biometrika*, Vol. I. (1901—2), pp. 307—309.

† Burkill, I. H. "On some variations in the number of stamens and carpels." *Journal of the Linnean Society*, Vol. XXXI. (1895), pp. 216—245.

‡ Dale, E. E. Unpublished results of study made in the Botanical Gardens of the University of Michigan.

§ MacLeod, J. "Over de correlatie tusschen het aantal meeddraden en het aantal stampers bij het Speenkruid (*Ficaria ranunculoides*)." *Botanisch Jaarboek*, Jaargang XI. (1899), pp. 91—107. "Cooperative investigations on plants. II. Variations and correlation in Lesser Celandine from divers localities." *Biometrika*, Vol. II. (1902—3), pp. 145—164, Miscellanea. "Change in organic correlation of *Ficaria ranunculoides*," *Biometrika*, Vol. I. (1901—2), pp. 125—128.

‖ Shull, G. H. "A quantitative study of variation in the bracts, rays and disk florets of *Aster shortii* Hook., *A. Novae-Angliæ* L., *A. puniceus* L. and *A. prenanthoides* Muhl. from Yellow Springs, Ohio." *American Naturalist*, Vol. XXXVI. (1902), pp. 111—152.

¶ Tower, W. L. "Variation in the ray-flowers of *Chrysanthemum leucanthemum* L. at Yellow Springs, Green County, Ohio, with remarks upon the determination of Modes." *Biometrika*, Vol. I. (1901—2), pp. 309—315.

many heads in full bloom, with comparatively few heads having gone to seed. There were thousands of buds on the plants, showing that this sample was gathered at the beginning of the flowering season.

On June 4, 1932, another sample of one thousand was taken at random from this same plot.

Another sample of one thousand heads was gathered from this plot on June 23, 1932, at which time there were scarcely any buds, while there were thousands of old heads in seed. A few days later there were not enough blooms for another sample. It seems plausible to assume that this sample was gathered at the very end of the flowering season for this particular plot.

The plants were growing very close together in an open place well elevated and well drained. The area was completely covered with daisies with a scattering of uncultivated rye. There was an extraordinary abundance of the species growing here.

The sample taken at the beginning of the season will be designated by Sample A, the one gathered in the middle of the season by Sample B, and the last by Sample C*.

Analysis of the Samples.

The following table gives the frequency distributions of the number of ray flowers per head for the three samples:

Number of ray flowers per head	Sample A Frequency	Sample B Frequency	Sample C Frequency	Sum of samples Frequency
11—12	0	0	3	3
13—14	131	158	139	428
15—16	242	239	241	722
17—18	238	263	240	741
19—20	229	206	232	667
21—22	146	117	128	391
23—24	10	7	17	34
25—26	3	6	0	9
27—28	0	4	0	4
29—30	1	0	0	1
Totals	1000	1000	1000	3000

Histograms in Fig. 1 show how the sample at the beginning of the flowering season compares with the sample at the close of the season. The general appearances of the two are the same while the ranges are nearly the same. Both rise abruptly for the class 13—14 ray flowers, have about the same heights for the three middle classes and drop suddenly for the class 23—24 ray flowers.

The significance of the means is

$$S = \frac{\text{Difference of Means}}{\text{Probable error of Difference of Means}} = \cdot 9,$$

which indicates that the difference of the two means is well within fluctuations due to random sampling. Vertical lines near the middle of Fig. 1 represent the means for the two distributions

* The author is indebted to W. H. Long, a student in the Forestry Department of the University of Michigan, for the last two counts.

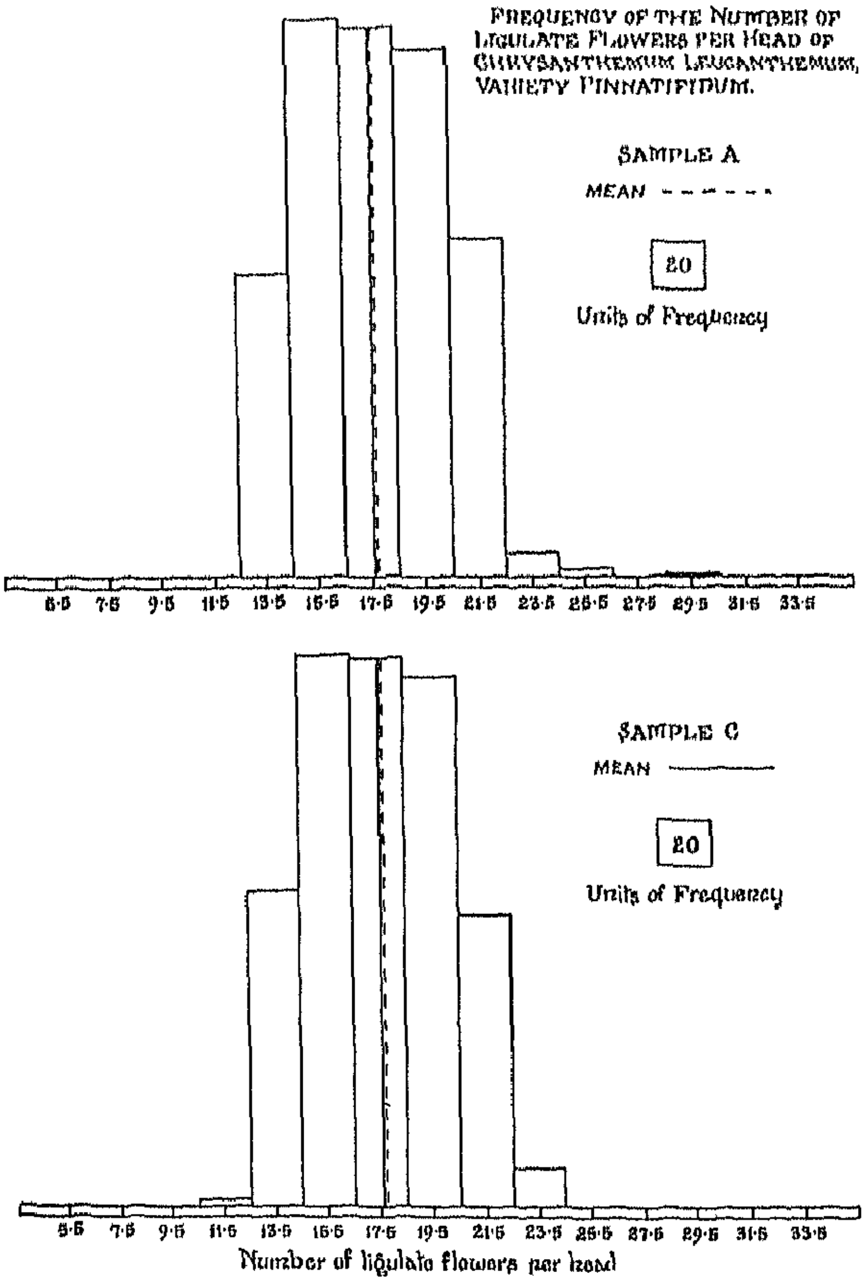

FREQUENCY OF THE NUMBER OF LIGULATE FLOWERS PER HEAD OF CHRYSANTHEMUM LEUCANTHEMUM, VARIETY PINNATIFIDUM.
SAMPLE A
MEAN
20
Units of Frequency
5·5
7·5
9·5
11·5
13·5
15·5
17·5
19·5
21·5
23·5
25·5
27·5
29·5
31·5
33·5
SAMPLE C
MEAN
20
Units of Frequency
Number of ligulate flowers per head

Fig. 1.

and show the small difference between them. The difference between the averages is not significant and is no doubt due to chance. The average number of ray flowers per head did not change appreciably from the beginning of the flowering season to the end.

The following table gives the chief characteristics* of the distributions of the number of ray flowers per head for the three samples, and for the sample made up of the sum of A, B and C.

Sample	Mean	Standard Deviation	Skewness	Median
A	17·630 ray flowers	2·64482 ray flowers	·237819	17·57 ray flowers
B	17·400 "	2·6844 "	·358392	17·29 "
C	17·522 "	2·62364 "	·108745	17·48 "
For sum	17·5167 "	2·6052 "	·217802	17·438 "

The means of the three samples differ by amounts which are less than three probable errors of the various differences of the means and hence are well within fluctuations due to random sampling. This fact clearly indicates the constancy in the average number of ray flowers per head. *These means are represented by the vertical lines near the middle of Fig. 2 (p. 264).*

Slight differences between the medians of the three samples exhibit also the constancy of the number of ray flowers produced by this species. Their differences are within sampling fluctuations. The standard deviations, which show how closely the items of the distributions are scattered about the means, indicate here the similarity of the three distributions and the small differences in the probable errors of the means. Differences between the standard deviations are not significant and indicate that the samples were no doubt taken at random from the same parent population.

The skewnesses for the distributions do not present any significant differences.

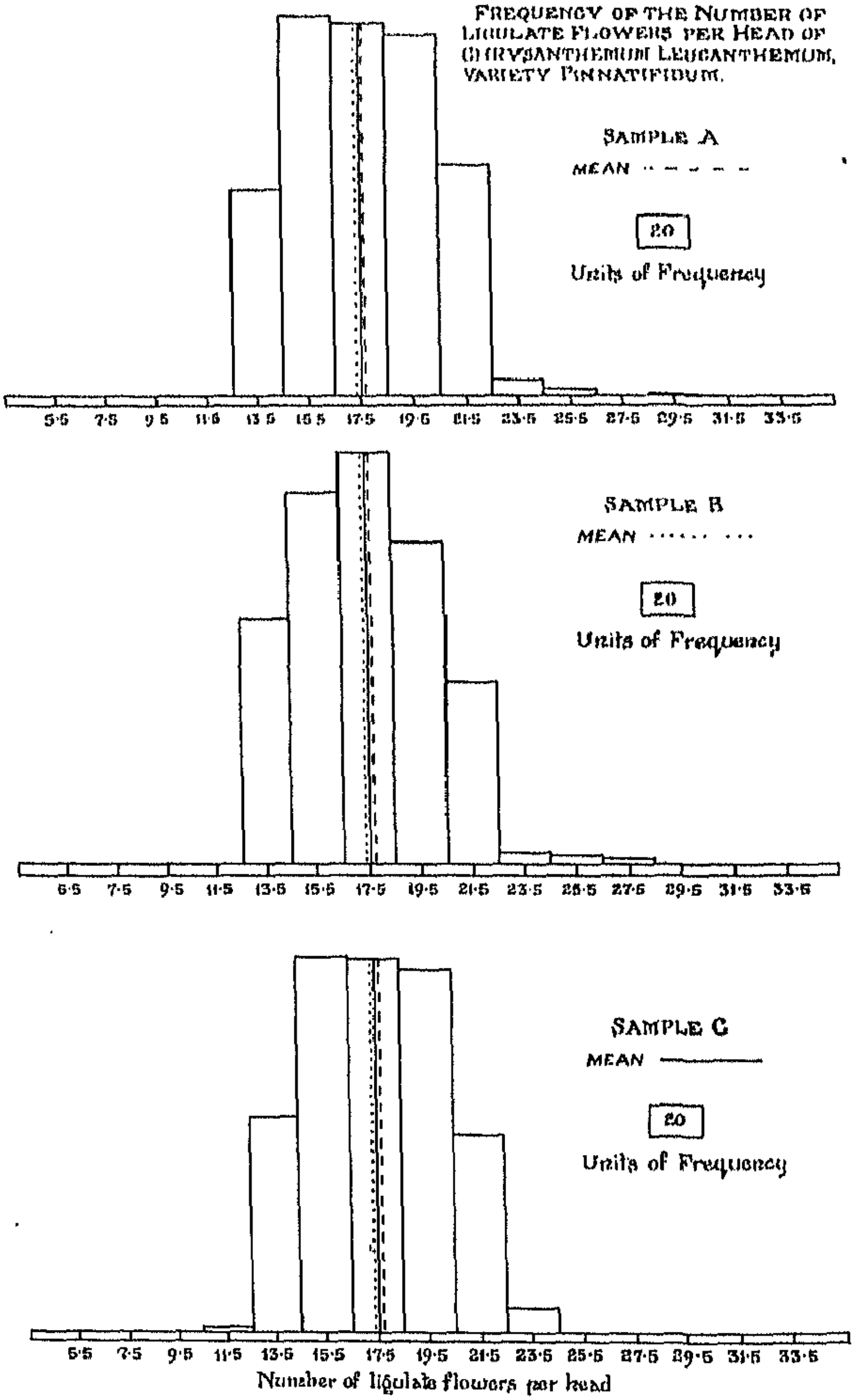
Fig. 2 (p. 264) exhibits frequencies of the samples taken at the beginning of the flowering season, at the middle of the season and at the end. All three distributions contain few heads which had less than 13 ray flowers and a very small percentage which had more than 25. The bulk of all three lies within five classes which have frequencies differing slightly. The central graph differs slightly from the other two but only by differences which are due to fluctuations in sampling from the same parent.

The above study for this species in the composite family clearly shows that the number of ligulate flowers per flower head remained constant throughout the flowering season, and that different climatic conditions which existed during the season did not affect the average number of ray flowers per head.

Analysis of entire Set of Ligulate Flowers.

Since the first three characteristics of the three samples did not differ significantly, a large sample was formed by combining all three distributions. The chief characteristics of this sample for the entire season are given in the table on p. 265. Fig. 3 (p. 265) exhibits the histogram for this distribution.

* The corrections for these characteristics were made according to the formula for correcting the moments for discrete variates. *Annals of Mathematical Statistics*, February 1930, p. 111.



FREQUENCY OF THE NUMBER OF LIGULATE FLOWERS PER HEAD OF CHRYSANTHEMUM LEUCANTHEMUM, VARIETY PINNATIFIDUM.
SAMPLE A
MEAN
20
Units of Frequency
5·5 7·5 9·5 11·5 13·5 15·5 17·5 19·5 21·5 23·5 25·5 27·5 29·5 31·5 33·5
SAMPLE B
MEAN
20
Units of Frequency
5·5 7·5 9·5 11·5 13·5 15·5 17·5 19·5 21·5 23·5 25·5 27·5 29·5 31·5 33·5
SAMPLE C
MEAN
20
Units of Frequency
5·5 7·5 9·5 11·5 13·5 15·5 17·5 19·5 21·5 23·5 25·5 27·5 29·5 31·5 33·5
Number of ligulate flowers per head

Fig. 2.

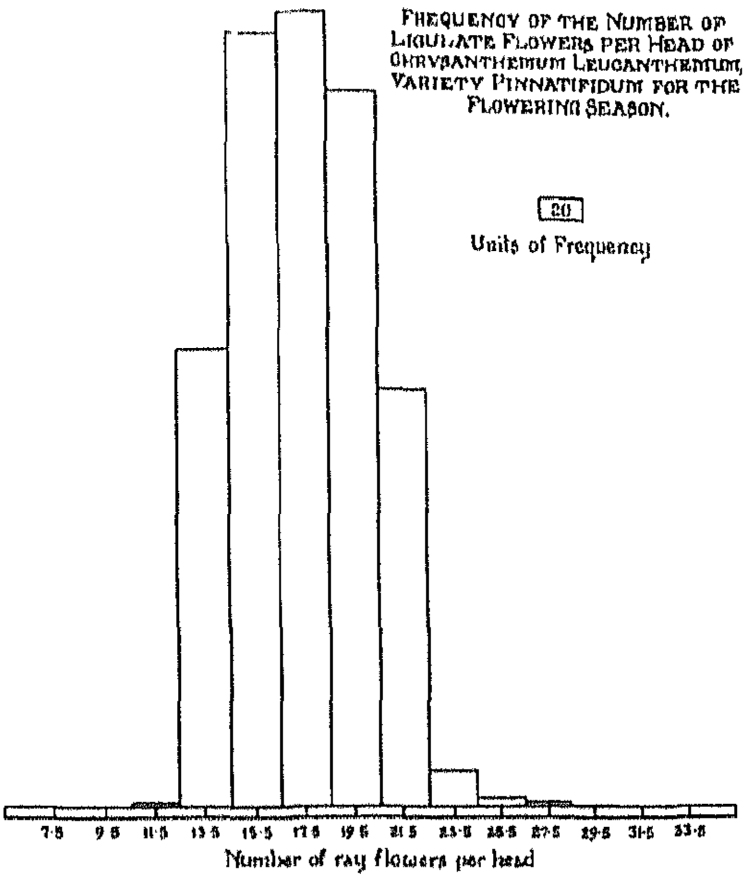


Fig. 3.

The following table gives the frequencies for the data before the grouping was made:

Number of ray flowers	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29
Frequency	3	142	286	332	390	374	367	348	319	335	56	22	12	5	4	4	0	1

The frequency polygon in Fig. 4 (p. 266) shows that there are modes at 16 and 21 as Ludwig found in his sample of this species*.

* Ludwig, F. "Ueber Variationskurven und Variationsflächen der Pflanzen." *Botanisches Centralblatt*, Bd. 64 (1895), S. 1–11.

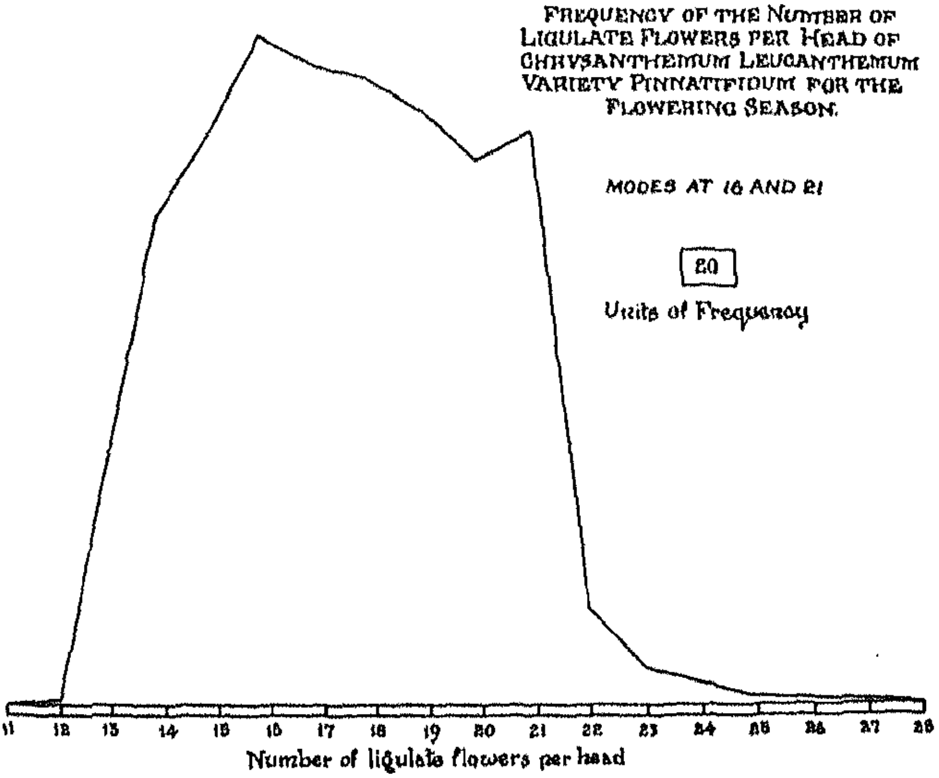


Fig. 4.

(ii) Strength of Vision in relation to Eye and Hair Pigmentation.

BY CATHERINE M. THOMPSON AND ETHEL M. ELDERTON

(WRITTEN UP BY K. PEARSON).

Many years ago Francis Galton raised the question* of whether it was not desirable to select candidates for the Civil Service not only by their intellectual powers and the capacity to satisfy a medical examiner, but, especially in the case of tropical appointments, by physical characters which do not enter into the customary medical inquiry. For example, was it more profitable to send young men of light or dark complexion to hot countries? What hair and eye pigmentation were more suited to India or again to Central Africa?

Last year my colleague, Professor Fawcett, asked me with a like end in view whether I knew of any data associating vision with pigmentation. Besides much data not yet reduced taken in the Anthropometric Laboratory at University College, London, I had on loan and in course of reduction data from the Cambridge Anthropometric Laboratory. This latter material offered in the case of some 1500 young men, between 18 and 22 years of age, the means of determining for one visual character, strength of sight, the relation to pigmentation. It did not, however, provide the effect of bright light of various intensities on modifying this relation of sight and pigmentation, and for this reason could scarcely be of service for Professor Fawcett's purpose. The results

* See *The Life, Letters and Labours of Francis Galton*, Vol. II. pp. 386—390, especially p. 395, and Vol. III. p. 220.

appeared of sufficient interest in themselves to deserve publication in the *Miscellanea* of this Journal. The material was extracted from the Cambridge data and tabled by Miss Catherine M. Thompson, *a student in the Department of Statistics in University College, London*. The constants were then worked out by Dr E. M. Elderton, and partly, under her supervision, by Miss Thompson. The Editor is solely responsible for the arrangement of the material and the conclusions drawn from it. He would state, however, that the Right Eye is known from other investigations to be more long-sighted than the Left.

The visual character taken was the distance at which diamond type could be read without glasses. That distance was measured in inches.

TABLE I. *Strength of Vision and Hair Tint.*

	Red	Fair	Brown	Dark	Jet Black	Total Population
A. Right Eye						
Number...	70	279	355	784	14	1502
Mean Distance... ...	54″·07	55″·19	56″·80	55″·42	53″·57	55″·62
Variability (S. D.) ...	16″·00	15″·36	15″·32	15″·16	14″·22	15″·29
Standard Error of Mean	1″·91	0″·92	0″·81	0″·54	3″·80	0″·39
B. Left Eye						
Number...	69	277	347·5	789·5	14	1497
Mean Distance... ...	51″·92	52″·68	54″·49	53″·91	54″·29	53″·73
Variability (S. D.) ...	15″·60	15″·31	14″·09	14″·66	14″·14	14″·71
Standard Error of Mean	1″·88	0″·92	0″·76	0″·52	3″·78	0″·38

Examining the standard errors, we see that no two eye-strengths are significantly different except in the case of the total populations, where the difference of Left and Right Eye strengths $=1''·891$ with a standard error of $0''·548$, or the difference is 3·45 times its standard error, which would only occur thrice in 10,000 trials, if Left and Right Eyes were samples of the same normal universe. It would be seen that the Right Eye for each hair-colour is stronger than the Left Eye except for Jet-Black Hairs, of which there were only fourteen. If we take the difference for the Right Eye between Fair and Brown, it is $1''·608$, with a standard error of $1''·227$. Thus the difference is only 1·31 times its standard error, and would be found to be as great or greater in 9·5 per cent. of trials. If we add all Right and Left Eyes together for Red and Fair, and again for Brown, Dark and Jet Black, we get:

	Mean distance	*Variance* (σ^2)
Fair and Red Hair ...	53″·752	239·733,774
Brown, Dark and Jet Black	54″·651	220·882,399

Excess Dark Hair $=0''·899$.

$$\text{Standard Error of Difference} = \left(\frac{239·733,774}{695} + \frac{220·882,399}{2304}\right)^{\frac{1}{2}}$$

$$=0''·6639.$$

Ratio of Difference to Standard Error $=1·354$.

Thus, such an excess in Dark-Hair vision might be expected to occur in about 8·8 per cent. of trials, if no such excess really existed. This is hardly better than the previous test for Right Eye only. No stress can be laid on it. At the same time the general run of stronger sight is the same for both eyes, and we have some little evidence that Red and Fair Hairs have weaker sight than Brown or Dark, although the differences are too small to be significant on the present numbers.

Table II gives the corresponding data for three eye-tints. There are three classifications: Right Eyes, Left Eyes, and both eyes when the Left Eyes have had their strength increased up to Right-Eye strength by the mean difference. Here we naturally find again the same significant difference between the total populations of Right Eyes and Left Eyes. If we consider the difference between the strengths of Light and Dark Right Eyes, it is 4″·607 with the standard error 2″·033, or the difference is 2·27 times its standard error, which would only occur 12 times in 1000 trials, if the Left Eyes and Right Eyes were samples from the same normal universe. On the other hand, comparing Fair and Dark Left Eyes the difference is only 2″·948, with a standard error of 2″·454, or the difference is only 1·20 times its standard error, and would occur in 11·5 per cent. of samples if the Fair and Dark Left Eyes were members of the same universe. The fact is that the Cambridge category "Light Eyes" was either badly defined or poorly distinguished; there are far too few cases in it. (Of course in more modern inquiries eye-scales of 12 to 16 tints are used.)

TABLE II. *Strength of Vision and Eye Tint.*

	Light	Medium	Dark	Total Population
A. Right Eye				
Number	27	992	499	1518
Mean Distance	59″·72	55″·84	55″·11	55″·67
Variability (S.D.) ...	9″·93	15″·24	15″·54	15″·28
Standard Error of Mean	1″·91	0″·48	0″·70	0″·39
B. Left Eye				
Number	27	985	502	1514
Mean Distance	56″·57	53″·77	53″·63	53″·77
Variability (S.D.) ...	12″·32	15″·02	14″·26	14″·73
Standard Error of Mean	2″·37	0″·48	0″·64	0″·38
C. Right Eye + Left reduced				
Number	54	1977	1001	3032
Mean Distance	59″·10	55″·76	55″·32	55″·67
Variability (S.D.) ...	11″·20	15″·30	15″·10	15″·18
Standard Error of Mean	1″·52	0″·34	0″·48	0″·28

It is at once obvious that there is no distinction in strength of vision between the categories of Medium and Dark Eyes. To throw more light on the subject the Left Eyes were modified to Right Eyes as explained above, and now from Class C the difference of strength between Fair and Dark Eyes is 3″·777, with a standard error of 1″·597, which gives the difference 2·36 times its standard error. Such a difference would occur only nine times in 1000 trials if Light and Dark Eyes belonged to the same visual universe, or, there is a very considerable degree of probability that Fair Eyes can see farther than Dark Eyes, i.e. the latter are more short-sighted.

These results throw, of course, very little light on the problem of the complexion suitable to tropical countries. But it is of interest to have the superior vision of the Right Eye confirmed, and further to learn that Light-eyed persons are more far-sighted than Dark-eyed; while on the other hand there is some evidence, but far from conclusive, that Dark-haired persons have stronger vision than Light-haired persons. According to this suggestion the strongest vision would be found in persons with dark hair and blue or grey eyes.

(iii) Note on the Sampling Distribution of $\sqrt{\beta_1}$, where the Population is Normal.

By P. WILLIAMS, B.A.

In a previous issue of this *Journal* Dr E. S. Pearson has published a table containing approximate 5 °/。 and 1 °/。 limits for the distributions of $\sqrt{\beta_1}$ and β_2 when sampling from a normal *population*. The results were based on the quadrature of empirical curves (Type VII and* Type IV of K. Pearson's system), the values of whose moment coefficients were obtained from expansions in series of inverse powers of n, the sample size, carried as far as terms in n^{-3}. Later work by Professor R. A. Fisher has made it possible to obtain for $\sqrt{\beta_1}$ exact values for the sampling moments up to μ_8†. These may be written as follows: (1), (2) and (3) are given in *Dr Pearson's note‡ and (4) in a paper by J. Pepper§.*

$$\mu_2(\sqrt{\beta_1})=\frac{6(n-2)}{(n+1)(n+3)}\qquad\qquad(1),$$

$$B_2(\sqrt{\beta_1})=\frac{\mu_4}{\mu_2^2}=3+\frac{36(n-7)(n^2+2n-5)}{(n-2)(n+5)(n+7)(n+9)}\qquad\qquad(2),$$

$$B_4(\sqrt{\beta_1})=\frac{\mu_6}{\mu_2^3}=15+\frac{540\{n^7+60n^6-131n^5-2708n^4-3620n^3+21352n^2+32943n-70070\}}{(n-2)^2(n+5)(n+7)(n+9)(n+11)(n+13)(n+15)}\qquad(3),$$

$$B_6(\sqrt{\beta_1})=\frac{\mu_8}{\mu_2^4}$$

$$=\frac{105(n+1)^3(n+3)^3\{n^6+171n^5+13893n^4+580401n^3-5131014n^2+14132208n-12932920\}}{(n-2)^3(n+5)(n+7)\ldots\ldots\ldots(n+21)}\quad(4).$$

In a short note Dr Pearson‡ has shown that the approximate expressions he had used for the moments differed only slightly from the true values; at $n=50$: $\sigma_{\sqrt{\beta_1}}$ was correct as far as the fourth decimal place, while the differences in $B_2(\sqrt{\beta_1})$ were as follows:

	$n=50$	$n=75$	$n=100$
True value	3·452	3·361	3·284
Value used in making tables	3·46	3·35	3·28

This disagreement is not large, but it seemed worth while to recalculate the corresponding Type VII curves, and also to make use of Professor Fisher's results to investigate the question of adequacy of approximation a little more fully. In the first place the Type VII curve was refitted for $n=50$ (the smallest value of n considered by Dr Pearson), using the true moment coefficients; the 5 °/。 and 1 °/。 limits were found to be identical, to three decimal places, with those previously given.

The Type VII curves have been used as an approximation to the unknown true distribution; their first four moment coefficients have the correct values‖, but their higher even moment *coefficients will of course be in error. The comparison in Table I (p. 270) is of interest.*

At $n=50$, B_4 and B_6 for the Type VII curve are so close to the true values that the approximation may be expected to be good. At $n=25$ the difference is especially large for B_6. *Since it is likely that a frequency curve with the correct first eight moment coefficients would be* nearer the true distribution than the Type VII curve, it was decided to make a further com-

* *Biometrika*, Vol. xxii. p. 248. The tables are reprinted in *Tables for Statisticians and Biometricians*, Part ii. Table XXXVII*bis*, p. 224. These limits correspond to the points beyond which the chances are ·05 and ·01 of a sample value falling, and are useful in forming a rapid judgment on the significance of departure from normality.

† *Proc. Roy. Soc.* A, Vol. 130 (1930), pp. 16—22.

‡ *Biometrika*, Vol. xxii. p. 423. § *Biometrika*, Vol. xxiv. p. 60.

‖ Since the Type VII curve is symmetrical, all its odd moment coefficients are necessarily correct.

TABLE I.

Sample size	True values from equations (3) and (4)		Values for Type VII curve with correct $\sigma_{\sqrt{\beta_1}}$ and $B_2(\sqrt{\beta_1})$	
	B_4	B_6	B_4	B_6
25	25·4803	297·0075	26·4371	359·2380
50	23·4203	267·4301	23·3040	269·8547

parison at $n=25$, by using one of the Hansmann system of symmetrical curves*. These curves make use of the moments as far as μ_8.

Taking the numerical values for B_4 and B_6 given in Table I, and also the values

$$\sigma_{\sqrt{\beta_1}}=\cdot 435\ 385, \qquad B_2=3{\cdot}578\ 325$$

obtained by putting $n=25$ in equations (1) and (2), it was found on applying Hansmann's rules, that his Type (i) curve is the appropriate one. Its equation takes the form

$$y=y_0\left\{\frac{1-x^2/q}{1+x^2/p}\right\}^{k} \qquad \text{...(5)},$$

where $x=\sqrt{\beta_1}/\sigma_{\sqrt{\beta_1}}$, and

$$\begin{cases} q=98{\cdot}0732, & p=0{\cdot}21540, \\ k=\ 5{\cdot}37496, & y_0=0{\cdot}42402. \end{cases}$$

On applying quadrature to this curve, the 5 % and 1 % limits for $\sqrt{\beta_1}$ are found to lie at 0·711 and 1·061 respectively. Using the Type VII curve the limits lie at 0·710 and 1·056. The former pair are presumably nearer the true values than the latter, but the difference is of no great importance. In fact as n increases above 25 and B_2 approaches the value of 3†, we may anticipate that the Type VII approximation to the unknown true sampling distribution will steadily improve. Dr Pearson's table has therefore been extended in Table II, using the Type VII approximation, except in the case of $n=25$ where Hansmann's curve has been used.

The interpretation of these limits is as follows. If, for example, a sample of 25 is drawn from a normal population, the chance is ·05 that $\sqrt{\beta_1}$ will lie above ·711; it is also ·05 that $\sqrt{\beta_1}$ will lie below $-$·711. Similarly the chance is ·01 that $\sqrt{\beta_1}$ will lie above 1·061, and ·01 that it will lie below $-$1·061. It is clear that for samples of 100 or less the sampling variation in $\sqrt{\beta_1}$ is very considerable so that it will be impossible to detect even large departures from normality in the population. The table however should be useful even if only to emphasise this fact.

In calculating the percentage limits of the Type VII curves it was possible to use the corresponding limits of the "t" distribution as tabled by Professor Fisher ‡. This we may do by noting that for the general Type VII curve

$$y=y_0\left(1+\frac{x^2}{a^2}\right)^{-m} \qquad \text{...(6)},$$

where

$$a^2=\frac{2\sigma^2 B_2}{B_2-3}, \qquad m=\frac{5B_2-9}{2(B_2-3)} \qquad \text{.................................(7)}.$$

* G. H. Hansmann, "On certain Non-normal Symmetrical Frequency Distributions," *Biometrika*, XXVI, pp. 129—95.

† The following table shows the changing values of B_2 as n increases:

n	10	15	20	25	30	40	50	60	70
$B_2(\sqrt{\beta_1})$	3·32	3·52	3·57	3·57	3·56	3·50	3·45	3·41	3·37

‡ *Statistical Methods for Research Workers*, Table IV, at end of work.

TABLE II.

Size of sample	Limits for $\sqrt{\beta_1}$	
	5 %	10 %
25	·711	1·061
30	·661	·982
35	·621	·921
40	·587	·869
45	·558	·825
50	·533	·787
60	·492	·723
70	·459	·673
80	·432	·631
90	·409	·596
100	·389	·567

If in Professor Fisher's notation n is the number of degrees of freedom, then (6) becomes the t *distribution on writing*

$$m=\tfrac{1}{2}(n+1) \quad\text{...(8)},$$

$$x/a=t/\sqrt{n} \quad\text{...(9)}.$$

Equations (1) and (2) with (7) give the appropriate values of a and m, the n with which to interpolate in the t tables is given by (8), and the resulting 5 % and 1 % values for t when inserted in (9) will give the required probability levels for $x=\sqrt{\beta_1}$.

It is hoped shortly to carry out a similar investigation regarding the sampling distribution of β_2.

(iv) Note on Dr Usher's Paper on Epicanthus (pp. 5—25 of this volume).

By KARL PEARSON.

The interpretation of the valuable material collected by Dr Usher is not easy. In his statistics he has excluded Figs. 24 and 25. Further "continuous" and "discontinuous" inheritance may be interpreted differently by different readers. The terms indeed lose much of their meaning, if we accept the view that epicanthus may disappear with growth; because by predicating a loss we can transfer a pedigree from the discontinuous to the continuous group. Further are we to reckon by "offspring" or by "parents"? Neglecting possible disappearance by growth, the Table on the following page represents the data presented by Dr Usher, and the reader is left to interpret the classification as he pleases.

Tabulated Data of Dr Usher's Epicanthus Paper.

Fig.	Number of Cases			Inheritance					
				Through Male		Through Female		Discontinuous	
	Male	Female	No sex given	Offspring	Fathers	Offspring	Mothers	Offspring	Parent
1	1	3	—	—	—	4	1	—	—
2	1	2	—	—	—	2	1	—	—
3	1	—	2	2	1	—	—	—	—
4	—	—	5	—	—	—	—	?5	?1
5	—	—	4	—	—	—	—	4	1
6	1	4	—	—	—	4	3	—	—
7	20	12	—	25	10	6	3	—	—
8	4	—	—	3	1	—	—	?1	?1
9	2	—	—	1	1	—	—	?1	?1
10	5	1	—	—	—	5	1	—	—
11	—	3	—	—	—	—	—	3	1
12	3	4	—	—	—	2	1	4	1
13	—	3	—	—	—	2	1	1	1
14	4	2	—	—	—	1	1	5	3
15	3	—	—	—	—	—	—	3	2
16	5	5	—	2	1	1	1	6	3
17	3	4	—	4	1	5	2	1	1
18	2	3	—	2	1	1	1	—	—
19	8	8	—	1	1	14	4	—	—
20	1	2	—	—	—	—	—	3	2
21	2	2	—	—	—	—	—	4	1
22	2	5	—	—	—	—	—	7	1
23	4	—	—	—	—	—	—	4	1
24*	—	1	—	—	—	—	—	1	1
25*	—	1	—	—	—	—	—	1	1
26	3	1	—	—	—	—	—	4	1
27	—	7	—	—	—	—	—	7	2
28	1	2	—	—	—	1	1	2	2
29	5	—	—	3	1	—	—	2	1
30	3	2	—	—	—	1	1	4	2
31	—	2	—	—	—	1	1	?1	?1
32	2	1	—	—	—	2	1	—	—
33	—	3	—	—	—	—	—	2	2
34	3	—	—	2	1	—	—	—	—
	80	83	11	45	10	52	24	68 (+?8)	30 (+?4)
	$\underbrace{\hspace{6em}}_{(183)}$								

* Not included by Dr Usher.

Vol. XXVII. Parts III and IV October, 1935

BIOMETRIKA

A JOURNAL FOR THE STATISTICAL STUDY OF BIOLOGICAL PROBLEMS

FOUNDED BY

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ISSUED BY THE BIOMETRIKA OFFICE
UNIVERSITY COLLEGE, LONDON
AND PRINTED AT THE
UNIVERSITY PRESS, CAMBRIDGE

Reprinted by offset-litho 1952

[Issued 31 October 1935]

VOLUME XXVII OCTOBER, 1935 PARTS III AND IV

BIOMETRIKA

STATISTICAL PRINCIPLES OF ROUTINE WORK IN TESTING CLOVER SEED FOR DODDER.

BY J. PRZYBOROWSKI AND H. WILEŃSKI.

(Plant Breeding and Agricultural Experimentation Department, University of Cracow, Poland.)

I. *Introductory.*

The testing of red clover and other small seeds for dodder is one of the important everyday tasks of a Seed Testing Station. As the result of such tests can only be probable and never certain, it is clear that the routine work of these Stations must have a sound statistical basis. The only other writer on this subject that we know of is Schindler*, but we wish to attack the problem from another standpoint.

The samples of clover seeds which are tested contain usually 100—200 gm., or over 50,000 single seed of clover. If in such a sample there are found several dodder seeds (the number varies from 0 to 36 per kg. according to the different countries) this would mean an unfavourable result of the test, and the seed would be considered unsatisfactory. Therefore in sampling clover we are in the position of making very numerous (over 50,000) and practically independent trials, and of counting those trials in which a definite event occurs. Each of these hypothetical trials consists in selecting at random a single seed to form the sample, the event being that the selected seed is a dodder seed.

It is known that if the number of independent trials, n, is very large and the probability, p, of an event, E, in these trials is constant and very small, then the number of occurrences of this event, k, follows the Poisson Law

$$P_k = e^{-m}\frac{m^k}{k!} \qquad\qquad (1),$$

where $m = np$, the mathematical expectation $\mathcal{E}(k) = m$ and the standard error $= \sqrt{m}$.

W. Bortkiewicz† was the first to call attention to the practical use of the Poisson Law. Since then it has been applied to numerous problems‡.

* J. Schindler: "Untersuchungen über Kleeseide-Verteilung in schwach seidehaltigen Kleesamen." *Landwirtschaftliche Versuchs-Stationen*, Bd. CVIII, Leipzig 1929.

† W. Bortkiewicz: *Das Gesetz der kleinen Zahlen*, Leipzig 1898.

‡ "Student": "On the Error of Counting with a Haemacytometer." *Biometrika*, Vol. v. 1907. K. Iwaszkiewicz and J. Neyman: "Counting Virulent Bacteria and Particles of Virus." *Acta Biologiae Experimentalis*, Vol. VI. Warsaw 1931. [The caution with which the Poisson Series must be applied is emphasised by L. Whitaker: *Biometrika*, Vol. x. pp. 36—71. In all cases of applying Poisson's Series the corresponding binomial ought to be worked out. This in the case of much of the data treated by Bortkiewicz, Mortara and others turns out to be a *negative* binomial, and the application of a Poisson

The purpose of this paper is twofold: (i) to show that we are justified in assuming that the number of dodder seeds in samples of clover does follow the Poisson Law and (ii) to construct rules, based on this assumption, which should be followed in routine work in order to minimise the chance of errors in tests for dodder. Some of these rules may have wider application in other sampling problems where the Poisson distribution holds good.

II. *Testing Goodness of Fit of the Poisson Series.*

When testing the assumption that the distribution of dodder in samples of clover seeds follows the Poisson Law we have to keep in mind the circumstances which may cause variations. In all practical cases we are justified in assuming that the probability of a randomly chosen seed being a dodder seed is very small. The only assumption which may be untrue is that of the independency of our "trials." In fact a sample of about 50,000 grains to be tested for dodder is never taken by selecting at random single seeds. On the contrary the samples which are analysed are usually made up of seeds which were in close proximity to each other in the sack. Therefore if the distribution of dodder in the whole mass is not random it will not follow the Poisson Law in the samples. So the problem before us consists really in testing whether the clover seed in ordinary trading conditions is properly mixed or homogeneous with regard to dodder. This has been tested experimentally for us by Broniewski. By means of his experiment we contrived to obtain a sack of clover seed containing a known number of dodder seeds, where the material was mixed to a degree similar to that obtained in seed trading. 100 kg. of red clover seeds were mixed with 2000 dodder seeds; the latter were dyed in order to make them easily distinguishable. In practice the homogeneity of seeds is increased by different processes of threshing, cleaning, etc., then the seed is transported in sacks, often on bad roads, and here the homogeneity may be diminished through continual shaking, the smaller dodder seeds slipping to the bottom. We have tried to allow for this in our experiments. After they were mixed and put into a sack, the seeds were carried in a four-wheeled waggon on a bad road over a distance of about 6 km. Then beginning from the top of the sack, 500 clover samples of about 100 gm. each were taken and examined for dodder. The result was that 986 dodder seeds were found. The small deviation from 1000, which was the expected number of dodder seeds in the 50 kg. analysed, may be regarded as the result of random sampling. The possibility should also be taken into consideration that in the course of the experiment a few dodder seeds may have remained unnoticed and therefore not included in the 986.

In Table I A the empirical results are compared with the theoretical numbers, given by the Poisson Series, assuming that the mathematical expectation $m=2$, i.e. the *a priori* known mean content of dodder per sample.

Series is of doubtful validity. The origin of negative binomials has been discussed by K. Pearson: *Biometrika*, Vol. XI, pp. 139—144 and especially by "Student": *Ibid.* Vol. XII, pp. 211—215. These papers should certainly be studied by those who simply apply the P, χ^2 test, without consideration of the binomial of which the Poisson Series is supposed to be the limit. ED.]

TABLE I.

Distribution of Dodder in 500 Samples of Clover Seeds. (Our Experiment.)

A. $m=2$			B. $m=1{\cdot}972$		
k	N_k	$N.P_k$	k	N_k	$N.P_k$
0	56	67·67	0	56	69·59
1	156	135·34	1	156	137·23
2	132	135·34	2	132	135·31
3	92	90·22	3	92	88·94
4	37	45·11	4	37	43·85
5	22	18·04	5	22	17·29
6	4	6·02 } 8·29	6	4	5·68 } 7·79
7	0	1·72	7	0	1·60
8	1	0·43	8	1	0·39
over 8	0	0·12	over 8	0	0·12
$n'=7,\ n=n'-1,\ \chi^2=8{\cdot}9160$ $P_{\chi^2}=0{\cdot}179$			$n'=7,\ n=n'-1$*, $\chi^2=8{\cdot}7599$ $P_{\chi^2}=0{\cdot}189$		

k = number of dodder seeds in a sample.
N_k = observed frequency.
$N.P_k$ = expected frequency.
n' = number of groups.
n = degrees of freedom.

In Table I B we have accepted m as unknown, i.e. we have considered these samples as drawn from material, the dodder content of which is entirely unknown to us; thus we estimated m from

$$m=\frac{1}{N}\Sigma kN_k \quad \text{.................................(2)},$$

where k is the number of dodder seeds in a 100 gm. sample; N_k the number of samples in which we have found k seeds; N the general number of the investigated samples. Here the approximate value of $m=986/500=1{\cdot}972$.

The diagram corresponds to the numbers given in Table I B.

It will be seen from both tables that the Poisson Law fits extremely well to our data. Thus we may conclude that if the ordinary trading conditions are similar to the conditions of our experiment, which we believe to be true, then the assumption that the frequency of dodder in samples of seeds follows the Poisson Law is generally justified.

* According to the view of R. A. Fisher, E. S. Pearson and J. Neyman the degrees of freedom in this case should be $n'-2$, and accordingly $P_{\chi^2}=0{\cdot}120$, i.e. *less* than P_{χ^2} for $m=2$, which is $0{\cdot}179$.

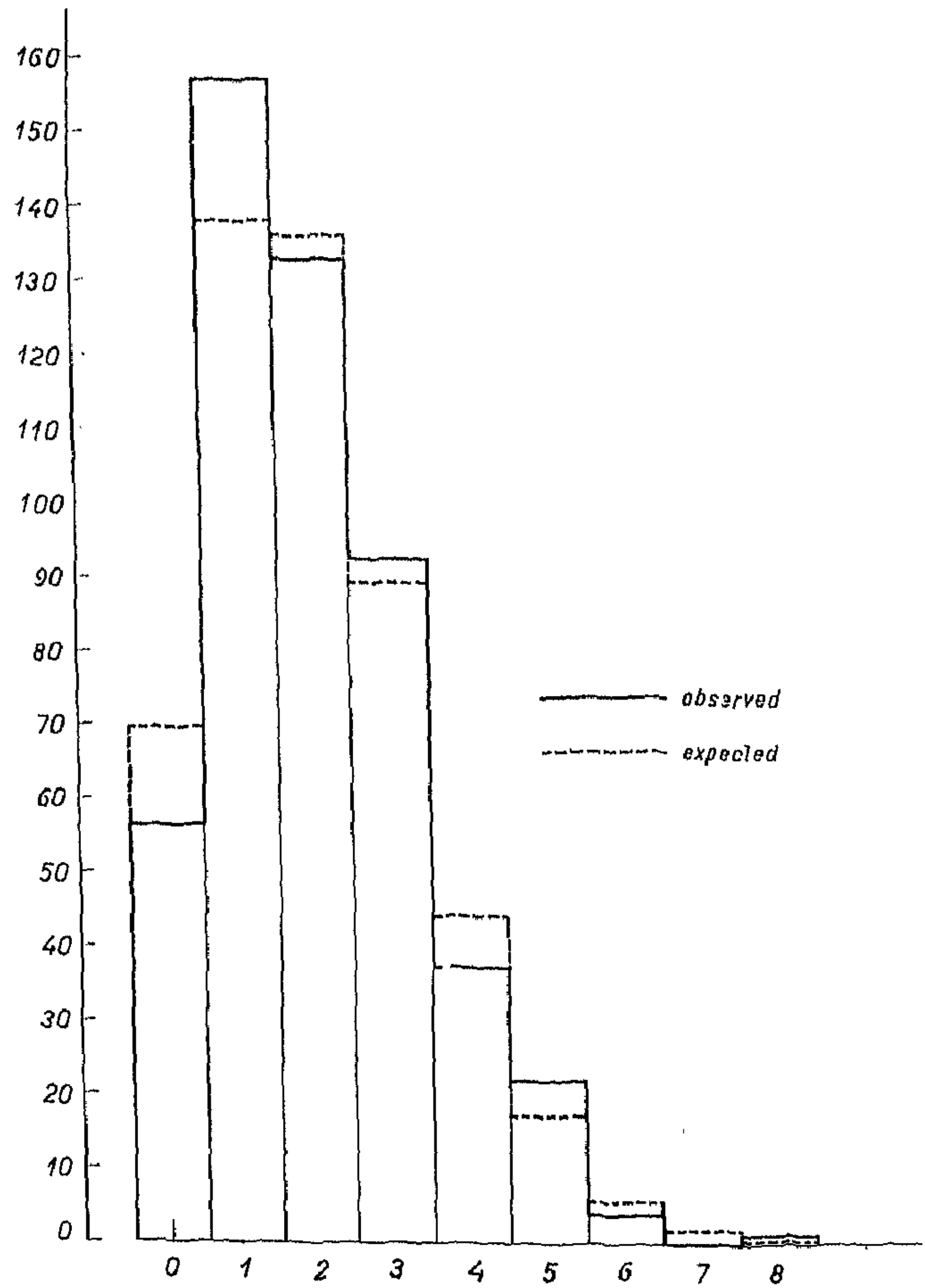

160
150
140
130
120
110
100
90
80
70
60
50
40
30
20
10
0
0 1 2 3 4 5 6 7 8
observed
expected

TABLE II.

Distribution of Dodder in Samples of Clover Seed. (Schindler's Experiments.)

1

k	N_k	$N.P_k$	
0	108	183·94	
1	205	183·94	
2	94	91·97	
3	26	30·66	
4	6	7·66	9·49
5	1	1·53	
over 5	0	0·30	
m	1		
n'	5		
χ^2	5·1990		
P_{χ^2}	0·270		

2

k	N_k	$N.P_k$	
0	599	606·53	
1	315	303·27	
2	74	75·82	
3	12	12·64	14·38
over 3	0	1·74	
m	0·5		
n'	4		
χ^2	0·0848		
P_{χ^2}	0·804		

3

k	N_k	$N.P_k$	
0	382	389·40	
1	111	97·35	
2	7	12·17	13·25
over 2	0	1·08	
m	0·25		
n'	3		
χ^2	5·0027		
P_{χ^2}	0·082		

4

k	N_k	$N.P_k$	
0	284	303·27	
1	170	151·63	
2	39	37·91	
3	7	6·32	7·19
over 3	0	0·87	
m	0·5		
n'	4		
χ^2	3·4863		
P_{χ^2}	0·328		

5

k	N_k	$N.P_k$	
0	795	774·64	
1	94	116·20	
2	11	8·71	9·20
over 2	0	0·45	
m	0·15		
n'	3		
χ^2	5·1260		
P_{χ^2}	0·078		

6

k	N_k	$N.P_k$	
0	447	452·42	
1	51	45·24	
2	2	2·26	2·34
over 2	0	0·08	
m	0·1		
n'	3		
χ^2	0·8477		
P_{χ^2}	0·656		

7

k	N_k	$N.P_k$
0	473	475·61
1	26	23·78
over 1	1	0·61
m	0·05	
n'	3	
χ^2	0·4709	
P_{χ^2}	0·815	

8

k	N_k	$N.P_k$	
0	295	303·27	
1	153	151·63	
2	44	37·91	
3	8	6·32	7·19
over 3	0	0·87	
m	0·5		
n'	4		
χ^2	1·3075		
P_{χ^2}	0·731		

9

k	N_k	$N.P_k$	
0	22	16·42	
1	20	41·04	
2	55	51·30	
3	43	42·75	
4	34	26·72	
5	10	13·36	
6	3	5·57	
7	4	1·99	8·41
over 7	0	0·85	
m	2·5		
n'	7		
χ^2	8·7617		
P_{χ^2}	0·189		

10

k	N_k	$N.P_k$	
0	0	1·08	
1	3	5·39	
2	13	13·48	
3	15	22·46	
4	33	28·07	
5	28	28·07	
6	24	23·40	
7	21	16·71	
8	10	10·44	
9	8	5·80	
10	0	2·90	10·90
11	4	1·32	
over 11	1	0·88	
m	5		
n'	10		
χ^2	7·0406		
P_{χ^2}	0·633		

11

k	N_k	$N.P_k$	
0	0	0·09	
1	0	0·66	
2	1	2·49	
3	4	6·22	
4	9	11·67	
5	16	17·50	
6	19	21·87	
7	19	23·44	
8	26	21·97	
9	19	18·31	
10	15	13·73	
11	14	9·36	
12	5	5·85	
13	6	3·38	
14	3	1·81	12·68
15	3	0·90	
over 15	1	0·74	
m	7·5		
n'	13		
χ^2	9·8060		
P_{χ^2}	0·633		

k = number of dodder seeds in a sample.
N_k = observed frequency.
$N.P_k$ = expected frequency.

n' = number of groups.
$n'-1$ = degrees of freedom.

Table II gives the results of fitting the Poisson Series to data published by Schindler. The results are often much better than in our case, but this is probably due to the fact that in Schindler's experiments special efforts were made to increase the homogeneity of the material. These efforts increase the agreement between the observed and expected frequencies but diminish the practical importance of the results.

Although we think that the homogeneity of material in our experiment was similar to that in ordinary trading conditions, it is useful to study the dodder distributions in samples drawn by the usual methods from stocks prepared for sale according to the ordinary trading treatment. One set of data of this kind is analysed below. (See p. 281.)

Lastly it should be noted that the independence of samples in Schindler's experiments as well as in ours, was by no means complete. However, the correlation could only be very slight and not have any significant influence on the numerical results. Therefore it will be assumed in the following sections that the number of dodder in samples of clover seed follows exactly the Poisson Law.

III. *Accuracy of the Analysis for Dodder.*

When given seed is analysed for dodder, one of the two following questions may be asked:

(i) Does the dodder content in the seed exceed a certain fixed standard?

(ii) What is the actual dodder content in the seed?

The solution of the first question involves what is called the test of a statistical hypothesis. The other question is an instance of another statistical problem, that of estimation.

(*a*) *Testing the Hypothesis concerning Dodder Content.* After examining a sample of seeds, we want to estimate the average number of dodder seeds in a unit of weight, e.g. in 1 kg. of the stock under consideration. Practically we are only concerned with the question whether that average content does or does not exceed the legal, customary or agreed standard. According to the trading customs accepted in various countries this usual standard varies from 0 to 36 dodder seeds per kg. The accepted standard, let us say, m_0 dodder seeds in 1 kg. of stock, is a limit between two classes of clover seed on the market. This limit will be called the tolerance limit of the true mean dodder content. Stocks containing m_0 or less dodder seeds in a unit of weight belong to one class, while those containing more than m_0 belong to the other class. In each case we are faced with the problem of deciding to which class the material under consideration belongs.

In the theory of testing statistical hypotheses, initiated by J. Neyman and E. S. Pearson*, we distinguish two kinds:

(1) a simple hypothesis which completely specifies the probability law,

(2) a composite hypothesis which specifies the probability law only partially.

* J. Neyman and E. S. Pearson. "On the Problem of the most efficient Tests of Statistical Hypotheses." *Phil. Trans. Roy. Soc.* 1933, Vol. 231 A. pp. 289—337.

It will be seen that when a composite hypothesis is given there are many different ways in which we may complete the specification of the probability law, and thus transform the given composite hypothesis into different simple ones. These hypotheses form a class of simple hypotheses "belonging" to the given composite one.

In our case the statistical hypothesis we test will concern the true mean number of dodder seeds in samples of clover seed. This number will be denoted by m; x will denote the number of dodder seeds in a sample. According to our previous convention it will be assumed that

$$p(x) = e^{-m}\frac{m^x}{x!} \qquad \text{(3)}$$

represents the frequency law of x, given m. Any hypothesis specifying the value of m, say $m = m_i$, will be a simple hypothesis, the probability law $p(x)$ being completely specified. On the other hand if a hypothesis states only that $m > m_i$ or $m \leqslant m_i$ or $m_i < m \leqslant m_j$, then the hypothesis does not specify the probability law completely and is a composite one. Simple hypotheses belonging to the composite which states that $m < 5$ will be all those which ascribe to m any value less than 5, e.g. $m = 1$, 1·5, ..., etc.

Denote by m_0 the conventional tolerance limit of the dodder content per 100 gm. of seeds. The practical question whether any given seeds contain more or less dodder than the tolerance limit allows may be reduced to the mathematical problem of testing one or other of the following statistical hypotheses. Either we may test the hypothesis $m \leqslant m_0$ or the hypothesis $m > m_0$.

The first of these hypotheses will be called the favourable hypothesis and denoted by F. The other hypothesis that $m > m_0$ will be called unfavourable and denoted by U. Obviously both F and U are composite hypotheses.

When testing statistical hypotheses one must bear in mind two kinds of possible errors: those of the first kind which consist in rejecting the hypothesis when it is true and those of the second kind which consist in accepting the hypothesis when it is false. Neyman and Pearson have developed a theory which allows us to choose from all possible criteria one which is able to control, in the most efficient way, both kinds of errors*. However, our present problem is so simple that there is no need to apply their theory, as the observations only supply us with one number, x, which is the number of dodder seeds in the sample, and which therefore is the only criterion to use.

If we were testing the hypothesis F, stating that $m \leqslant m_0$, we should reject the hypothesis if x, the number of dodder seeds in the sample, is "too large." On the other hand if x is not "too large" we should accept F. If it were the hypothesis U which we were testing we should proceed the other way round and reject U whenever x is "sufficiently small," say $x \leqslant a$. Here we shall only use the Neyman-

* See J. Przyborowski et H. Wileński: "Sur les erreurs de la première et de la seconde catégorie dans la Vérification des Hypothèses concernant la loi de Poisson." *Comptes rendus des séances de l'Académie des Sciences*, Vol. cc. p. 1460. Paris, 1935.

Pearson theory in order to give an exact meaning to the words in inverted commas, namely, "too large" and "sufficiently small." We shall have to take into consideration the probability of both kinds of errors which would correspond to any number which could be chosen to serve as a conventional limit between the values of x which are "too large" and the values which are not "too large."

In the following we shall discuss the problem, assuming that we are testing the hypothesis U. As it was stated this hypothesis will be rejected, and thus the seed will be considered satisfactory, whenever x is "sufficiently small." Denote by a any number which could be used as a conventional limit between the values of x which are "sufficiently small" and those which are not. This will be called the tolerance limit of dodder content in the sample. If a sample contains a dodder seeds or less, the hypothesis U will be rejected.

Let us estimate the probability, say $P_1(m_0, a)$, of first kind errors corresponding to the fixed values of a and m_0. Denote by $p(m)$ the probability law *a priori* concerning the mean number, m, of dodder seeds per sample. Obviously

$$P_1(m_0, a) = \int_{m_0}^{\infty} P\{(x \leqslant a) \mid m\} \, p(m) \, dm \quad \text{..................(4)},$$

where $P\{(x \leqslant a) \mid m\}$ is the probability corresponding to the fixed value of m of having a dodder seeds, at the most, in a sample. In the above formula $p(m)$ is unknown. Therefore the actual value of $P_1(m_0, a)$ is unknown. However, we can find a value which $P_1(m_0, a)$ never exceeds.

Obviously for any value $m > m_0$ we shall have

$$P\{(x \leqslant a) \mid m\} < P\{(x \leqslant a) \mid m_0\} \quad \text{........................(5)}.$$

Therefore

$$P_1(m_0, a) \leqslant P\{(x \leqslant a) \mid m_0\} \int_{m_0}^{\infty} p(m) \, dm \leqslant P\{(x \leqslant a) \mid m_0\} \quad \text{........(6)}$$

as the integral

$$\int_{m_0}^{\infty} p(m) \, dm \leqslant 1 \quad \text{..................................(7)},$$

whatever the positive number m_0 and whatever the unknown function $p(m)$. It is thus seen that though we are not able to calculate the actual value of the probability of first kind errors, we are able to determine the upper bound of this probability. This is given by

$$P\{(x \leqslant a) \mid m_0\} = \sum_{x \leqslant a} e^{-m_0} \frac{m_0^x}{x!} = \frac{1}{a!} \int_{m_0}^{\infty} x^a e^{-x} dx \quad \text{..............(8)}.$$

If in fixing the rules of testing the hypothesis U we choose the values of m_0 and a so that $P\{(x \leqslant a) \mid m_0\} = \epsilon$, where ϵ is any small number, say ·05 or ·01, chosen in advance, then we may be certain that in the long run the hypothesis tested will be wrongly rejected with a relative frequency not exceeding ϵ.

Now let us turn to the errors of the second kind which consist in stating that the seed tested is not satisfactory, whereas in fact the average amount of dodder

it contains does not exceed say $m' < m_0$. Denote by $P_{\text{II}}(m', a)$ the probability of such an error. Using all previous notation we shall have

$$P_{\text{II}}(m', a) = \int_0^{m'} P\{(x > a) \mid m\}\, p(m)\, dm \quad \text{.................(9)},$$

where $P\{(x > a) \mid m\}$ is the probability, corresponding to the fixed value of m, of having more than a dodder seeds in a sample. It is easily seen that owing to the function $p(m)$ being unknown, we are not able to obtain the actual value of $P_{\text{II}}(m', a)$. However, as for $m \leqslant m'$,

$$P\{(x > a) \mid m\} \leqslant P\{(x > a) \mid m'\} \quad \text{.......................(10)},$$

we may write again

$$P_{\text{II}}(m', a) \leqslant P\{(x > a) \mid m'\} \int_0^{m'} p(m)\, dm \leqslant P\{(x > a) \mid m'\} \quad \text{......(11)},$$

as the integral $\int_0^{m'} p(m)\, dm$ cannot exceed unity.

The formulae (6) and (11) give us the limits which the probabilities of first and second kind errors cannot exceed. However, it will be noticed that in unfavourable cases *these limits may be approached by the corresponding probabilities* as closely as desired, and as in arranging the rules for routine work we have to think of the most unfavourable conditions, we may deal with the upper limits as if they were the values of the probabilities themselves.

(*b*) *Numerical Illustrations.* The values of $P\{(x \leqslant a) \mid m_0\}$ and $P\{(x > a) \mid m'\}$ are given in Tables III and IV respectively*. The use of these tables may be illustrated by the following examples:

Example I. Suppose the tolerance limit m_0 of dodder per sample is fixed at $m_0 = 6$. In other words we assume that if the true average content of dodder per sample in the seed tested exceeds $m_0 = 6$, then it is agreed to consider such a seed unsatisfactory. Suppose further that the rules of testing are set up as follows: the Buyer is willing to accept the seed only when the random sample contains dodder seeds not exceeding, say, $a = 5$. In other cases the Buyer will refuse the seed. Thus the tolerance limit for the true mean dodder content is $m = 6$ and the tolerance limit of the dodder content in the sample is $a = 5$. We now turn to Table III in order to judge whether the interests of the Buyer are properly protected. We find that the value of $P\{(x \leqslant a) \mid m_0\}$ corresponding to these limits is ·446. This means that if the above rules are followed, then unsatisfactory seed will be accepted by the Buyer, with a relative frequency which, in unfavourable cases, may be as large as 44 per cent. It is obvious therefore that if the tolerance limit of the true dodder content is really $m_0 = 6$, the tolerance limit for the dodder content in the sample, fixed at $a = 5$, does not present any real protection to the interests of the Buyer.

* These were calculated using the *Tables of the Incomplete Γ-Function*, edited by Karl Pearson and published by the Office of Biometrika, University College, London. Reissue 1934.

TABLE III.

Upper Limits of the Probability of First Kind Errors: $P\{(x \leq a) \mid m_0\}$.

a / m_0	0	1	2	3	4	5	6	7	8	9	10	a / m_0
0·1	·905											0·1
0·5	·607											0·5
1	·368	·736										1
2	·135	·406	·677									2
3	·050	·199	·423	·647								3
4	·018	·092	·238	·433	·629							4
5	·007	·040	·125	·265	·440	·616						5
6	·002	·017	·062	·151	·285	·446	·606					6
7	·001	·007	·030	·082	·173	·301	·450	·599				7
8	—	·003	·014	·042	·100	·191	·313	·453	·593			8
9	—	·001	·006	·021	·055	·116	·207	·324	·456	·587		9
10	—	—	·003	·010	·029	·067	·130	·220	·333	·458	·583	10
11	—	—	·001	·005	·015	·038	·079	·143	·232	·341	·460	11
12	—	—	·001	·002	·008	·020	·046	·090	·155	·242	·347	12
13	—	—	—	·001	·004	·011	·026	·054	·100	·166	·252	13
14	—	—	—	—	·002	·006	·014	·032	·062	·109	·176	14
15	—	—	—	—	·001	·003	·008	·018	·037	·070	·118	15
16	—	—	—	—	—	·001	·004	·010	·022	·043	·080	16
17	—	—	—	—	—	·001	·002	·005	·013	·026	·049	17
18	—	—	—	—	—	—	·001	·003	·007	·015	·030	18
19	—	—	—	—	—	—	·001	·002	·004	·009	·018	19
20	—	—	—	—	—	—	—	·001	·002	·005	·011	20
21	—	—	—	—	—	—	—	—	·001	·003	·006	21
22	—	—	—	—	—	—	—	—	·001	·002	·004	22
23	—	—	—	—	—	—	—	—	—	·001	·002	23
24	—	—	—	—	—	—	—	—	—	—	·001	24
25	—	—	—	—	—	—	—	—	—	—	·001	25

a = tolerance limit for the dodder content in the sample.
m_0 = " " " true mean dodder content.

Looking at Table III it will be seen that keeping the tolerance limit of the true mean dodder content at $m_0 = 6$, we may procure a reasonable level of safety for the Buyer if we fix the tolerance limit of dodder content in the sample at $a = 2$ or even $a = 1$. This means that the seed will be accepted as satisfactory whenever the dodder content in the sample does not exceed 2 or 1 seeds. Even with such precautions the Buyer may get unsatisfactory goods. This, however, will happen but rarely and with a relative frequency not exceeding ·06 in the one case and ·02 in the other.

Example II. We have considered the problem of tolerance limits from the point of view of the Buyer, now let us consider it from the Seller's.

Assume that the tolerance limit for dodder content in the sample has been fixed at $a = 2$ to suit the interests of the Buyer who is not willing to accept seed with a true mean content of dodder exceeding $m_0 = 6$ seeds per size of sample. The Seller will naturally be interested in the chance that the seed, which actually contains less than 6 dodder seeds per sample, should successfully pass the test.

The probabilities of second kind errors, i.e. that the seed will be unjustly rejected, are given in Table IV, where a is again the tolerance limit for the sample and m' the assumed true mean content of dodder in the seed tested.

TABLE IV.

Upper Limits of the Probability of Second Kind Errors: $P\{(x>a)\mid m'\}$.

m' / a	·05	·1	·2	·25	·5	1	2	2·5	5	10	20	25	m' / a
0	·049	·095	·181	·221	·393	·632	·865	·918	·993	—	—	—	0
1	·001	·005	·018	·027	·090	·264	·594	·713	·960	·999	—	—	1
2	—	—	·001	·002	·014	·080	·323	·456	·875	·997	—	—	2
3	—	—	—	—	·002	·019	·143	·242	·735	·990	—	—	3
4	—	—	—	—	—	·004	·053	·109	·560	·971	—	—	4
5	—	—		—	—	·001	·017	·042	·384	·933	—	—	5
6	—	—	—	—	—	—	·005	·014	·238	·870	—	—	6
7	—	—	—	—	—	—	·001	·004	·133	·780	·999	—	7
8	—	—	—	—	—	—	—	·001	·068	·667	·998	—	8
9	—	—	—	—	—	—	—	—	·032	·542	·995	—	9
10	—	—	—	—	—	—	—	—	·014	·417	·989	—	10
11	—	—	—	—	—	—	—	—	·005	·303	·979	·999	11
12	—	—	—	—	—	—	—	—	·002	·208	·961	·997	12
13	—	—	—	—	—	—	—	—	·001	·136	·934	·994	13
14	—	—	—	—	—	—	—	—	—	·084	·895	·988	14
15	—	—	—	—	—	—	—	—	—	·049	·843	·978	15
16	—	—	—	—	—	—	—	—	—	·027	·779	·962	16
17	—	—	—	—	—	—	—	—	—	·014	·703	·940	17
18	—	—	—	—	—	—	—	—	—	·007	·619	·908	18
19	—	—	—	—	—	—	—	—	—	·003	·530	·866	19
20	—	—	—	—	—	—	—	—	—	·002	·441	·815	20
21	—	—	—	—	—	—	—	—	—	·001	·356	·753	21
22	—	—	—	—	—	—	—	—	—	—	·279	·682	22
23	—	—	—	—	—	—	—	—	—	—	·220	·606	23
24	—	—	—	—	—	—	—	—	—	—	·157	·527	24
25	—	—	—	—	—	—	—	—	—	—	·112	·447	25
26	—	—	—	—	—	—	—	—	—	—	·078	·371	26
27	—	—	—	—	—	—	—	—	—	—	·053	·300	27
28	—	—	—	—	—	—	—	—	—	—	·034	·237	28
29	—	—	—	—	—	—	—	—	—	—	·022	·182	29
30	—	—	—	—	—	—	—	—	—	—	·014	·137	30
31	—	—	—	—	—	—	—	—	—	—	·008	·100	31
32	—	—	—	—	—	—	—	—	—	—	·005	·072	32
33	—	—	—	—	—	—	—	—	—	—	·003	·050	33
34	—	—	—	—	—	—	—	—	—	—	·001	·034	34
35	—	—	—	—	—	—	—	—	—	—	·001	·023	35
36	—	—	—	—	—	—	—	—	—	—	—	·015	36
37	—	—	—	—	—	—	—	—	—	—	—	·009	37
38	—	—	—	—	—	—	—	—	—	—	—	·006	38
39	—	—	—	—	—	—	—	—	—	—	—	·003	39
40	—	—	—	—	—	—	—	—	—	—	—	·002	40
41	—	—	—	—	—	—	—	—	—	—	—	·001	41
42	—	—	—	—	—	—	—	—	—	—	—	·001	42

a = tolerance limit for dodder content in a sample.

m' = assumed true mean dodder content per size of sample.

TABLE IV *(continued)*.

$m' = 50$	
a	
29	·999
30	·998
31	·997
32	·996
33	·993
34	·989
35	·984
36	·976
37	·966
38	·953
39	·935
40	·914
41	·888
42	·856
43	·820
44	·779
45	·733
46	·683
47	·630
48	·575
49	·518
50	·462
51	·408
52	·355
53	·304
54	·258
55	·216
56	·178
57	·145
58	·117
59	·092
60	·072
61	·056
62	·042
63	·032
64	·024
65	·017
66	·013
67	·009
68	·006
69	·004
70	·003
71	·002
72	·001
73	·001

$m' = 100$			
a		a	
71	·999	103	·358
72	·998	104	·322
73	·997	105	·287
74	·996	106	·255
75	·995	107	·225
76	·993	108	·196
77	·990	109	·171
78	·987	110	·147
79	·983	111	·126
80	·978	112	·107
81	·971	113	·091
82	·964	114	·076
83	·954	115	·063
84	·943	116	·052
85	·930	117	·043
86	·915	118	·035
87	·897	119	·028
88	·877	120	·023
89	·854	121	·018
90	·829	122	·014
91	·802	123	·011
92	·772	124	·009
93	·740	125	·007
94	·705	126	·005
95	·669	127	·004
96	·632	128	·003
97	·593	129	·002
98	·554	130	·002
99	·514	131	·001
100	·474	132	·001
101	·434	133	·001
102	·396		

$m' = 200$			
a		a	
158	·999	203	·398
159	·998	204	·371
160	·998	205	·345
161	·997	206	·320
162	·997	207	·295
163	·996	208	·271
164	·995	209	·249
165	·994	210	·227
166	·992	211	·207
167	·991	212	·188
168	·989	213	·170
169	·986	214	·153
170	·983	215	·137
171	·980	216	·122
172	·976	217	·109
173	·972	218	·097
174	·966	219	·086
175	·961	220	·075
176	·954	221	·066
177	·946	222	·058
178	·938	223	·050
179	·928	224	·044
180	·918	225	·038
181	·906	226	·032
182	·893	227	·028
183	·879	228	·024
184	·865	229	·020
185	·848	230	·017
186	·830	231	·015
187	·811	232	·012
188	·791	233	·010
189	·769	234	·009
190	·747	235	·007
191	·723	236	·006
192	·699	237	·005
193	·674	238	·004
194	·647	239	·003
195	·621	240	·003
196	·593	241	·002
197	·566	242	·002
198	·537	243	·001
199	·510	244	·001
200	·481	245	·001
201	·453	246	·001
202	·425	247	·001

It will be seen that if the seed tested contains, on the average, as many as $m' = 5$ grains per sample, then the probability of the seed not being passed by the test is equal to ·875. In other words, if the interests of the Buyer are met by choosing the sample tolerance limit $a = 2$, then in 87·5 per cent. of cases the seed offered will be found unsatisfactory, showing more than two dodder seeds in a

sample, when in fact it contains $m' = 5$ dodder seeds per sample, thus satisfying the requirements of the Buyer. A reasonable chance of passing the test successfully corresponds to $m' = \cdot 5$. Here the probabilities of an unjust rejection of the seed is ·014.

Thus in order to secure a reasonable chance of his goods passing the test, the Seller must aim at a dodder content about 12 times smaller than that which is fixed as the tolerance limit $m_0 = 6$.

The position is most unsatisfactory, as whenever the test is efficiently protecting the interest of one of the contracting parties, the interests of the other are seriously damaged.

Now let us assume that the samples tested are, say three times, larger so that the tolerance limit of the true mean dodder content is correspondingly increased to $m_0 = 18$. It will be seen that a reasonable protection of the Buyer's interests will be attained if the tolerance limit of the dodder content in the sample were fixed at, say, $a = 10$. Doing so we may be certain that the corresponding probability of first kind of error will not exceed ·03. (See Table III.) On the other hand, looking at Table IV, we see that the seed, whose true mean dodder content $m' = 5$, will have a reasonable chance of passing the test, the upper limit of the probability of second kind error being ·014. Now the ratio $m_0/m' = 3{\cdot}6$, and thus the gap between the requirement of the Buyer and the safety level of the Seller is shortened very considerably. It will be seen that the use of larger samples is most beneficial both to the Seller and the Buyer. Unfortunately this means a larger expenditure on analysis. However, we shall make further mention of this point below.

Example III. We shall here call special attention to the case when the tolerance limit of dodder content in the sample is fixed at the lowest possible value $a = 0$. It is interesting to see exactly what sort of protection this provides to the interests of the Buyer. Looking at Table III we find that if $m_0 = 3$, then, on the average, one sample out of 20 would be totally free from dodder. The test applied will thus give a favourable result and the seed will be accepted. We see therefore that statements like "dodder free" if they are based upon analysis of only one sample of usual size may be frequently misleading. The seed which is described as "dodder free" is quite likely to contain as many as two dodder seeds per size of the sample tested, it is also quite possible that it contains three; in fact we should only be surprised when the true mean content is as large or larger than four.

If the size of the sample tested is large, then a dodder content of 2 or 3 grains per sample could be perhaps considered as negligible. But in the ordinary size of sample such amounts would be disappointing, especially in a seed which has been qualified as dodder free.

(c) *Upper Confidence Limits of Dodder Content.* The last example leads us to the new form of problem of estimation which has been started by R. A. Fisher* and

* R. A. Fisher: *Proc. Cam. Phil. Soc.* Vol. XXVI. part 4.

then developed by J. Neyman*, namely the problem of confidence limits for a given unknown parameter. In our case the role of this parameter is played by the true mean dodder content per size of sample in a given consignment of seed.

We think that the best way of expressing the results of an analysis for dodder content consists in stating the limit which presumably is not exceeded by the true mean dodder content, e.g. the seed does not contain more than, say, 4 grains of dodder per kg. The limit stated, viz. 4, is what is called the upper confidence limit. J. Neyman has shown how statements of this kind can be made so that the probability of their being correct will not be smaller than a limit fixed in advance. This limit of probability has been called "the confidence coefficient." If it is said that a seed testing station works at a confidence coefficient, say $\alpha = \cdot 99$, it means that its verdicts concerning the quality of the seed in the form stated above may be wrong only with a relative frequency not exceeding $1-\alpha$ or one in a hundred.

The upper confidence limits, corresponding to the chosen confidence coefficient α, are found as follows: denote by a any integer number of dodder seeds which may be found in a sample, and find such a value of $m = m_a$ that

$$P\{(x > a) \mid m_a\} = \alpha \quad \text{.............................(12).}$$

The numbers m_a thus defined are the upper confidence limits. Suppose, in fact, that the above equation is solved for any $a = 0, 1, 2, \ldots$, and suppose we make a rule of stating that the true mean dodder content, m, in seed does not exceed m_a whenever the analysis of a sample gives $x = a$. It is easy to see that the probability of an error in such a statement never exceeds the limit $1-\alpha$.

This may be proved as follows†: we notice first that an error in estimating the dodder content by stating the upper confidence limit may occur only if the actual dodder content m and the dodder content in the sample satisfy certain inequalities. Namely, we shall be wrong whenever, say,

$$m_a < m \leqslant m_{a+1} \quad \text{................................(13)}$$

and

$$x \leqslant a \quad \text{.....................................(14)}$$

happen at the same time.

Let us estimate the probability of the simultaneous occurrence of the above two inequalities. For this purpose we notice that

$$P\{(x \leqslant a) \mid m\} = 1 - P\{(x > a) \mid m\} \quad \text{....................(15).}$$

If $m_a < m$, then obviously

$$P\{(x > a) \mid m\} > P\{(x > a) \mid m_a\} = \alpha \quad \text{....................(16).}$$

Thus we shall have

$$P\{(x \leqslant a) \mid m\} < 1 - \alpha \text{............................(17).}$$

This being true for any m satisfying (13) and for any a, we conclude that whatever the unknown frequency distribution of m, the probability of having (13) and (14)

* J. Neyman: *Journal R. Statistical Soc.* Vol. XCVII. pp. 589—598.

† See T. Matuszewski, J. Neyman and J. Supińska: *Supplement to the Journal R. Statistical Soc.* Vol. II. part 1.

simultaneously does not exceed $1-\alpha$. This is another way of saying that the probability of errors in statements $m \leqslant m_a$ made when we get, in the sample, exactly $x=a$ dodder seeds, does not exceed $1-\alpha$.

The upper confidence limits for the mean dodder content corresponding to different confidence coefficients, α, are given in Table V*. The use of this table may be illustrated in the following example:

Example IV. Suppose in three different samples of 100 gm. of clover seeds the amount of dodder found was $x_1=0$, $x_2=3$ and $x_3=10$ respectively. The statement concerning the true mean dodder content in the seeds tested must depend upon the caution of the testing station. If it be considered that errors in statements occurring, on the average, once in ten times may be admitted, then the confidence coefficient chosen must be ·9. In this case the statements concerning the true dodder content per 100 gm. of seeds in the consignments tested, should be

$$m_1 \leqslant 2{\cdot}3,\ \ m_2 \leqslant 6{\cdot}7,\ \ m_3 \leqslant 15{\cdot}4.$$

(In order to obtain the estimates per 1 kg. we have only to multiply the above figures by 10.)

However, it may be considered that this frequency of errors, i.e. 1 in 10, is too considerable. Then the confidence coefficient chosen should be larger. It may be desired to commit errors as seldom as one in a hundred, then $\alpha = {\cdot}99$ and our statements about the dodder content in the same consignments would be

$$m_1 \leqslant 4{\cdot}6,\ \ m_2 \leqslant 10{\cdot}0,\ \ m_3 \leqslant 20{\cdot}2.$$

We notice that the above example suggests again that the reliability of tests for dodder requires the use of larger samples. In fact, even when no dodder seeds are found in a sample, if we do not want to risk an error more than once in a hundred statements, we should only state that the true dodder content in the seed does not exceed 4·6 grains per size of sample. If this is large then the accuracy may be sufficient; if, however, the sample is small then it may be considered unsatisfactory.

The question of the size of the sample may be treated more accurately in the following way:

Fix any confidence coefficient, say $\alpha = {\cdot}99$, and suppose that the samples in the above examples contained 100 gm. of seeds. The result of the test of the first sample may be stated by saying that in the seed analysed there was probably no more than 46 grains of dodder per 1 kg.

Now assume that the sample contained 5 kg. of seed, and that again no dodder was found. Our estimate per size of sample will be the same, i.e. not more than 4·6 grains on the average. But if we turn from the size of sample to the standard unit of 1 kg., the statement will change. We shall state, in fact, that in the consignment there is presumably no more than ·9 grain of dodder per 1 kg. In this way it is easy to calculate that if for the sample which proves dodder free it should be sufficient to state that the amount of dodder per 1 kg. of seed does not exceed 1 grain, then the size of this sample should be as large as 4·6 kg.

* This was again calculated using the *Tables of the Incomplete Γ-Function*.

TABLE V.

Upper Confidence Limits, m_α, for the True Mean Dodder Content.

α / a	0·9	0·95	0·98	0·99	0·995	0·999	α / a
0	2·3	3·0	3·9	4·6	5·3	6·9	0
1	3·9	4·7	5·8	6·6	7·4	9·2	1
2	5·3	6·3	7·5	8·4	9·3	11·2	2
3	6·7	7·8	9·1	10·0	11·0	13·1	3
4	8·0	9·2	10·6	11·6	12·6	14·8	4
5	9·3	10·5	12·0	13·1	14·2	16·5	5
6	10·5	11·8	13·4	14·6	15·7	18·1	6
7	11·8	13·1	14·8	16·0	17·1	19·6	7
8	13·0	14·4	16·2	17·4	18·6	21·2	8
9	14·2	15·7	17·5	18·8	20·0	22·7	9
10	15·4	17·0	18·8	20·2	21·4	24·1	10
11	16·6	18·2	20·1	21·5	22·8	25·6	11
12	17·8	19·4	21·4	22·8	24·1	27·0	12
13	19·0	20·7	22·7	24·1	25·5	28·4	13
14	20·1	21·9	24·0	25·5	26·8	29·9	14
15	21·3	23·1	25·2	26·7	28·2	31·2	15
16	22·5	24·3	26·5	28·0	29·5	32·6	16
17	23·6	25·5	27·8	29·3	30·8	34·0	17
18	24·8	26·7	29·0	30·6	32·1	35·4	18
19	25·9	27·9	30·2	31·9	33·4	36·7	19
20	27·0	29·1	31·5	33·1	34·7	38·0	20
21	28·2	30·2	32·7	34·4	36·0	39·4	21
22	29·3	31·4	33·9	35·6	37·2	40·7	22
23	30·5	32·6	35·1	36·8	38·5	42·0	23
24	31·6	33·8	36·3	38·1	39·8	43·3	24
25	32·7	34·9	37·5	39·3	41·0	44·6	25
26	33·8	36·1	38·7	40·5	42·3	45·9	26
27	35·0	37·2	39·9	41·8	43·5	47·2	27
28	36·1	38·4	41·1	43·0	44·7	48·5	28
29	37·2	39·5	42·3	44·2	46·0	49·8	29
30	38·3	40·7	43·5	45·4	47·2	51·1	30
31	39·4	41·8	44·7	46·6	48·4	52·3	31
32	40·6	43·0	45·8	47·8	49·7	53·6	32
33	41·7	44·1	47·0	49·0	50·9	54·9	33
34	42·8	45·3	48·2	50·2	52·1	56·2	34
35	43·9	46·4	49·4	51·4	53·3	57·4	35
36	45·0	47·5	50·6	52·6	54·6	58·7	36
37	46·1	48·7	51·7	53·8	55·8	59·9	37
38	47·2	49·8	52·9	55·0	57·0	61·2	38
39	48·3	50·9	54·0	56·2	58·2	62·4	39
40	49·4	52·1	55·2	57·4	59·4	63·7	40
41	50·5	53·2	56·4	58·5	60·6	64·9	41
42	51·6	54·3	57·5	59·7	61·8	66·1	42
43	52·7	55·5	58·7	60·9	63·0	67·4	43
44	53·8	56·6	59·8	62·1	64·2	68·6	44
45	54·9	57·7	61·0	63·2	65·4	69·8	45
46	56·0	58·8	62·1	64·4	66·5	71·1	46
47	57·1	59·9	63·3	65·6	67·7	72·3	47
48	58·2	61·1	64·4	66·8	68·9	73·5	48
49	59·3	62·2	65·6	67·9	70·1	74·7	49
50	60·4	63·3	66·7	69·1	71·3	76·0	50

a = dodder content in a sample.

α = confidence coefficient.

IV. *Practical Conclusions.*

We have seen above that if the tolerance limit for the true mean dodder content is comparatively low we have to face the fact that the method of testing is liable to greatly injure the interests either of the Buyer or of the Seller or both. This position is improved if the tolerance limit for the true dodder content is high. In order to work with high tolerance limits we have either to consider clover seed containing larger amounts of dodder as satisfactory, which is impossible, or to work with larger samples.

It appears that it is extremely difficult to give reliable statements accurately describing the dodder content in seed, if these statements are to apply to small quantities of seeds such as a sack, as the size of the sample taken out of each sack which would provide sufficiently accurate information would have to be so large that the analysis would be too costly. A possible issue seems to consist in estimating the dodder content in larger quantities of seed such as 100 sacks.

If such an amount of seed is presented and there are grounds to believe that it is homogeneous, a seed station may undertake the analysis of the whole. Comparatively small samples may be taken out of each sack and analysed separately. The data obtained must be used for two purposes: (i) to test the hypothesis that the 100 sacks present, in fact, homogeneous seed, and (ii) to estimate the dodder content in the seed. If the result of (i) is favourable and the seed is proved to be homogeneous there will be only one estimate concerning the whole consignment. If however the result of (i) is unfavourable, it will be necessary to estimate the dodder content separately for each sack.

The homogeneity test, mentioned in (i), is carried out in exactly the same way as that which led us to Table I B. If the seed proves homogeneous then the single estimate of the true mean dodder content will be obtained by combining all 100 samples from the separate sacks into one large sample. As the sum of a number of variables each following the Poisson Law of Frequency itself follows the same Law all our tables will be valid to judge the accuracy of this estimate. Practically the work will consist in summing up the amount of dodder found in separate samples and in referring to Table V to get the upper confidence limit for the mean dodder content in the whole consignment per weight equal to the size of the 100 samples combined. Dividing the upper confidence limit thus obtained by 100 we shall obtain the confidence limit of mean dodder content per size of one sample.

Obviously, even if the requirements as to the purity of clover seed are very strict, owing to the large number of partial samples from separate sacks, when dealing with the combined estimate we shall have to work with high tolerance limits. Thus the interests of both Buyer and Seller will be protected as far as possible.

It must be emphasised that the final verdict concerning the dodder content will apply to the whole stock of 100 sacks of seed. Among those 100 sacks there

will be several showing apparently smaller dodder content and others showing larger. However, it would not be right to conclude that these sacks differ in reality in their mean dodder content, as formerly it has been found that, when analysing the distribution of dodder content in samples from single sacks, the variation is no more than what should be expected from purely random causes when dealing with perfectly homogeneous material. In other words, we shall have to assume that if a sample from one sack contained less dodder than that from another, this difference is due solely to random variation and not to a real difference in the dodder content of the two sacks.

Assume now another eventuality: namely, that the χ^2 test showed a marked heterogeneity of seed in the 100 sacks presented for analysis. In this case the Seed Station should withhold from a general verdict concerning the whole stock, as this should be considered as heterogeneous. Separate estimates of the dodder content should be given for single sacks. These, of course, being based on the analysis of small samples, will be generally rather inaccurate, but we do not think there are any means of improving the position. It will perhaps be possible to increase the samples from single sacks, but this involves considerable expense.

Let us illustrate the above procedure by an example. A stock of 100 sacks of clover seed has lately been analysed in the seed testing laboratory in our Department. A sample of 100 gm. of seed was taken out of each sack and analysed for dodder. Columns 1 and 2 of Table VII state the results of the analysis. It will be seen that 76 samples proved to be dodder free, 22 others contained 1 dodder seed, etc. According to the principles stated above we tested the hypothesis that

TABLE VII.

Distribution of Dodder Content in Samples from 100 Separate Sacks.

k	N_k	$N.P_k$	
0	76	77·11	
1	22	20·05	
2	2	2·61	2·84
over 2	0	0·23	
$m=0·26,\ n'=3,\ \chi^2=0·4541,\ P_{\chi^2}=0·821$			

k = number of dodder seeds in a sample.
N_k = observed frequency.
$N.P_k$ = expected frequency.
n' = number of groups.
$n'-1$ = degrees of freedom*.

* According to the view of R. A. Fisher, E. S. Pearson and J. Neyman the degrees of freedom in this case should be $n'-2$ and $P_{\chi^2}=·500$.

the stock analysed was homogeneous. Column 3 of Table VII gives the frequencies calculated from the fitted Poisson Series. It will be seen that the fit is almost perfect, $P = \cdot 821$ (or $\cdot 50$, if degrees of freedom $= 1$). Thus we can conclude that there are no difficulties in formulating a verdict concerning the whole stock of seed as the differences between dodder content in samples from single sacks will be ascribed solely to individual random variation.

In all the 100 samples of 100 gm. each 26 dodder seeds have been found; thus the dodder content is 2·6 dodder seeds per kg. Choosing the confidence coefficient ·99 we find from Table V that the upper confidence limit of the true dodder content is 40·5 per 10 kg. or 4·05 per 1 kg. If the agreed tolerance limit for true dodder content is, say, 5 dodder seeds per kg., then the tested stock will be considered as satisfactory. If, however, the agreed tolerance limit was only 1 dodder seed per kg., naturally the stock will be rejected. In Table IV we can find the upper bound of the probability of an error in this last judgment. This is lower than ·001. The Seller will probably not question this judgment and will not propose to increase the sample. However, if the tolerance limit is just below the confidence limit of the true dodder content, for instance say, 3·5, then perhaps it would be worth while to repeat the sampling and base the final verdict on larger data.

It will be seen that in the procedure which has been recommended above there are several points which are not completely specified. We deliberately refrain from propounding any definite rules. For instance, such questions as fixing the tolerance limits for the true mean dodder content and those of the confidence coefficients with which the seed testing stations should work, as well as the question of when the fit of the Poisson Series to the distribution of dodder seeds in samples from a number of sacks should be considered as showing heterogeneity of the material, etc., these should be answered by the Dodder Committee of the International Seed Testing Association, assisted by representatives of the seed traders. We think, however, that the work of this Committee should follow the methods indicated in the present paper. Among other matters it will be necessary to give an adequate definition to the term "dodder free" and to revise the meaning of the term "absolutely dodder free." We hope that the results presented above prove sufficiently that the present use of these terms may often be misleading.

I should like to add a word of thanks to Dr J. Neyman for his kind assistance.

A STUDY OF PREDYNASTIC EGYPTIAN SKULLS FROM BADARI BASED ON MEASUREMENTS TAKEN BY MISS B. N. STOESSIGER AND PROFESSOR D. E. DERRY.

By G. M. MORANT, D.Sc.

1. *Introduction.* In the season 1924—25 an expedition of the British School of Archaeology in Egypt, under the direction of Sir Flinders Petrie, excavated predynastic sites in the neighbourhood of Badari, 30 miles south of Asyut. The remains found could be assigned to an earlier date than any previously discovered belonging to the same era. Fifty-nine skulls were preserved and the majority of these were measured by Professor Derry in Cairo. They were subsequently sent by Sir Flinders Petrie to Professor Karl Pearson, then Director of the former Biometric Laboratory at University College, London, and the study of them by Miss Stoessiger (now Mrs Clapham) was published in 1927 *. In the seasons 1928—30 Mr Brunton, who in 1924 had worked at Badari, leading a British Museum expedition, excavated more human remains associated with the Badari culture. A few of these came from Badari, but most are from a neighbouring site near Mostagedda. Eighty-three of the skulls were sent to Cairo and measured there by Professor Derry. He kindly placed his measurements of both series at the disposal of the present writer and those of the 1928—30 series are given in the appended tables. The objects of this paper are to compare the descriptive data provided by the two observers and to estimate from them the nature and racial affinities of the population represented by the two series of crania.

2. *A Comparison of Measurements of the* 1924—25 *Badari Series taken by Miss Stoessiger and Professor Derry.* In Miss Stoessiger's paper individual measurements are given of 59 skulls from Badari and unpublished measurements taken by Professor Derry are available for 53 of these. We thus have an unusual opportunity of comparing readings on the same specimens obtained by two observers working in different laboratories and following different techniques. Miss Stoessiger used the biometric technique without modification, and Professor Derry adopted that of the Monaco Congress † with some modifications and additions which he explained to the present writer. The measurements which are defined in these two schemes in ways which may be expected to give closely similar or identical results are the

* "A Study of the Badarian Crania recently excavated by the British School of Archaeology in Egypt." *Biometrika*, Vol. XIX. pp. 110—150.

† He refers to the English translation of the Monaco report published by Dr W. L. H. Duckworth in 1913 as: *International Agreements for the Unification (a) of Craniometric and Cephalometric Measurements, (b) of Anthropometric Measurements to be made on the Living Subject* (Cambridge University Press).

following (the biometric index letters being given first in the brackets, followed by the numbers of the Monaco report):

(i) Maximum calvarial length from the glabella in the median sagittal plane (L, 1).

(ii) Maximum calvarial breadth (B, 3). The two definitions differ since the biometric B is the maximum transverse diameter on the parietal bones, while the Monaco breadth is the maximum transverse whether found on the parietal bones or not. For Egyptian crania, however, the maximum breadth obtainable falls almost invariably on the parietal bones.

(iii) Minimum frontal breadth (B', 5).

(iv) Basio-bregmatic height (H', 4 a).

(v) Maximum horizontal circumference above the supra-orbital ridges and passing through the ophryon (U, 23 a).

(vi) Minimum arc from nasion to bregma (S_1).

(vii) Minimum arc from bregma to lambda (S_2).

(viii) Minimum arc from lambda to opisthion (S_3).

(ix) Total sagittal arc from nasion to opisthion (S, 22).

(x) Maximum bizygomatic breadth (J, 8).

(xi) Breadth between lowest points on zygomatico-maxillary sutures (GB).

(xii) Chord from nasion to basion (LB, 9).

(xiii) Chord from nasion to alveolar point ($G'H$, 12).

(xiv) Nasal height from nasion to point furthest removed from it on the margin of the left pyriform aperture (NH, L, Professor Derry took this—the Frankfurt—nasal height and not the Monaco measurement 13).

(xv) Maximum breadth of the pyriform aperture (NB, 14).

(xvi) Breadth of right orbit from the dacryon (O_1', R, 16). Professor Derry writes: "Where the lacrymal bone is broken, or missing, I use the angle formed by the union of frontal and frontal process of the maxilla, which I believe is more easily determined, and therefore more exact, than many other so-called 'points' on a skull." The biometric practice is to omit the measurement if the position of the dacryon is uncertain owing to defect of the lacrymal bone. Professor Derry gives the orbital breadth on both sides, and Miss Stoessiger that on the right side only.

(xvii) Maximum heights of orbits perpendicular to their breadths (O_2, R and O_2, L, 17).

(xviii) Profile angle, i.e. the angle between the line joining nasion to alveolar point and the Frankfurt horizontal plane ($P\angle$).

(xix) Bicondylar breadth of the mandible (w, 25).

(xx) Symphyseal height of the mandible, i.e. the maximum chord from the tip of the process between the central incisors (intradental) to the lower border of the bone in the symphyseal plane (h_1, 29).

(xxi) Minimum breadth of the left ascending ramus of the mandible (*rb'*, 28 *a*).

It is not suggested that the biometric and Monaco definitions of all the above measurements accord exactly, and, indeed, those of the latter scheme are not full enough in many cases to decide this point at all definitely. It may be hoped, however, that the two sets of readings will be found in close accordance for these 21 characters. Agreement would not be expected in the case of the following measurements owing to differences in definition.

(xxii) Auricular height. Professor Derry writes: "I take this with the skull on the craniophor and with the lower margin of the left orbit on the same horizontal plane as the upper margin of the left auditory meatus. At times this is the same as the upper surface of the supports in the meatus, but more often it is above these owing to the slope of the bony roof of the meatus. In any case I place the point of the scriber against the margin of the meatus and then move it to the orbit and rotate the skull until the lower edge is at the same level. The sliding arm on the craniophor scale is then brought down until it just touches the skull lightly." The biometric practice is to make contact between the ear-rods of the craniophor and the poria in all cases and hence measurements obtained in the two ways cannot be expected to agree exactly. For the six comparisons which can be made Professor Derry's reading is greater than Miss Stoessiger's *OH* in all cases and the differences are between 1·4 and 3·2 mm.

(xxiii) Transverse cranial arc between "the most prominent point on the posterior root of the left zygoma, exactly above the auditory aperture" and the same point on the right passing through the bregma (23). This must always be considerably less than the biometric vertical arc from porion to porion (Q') and in the case of five skulls for which both are available the differences range from 11·5 to 18·0.

(xxiv) Chord from basion to prosthion (10). This must be greater, almost invariably, than the chord from basion to alveolar point, defined to be the lowest point on the process between the central incisors (*GL*). Comparisons between the two measurements can be made in the case of 47 skulls: 41 show differences of the sign expected (the greatest being 3·5), two show zero differences and the remaining four differences of the opposite sign (the greatest being 1·2).

(xxv) Interorbital breadth. Professor Derry writes: "This is the actual width between the medial orbital walls when the two arms of the callipers are placed within the orbits and in contact with the medial walls." His definition differs from that of the Monaco technique (15) and from the biometric breadth between the dacrya (*DC*). Comparison with the latter can be made in six cases and the largest difference found is 1·3.

(xxvi) Bigonial breadth (26). According to the Monaco definition this measurement is taken on the external surface of the jaw between the bilateral points "at which the angle is formed between the lower margin of the body of the

jaw and the posterior border of its ascending ramus." This is practically the same definition as that of the biometric measurement w_2 which has been found to give unsatisfactory results in practice. The three differences (D. − S.) available are − 1·0, − 1·2 and + 2·3.

(xxvii) Maximum breadth of ramus (28 *b*). There is no biometric measurement similar to this.

(xxviii) Mandibular angle (32). This and the two following mandibular measurements were taken by Professor Derry on a mandible board, which he has described*, and which is precisely similar in construction to the instrument used by Miss Stoessiger. But the adjustment of the bone on the board was not the same in the two cases and hence the measurements are not comparable. The few differences available confirm this supposition. Professor Derry's mandibular angle is similar to the biometric $M\angle$.

(xxix) Projective length of corpus: similar to biometric c_pl.

(xxx) Projective length of ramus: similar to biometric rl.

The differences between the readings of the two observers for the same skulls may now be examined, in the case of the first 21 of the above measurements, to test whether the agreement is sufficiently close to be of no account in making comparisons with other series. Data are given in Table I for combined male and female specimens. Professor Derry took eight of the measurements in question on a few only of the Badari 1924—25 skulls which are sufficiently complete to provide readings for them. The information regarding personal equation available for these eight measurements in the table is obviously inadequate, fewer than 10 differences being given for each. As far as they go, however, the constants indicate that there is a satisfactory agreement for all except $P\angle$ and w_1. The definitions of the profile angle used by the two observers were the same, but there is reason to believe (see under Auricular height, measurement (xxii) above) that the adjustment of the skull in the craniophor was not made in the same way. This probably accounts for the rather large differences found and it will be safest to conclude that Professor Derry's profile angles are not comparable with Miss Stoessiger's. The same conclusion must be accepted in the case of the bicondylar breadth of the mandible (w_1). Much larger numbers of differences are available for the remaining 14 measurements—counting O_2, R and O_2, L as different measurements—in Table I. Several of the maximum individual differences in column 8 are considerably larger than those usually found between readings given by two workers in the same laboratory, and only four of the mean differences in the next column can be considered to differ insignificantly from zero (i.e. by less than three times their probable errors). The absolute sizes of the mean differences are of more importance, however, than the significance of their divergences from zero. They might be usefully compared with the probable errors of the means for the series, but these are as yet unknown. An approximation to them can be made by supposing that

* "A New Mylometer," *Journal of Anatomy and Physiology*, Vol. XLVIII. (1914), pp. 480—481.

the standard deviations for the Badari male skulls are the same as for longest Egyptian series available, viz. the E series of 26th—30th Dynasty skulls from Gizeh described by Professor Karl Pearson and Miss A. G. Davin*. The male Egyptian E standard deviations are in column 6 of the table and by using them the presumed probable errors of means for 35 Badari skulls are obtained and these are given in the last column. The numbers on which Miss Stoessiger's male means of cranial measurements of the Badari series are based actually range from 32 to 36. A comparison of the constants in columns 4 and 7 of Table I shows

TABLE I.

Data relating to the Differences between Miss Stoessiger's (S.) and Professor Derry's (D.) Measurements of the 1924—25 Badari skulls.

Character	No. of comparisons	Maximum difference (D. − S.)	Mean difference (D. − S.)	Standard deviation of differences	Male σ's of Egyptian E series	Presumed probable error of mean for 35 skulls
L	53	− 2·0	− ·32 ± ·060	0·65 ± ·043	5·72 ± ·09	·65
B	52	− 3·5	+ ·02 ± ·068	0·73 ± ·048	4·70 ± ·08	·54
B'	53	+ 2·5	+ ·40 ± ·067	0·72 ± ·047	4·05 ± ·06	·46
H'	52	+ 3·0	− ·34 ± ·065	0·70 ± ·046	5·03 ± ·08†	·57
U	53	+ 0·5	+ 1·29 ± ·200	2·16 ± ·142	13·77 ± ·22	1·57
S_1	8	− 1·0	− ·06	—	—	—
S_2	8	+ & − 2·0	+ ·13	—	—	—
S_3	8	− 2·0	− ·37	—	—	—
S	8	− 2·5	− ·31	—	—	—
J	33	+ 2·5	− ·03 ± ·087	0·74 ± ·061	4·57 ± ·08	·52
GB	45	− 5·4	− ·43 ± ·145	1·44 ± ·102	4·67 ± ·08	·53
LB	53	+ & − 1·0	− ·15 ± ·036	0·39 ± ·026	3·97 ± ·06	·45
$G'H$	47	− 2·5	− ·79 ± ·064	0·65 ± ·045	4·15 ± ·07	·47
NH, L	47	− 2·3	− ·64 ± ·064	0·65 ± ·045	2·92 ± ·05	·33
NB	47	− 2·1	+ ·07 ± ·053	0·54 ± ·038	1·77 ± ·03	·20
O_1', R	46	− 2·9	− ·76 ± ·075	0·75 ± ·053	1·67 ± ·03‡	·19
O_2, R	45	+ 3·2	− ·23 ± ·071	0·71 ± ·050	1·91 ± ·03	·22
O_2, L	44	− 1·4	− ·49 ± ·050	0·40 ± ·035	1·88 ± ·03	·21
$P\angle$	6	− 2°·5	− °·58	—	—	—
w_1	3	+ 2·8	+ 1·37	—	—	—
h_1	2	− 0·4	− ·10	—	—	—
rb'	5	+ 0·2	+ ·02	—	—	—

that the ratio of the mean difference between the readings of the two observers to the presumed probable error of the male mean for the series ranges from ·04 to 4·0 (!). If the former constant is less than 0·7 of the latter the effect of personal equation on mean values may be considered negligible, and this is so for L, B, H', J, LB and NB; if the ratio is between 0·7 and 1·4 (B', U, GB and O_2, R) the "errors" will affect the comparisons appreciably but probably not render them

* "On the Biometric Constants of the Human Skull." *Biometrika*, Vol. xvi. (1924), pp. 328—363.

† For the vertical height from the basion (H) in place of the basio-bregmatic height (H').

‡ For the "curvature" (O_1, R) in place of the dacryal (O_1', R) orbital breadth.

misleading; if the ratio is greater than 1·4 ($G'H$, NH, L, O_1', R and O_3, L) it may be supposed quite unsafe to neglect the consideration of personal equation in comparing means given by the two observers for a series made up by approximately 35 specimens. As measurements of variability will also be wanted for the series, it is necessary to consider the standard deviations, as well as the means, of the distributions of differences between the two sets of readings for the same skulls. These are given in Column 5 of Table I and they may be compared with the Egyptian E male standard deviations in the following column. If the former constant is less than 0·2 of the latter in the case of a particular character the agreement may be supposed satisfactory, and this is so for L, B, B', H', U, J, LB and $G'H$; if the ratio is between 0·2 and 0·4 (GB, NH, L, NB, O_3, R and O_3, L) the agreement is less satisfactory, and if greater than 0·4 (O_1', R) decidedly unsatisfactory. By applying these arbitrary tests to the means and standard deviations of the differences between Miss Stoessiger's and Professor Derry's readings on the same skulls, it is possible to group the measurements for which they used similar or identical definitions in the following way:

(*a*) those for which agreement is sufficiently close to make it possible to neglect the effect of personal equation—L, B, H', J, LB, and probably S_1, S_2, S_3, S, h_1 and rb', though few data for these last six from which estimates of their accuracy can be obtained are available;

(*b*) those for which agreement is less satisfactory, so that there is a danger of personal equation affecting comparisons between constants for series appreciably, though not to a marked extent—B', U, GB, NB, O_3, R;

(*c*) those for which comparisons between the constants will be definitely invalid owing to personal equation—$G'H$, NH, L, O_1', R, and O_3, L, and probably $P\angle$ and w_1, though few data for these last two are available.

There can be no doubt that personal equation would be of much less account if measurements of two workers in the former Biometric Laboratory were being compared than in the case of the present comparison, but this might have been anticipated.

3. *The Sexing of the Badari Series.* Three estimates of the sexes of the adult 1924—25 Badari skulls are available. The first of these to be published is that of Miss Stoessiger's paper in which it is said that the specimens were sexed by Professor Karl Pearson and the present writer. There are 36 males and 22 females distinguished, and 53 of these have been sexed by Professor Derry. He is willing to accept the determination given in the paper in *Biometrika* in 47 cases, but differences must be recorded for the remaining six which are: Nos. 5394, S. ♂?, D. ♀; 5435, S. ♂, D. ♀; 5438, S. ♂, D. ♀; 5743, S. ♂, D. ♀?; 6004, S. ♂, D. ♀; 5383, S. ♀, D. ♂. The third sexing of the series is that given in the report on the excavations* from which the following is quoted (pp. 19—20):

* *The Badarian Civilisation and Predynastic Remains near Badari.* By Guy Brunton and Gertrude Caton-Thompson, British School of Archaeology in Egypt and Egyptian Research Account, Thirtieth Year, 1924. London, 1928.

"There was no sign that the bodies had been preserved in any way. The skin was sometimes still visible, occasionally well preserved, but the internal organs had always disappeared....As regards segregation of sexes in cemeteries, a curious fact emerges. Female graves were not placed apart; where there are graves of women, there are also graves of men alongside. Certain areas, however, were reserved for men....The most easily observed feature was the hair....It is interesting to note that in no case could a beard or moustache be detected."

The method of sexing used is not described, but it appears from the remarks on individual skeletons that the length and condition of the hair was taken as one criterion, and the nature of the grave furniture also gave some guidance presumably. Sexes are given in the report (A.R.) for 42 of the skulls sexed both in Miss Stoessiger's paper (S.) and by Professor Derry (D.). It is found that for 24 of these, all three estimates agree; for 30, A.R. agrees with S.; for 35, A.R. agrees with D., and for seven A.R. differs from both S. and D. The last seven are Nos. 5411, A.R. ♀ "long wavy black hair arranged in twisted tresses"; 5745, A.R. ♀ "hair thick, black and wavy, bracelet"; 5815, A.R. ♀ "dark, slightly wavy hair"; 5359, A.R. ♂; 5360, A.R. ♂; 5368, A.R. ♂; 5370, A.R. ♂ "in male cemetery." A re-examination of these seven skulls suggests that an anatomist would have little hesitation in assigning sexes to them opposite to those given in the report, in spite of the fact that the sexing of the Badari skulls is particularly difficult owing to the fact that the type is a feeble one showing little muscular development. It is unfortunate that there is no closer agreement between the three estimates, and the effect of the discordance on means must at least be examined. Of the measurements which have been found to be least affected by personal equation, the three major calvarial diameters are the ones most suitable to use for this purpose and means for them are given in the following table, computed only for the 42 skulls for which all three estimates of sex are available and using Miss Stoessiger's measurements in every case.

Sexing	Sex	L	B	H'*
Stoessiger	♂	183·4 (27)	131·2 (27)	134·0 (26)
Derry	♂	183·5 (23)	131·9 (23)	134·5 (22)
Archaeological Report	♂	182·2 (24)	131·6 (24)	133·6 (23)
Stoessiger	♀	177·2 (15)	131·1 (15)	128·9 (15)
Derry	♀	178·3 (19)	130·2 (19)	129·4 (19)
Archaeological Report	♀	179·7 (18)	130·6 (18)	130·3 (18)

The means are seen to be in better accordance than might have been anticipated. For L and H' those given by Miss Stoessiger and Professor Derry agree better than either set does with those based on the sexing in the Archaeological

* H' is not given for one skull which is male according to all three estimates.

Report, but for B the last two are in closest agreement. In the case of two of the estimates, a similar comparison can be made based on 53 skulls, and the means are:

Sexing	Sex	L	B*	H'†
Stoessiger Derry ...	♂	182·2 (33) 182·2 (29)	130·9 (33) 131·5 (29)	133·1 (32) 133·4 (28)
Stoessiger Derry ...	♀	177·0 (20) 177·9 (24)	130·2 (19) 129·6 (23)	128·9 (20) 129·3 (24)

The pairs of means here are actually all closer together than the corresponding ones based on the 42 skulls. We are comparing two anatomical estimates of sex which almost certainly differ owing to the interchange of a few doubtfully male or doubtfully female specimens, and the way in which these are assigned does not appear to influence the means greatly. But still the greatest observed differences between the means are of an order which would have indicated unreliable measurements if they had been found as mean differences between the readings given by two different observers (cf. data in Table I). The personal equation in sexing is hence a factor which is likely to affect comparisons between series appreciably. The sexes given in Miss Stoessiger's paper were accepted in making the comparisons in the following section of the present paper since they are given for all the specimens, while the other two records only give sexes for parts of the total 1924—25 series. It is not suggested that the former sexing is more accurate than the others when available. For the 1928—30 series, we have only Professor Derry's determinations to follow.

4. *A Comparison of the* 1924—25 *and* 1928—30 *Badari Series.* Constants are given in Table II based on Miss Stoessiger's measurements of the 1924—25 series and Professor Derry's of the 1928—30 series. The data there only relate to those characters for which it has been shown that comparisons between the measurements of the two observers are not likely to be vitiated to a marked extent owing to their personal equations‡. Standard deviations have not been calculated for the series excavated on the two occasions, since it is clear that their means are in close agreement, but a non-metrical comparison of their distributions suggests that they have approximately the same variability as one another and as the combined sample in the case of every character. The standard deviations of the last were calculated and the probable errors of the means are given (= p.e. M_p, say). The probable errors of the means of the two component samples will certainly

* B is not given for one skull which is female according to both estimates.

† H' is not given for one skull which is male according to both estimates.

‡ All such are included in Table II (see p. 302 below) except the height of the right orbit (O_2, R). As the height of the left orbit was found to be an unreliable measurement, it appeared safer to class O_2, R with those of that kind and data for it are given in Table III.

be larger than these, but their differences may be compared with $\sqrt{2 \times (\text{p.e. } M_p)^2}$ to give an estimate of whether they are significant or not. The difference divided by this quantity is greater than 3 in the case of the male $100\,H'/L$ (3·4), H' (3·6) and J (3·9), and the female S (3·1), S_2 (3·3), NB (3·8), $100\,B/H'$ (3·9), GB (4·5), B (4·7), $100\,H'/L$ (4·7) and L (4·9). The ratio is greatest for the last comparison, and if it is assumed that for this the standard deviations of the component distributions are the same as for the pooled distribution—which is a reasonable assumption—then the female mean L for the 1924—25 series will be $176{\cdot}7 \pm {\cdot}52$ and for the 1928—30 $174{\cdot}4 \pm {\cdot}43$. The difference between these is 3·4 times its probable error. It will be safe to suppose that the differences between the mean measurements of the two separate series are all insignificant, except possibly in the case of the female values for L, B, GB and $100\,H'/L$, and no clearly significant differences are found for these. The conclusion that the Badari series excavated on the two occasions represent the same racial type appears to be sufficiently justified and hence they have been pooled.

This conclusion might be invalidated if clearly significant differences were found for any characters not in Table II. Table III gives the means for those characters for which identical or very similar definitions were used by the two observers, but for which their readings on the same skulls were found to differ appreciably or markedly. It will not be legitimate to pool data for these characters. The "divergences expected" in the last column of the table are those actually found (see Table I) between the mean measurements of the two observers for the same skulls. The constants for the 1924—25 and 1928—30 series in Table III are still in close agreement, and it will be safe to assume that none indicate possible differentiation of the types—paying due regard to the divergences expected owing to personal equation in comparing the means—except possibly in the case of the upper facial ($G'H$) and nasal (NH, L) heights and indices, and in the case of the bicondylar breadth of the mandible (w_1). The divergence expected in the case of the last character is only based on the comparison of measurements of three specimens, and hence it is quite likely to be misleading, and Miss Stoessiger's and Professor Derry's readings may be directly comparable. But the measures of personal equation in the case of the other characters which are showing larger differences than would have been anticipated are based on adequate numbers of comparisons. The standard deviations were calculated for the two series separately giving the results provided in Table III, p. 303, the means for the 1928—30 series being given values which they would presumably have had if measured by Miss Stoessiger.

The difference between the male means for $G'H$ is here 3·4 times its probable error and the difference between the female means for NH, L is 3·8 times its probable error, while the other two differences are quite insignificant. It appears that the 1928—30 type had a significantly higher face than the 1924—25 type, though part of the observed difference may possibly be due to the fact that the series were sexed by different observers. The discordance is not sufficiently marked

TABLE II.

Constants from Miss Stoessiger's (S.) and Professor Derry's (D.) Measurements of different Badari Series.

Measurements		Means						Variabilities: pooled (S. & D.)			
		♂			♀			Standard Deviations		Coefficients of Variation	
		1924—25 (S.)	1926—30 (D.)	Pooled (S. & D.)	1924—25 (S.)	1926—30 (D.)	Pooled (S. & D.)	♂	♀	♂	♀
Least affected by personal equation	L	182·3 (36)	182·6 (45)	182·5 ± ·44 (81)	176·7 (22)	174·4 (33)	175·3 ± ·33 (55)	5·88 ± ·31	3·65 ± ·23	3·22 ± ·17	2·06 ± ·13
	B	130·8 (36)	132·2 (45)	131·6 ± ·41 (81)	130·3 (21)	127·6 (33)	128·7 ± ·41 (54)	5·47 ± ·29	4·47 ± ·29	4·16 ± ·22	3·47 ± ·23
	H'	132·9 (34)	135·1 (43)	134·1 ± ·43 (77)	129·1 (22)	129·9 (33)	129·6 ± ·37 (55)	5·57 ± ·30	4·08 ± ·26	4·16 ± ·23	3·15 ± ·20
	LB	99·3 (35)	101·2 (41)	100·3 ± ·34 (76)	96·1 (22)	96·5 (33)	96·3 ± ·31 (55)	4·42 ± ·24	3·41 ± ·22	4·40 ± ·24	3·34 ± ·23
	S_1	127·0 (35)	126·9 (44)	126·9 ± ·42 (79)	123·2 (22)	122·3 (32)	122·7 ± ·37 (54)	5·52 ± ·30	4·04 ± ·26	4·35 ± ·23	3·23 ± ·21
	S_2	129·1 (35)	130·0 (44)	129·6 ± ·50 (79)	128·5 (22)	125·2 (32)	126·5 ± ·70 (54)	6·64 ± ·36	7·65 ± ·50	5·13 ± ·28	6·07 ± ·40
	S_3	115·7 (34)	113·9 (44)	114·7 ± ·54 (78)	111·7 (22)	112·3 (32)	112·1 ± ·48 (54)	7·09 ± ·38	5·22 ± ·34	6·18 ± ·34	4·66 ± ·30
	S	372·0 (35)	370·8 (44)	371·3 ± ·90 (79)	363·4 (22)	359·8 (32)	361·3 ± ·82 (54)	11·88 ± ·64	8·95 ± ·58	3·20 ± ·17	2·48 ± ·16
	J	122·5 (32)	124·9 (29)	123·8 ± ·44 (61)	117·7 (13)	118·0 (26)	117·9 ± ·50 (39)	5·00 ± ·31	4·67 ± ·36	4·12 ± ·25	3·96 ± ·30
	$100\ B/L$	71·8 (36)	72·4 (45)	72·1 ± ·24 (81)	73·8 (21)	73·2 (33)	73·4 ± ·27 (54)	3·17 ± ·17	2·89 ± ·19	—	—
	$100\ H'/L$	73·1 (34)	74·1 (43)	73·7 ± ·21 (77)	73·1 (22)	74·5 (33)	73·9 ± ·21 (55)	2·77 ± ·15	2·33 ± ·15	—	—
	$100\ B/H'$	98·3 (34)	98·2 (43)	98·2 ± ·41 (77)	101·0 (21)	98·4 (33)	99·4 ± ·47 (54)	5·30 ± ·29	5·08 ± ·33	—	—
	h_1	32·6 (34)	32·5 (38)	32·5 ± ·24 (72)	31·5 (18)	30·8 (29)	31·1 ± ·22 (47)	3·01 ± ·17	2·25 ± ·16	9·26 ± ·52	7·23 ± ·51
	rb'	33·6 (39)	34·2 (40)	33·9 ± ·17 (79)	32·0 (23)	32·3 (33)	32·2 ± ·21 (56)	2·28 ± ·12	2·37 ± ·15	6·72 ± ·36	7·36 ± ·47
More affected by personal equation	B'	91·1 (36)	91·1 (45)	91·1 ± ·29 (81)	89·4 (22)	88·9 (34)	89·1 ± ·37 (56)	3·92 ± ·21	4·16 ± ·27	4·30 ± ·23	4·67 ± ·30
	U	501·3 (36)	499·0 (44)	500·0 ± 1·11 (80)	489·0 (22)	485·6 (33)	487·0 ± ·81 (55)	14·74 ± ·79	8·90 ± ·57	2·95 ± ·16	1·83 ± ·12
	GB	94·6 (34)	94·4 (42)	94·5 ± ·39 (76)	89·6 (18)	92·4 (33)	91·4 ± ·44 (51)	4·98 ± ·27	4·62 ± ·31	5·27 ± ·29	5·06 ± ·34
	NB	24·9 (34)	24·8 (42)	24·8 ± ·14 (76)	23·6 (20)	24·3 (34)	24·0 ± ·13 (54)	1·80 ± ·10	1·43 ± ·09	7·26 ± ·40	5·96 ± ·39

Character	Means				Standard Deviations			
	♂		♀		♂		♀	
	1924—25	1928—30	1924—25	1928—30	1924—25	1928—30	1924—25	1928—30
G'H	67·1 ± ·46 (34)	[69·3] ± ·45 (43)	64·8 ± ·63 (20)	[66·0] ± ·33 (34)	3·95 ± ·32	4·41 ± ·32	4·18 ± ·45	2·83 ± ·23
NH, L	48·4 ± ·32 (34)	[49·4] ± ·36 (43)	45·6 ± ·42 (20)	[47·4] ± ·23 (34)	2·78 ± ·23	3·47 ± ·25	2·81 ± ·30	1·99 ± ·16

TABLE III.

Mean Measurements of two Badari series, from Readings by Miss Stoessiger (S.) and Professor Derry (D.), between which Agreement is not expected owing to Personal Equation.

Character	♂		♀		Divergence expected (S. − D.)
	1924—25 (S.)	1928—30 (D.)	1924—25 (S.)	1928—30 (D.)	
$G'H$	67·1 (34)	68·5 (43)	64·8 (20)	65·2 (34)	+·8
NH, L	48·4 (34)	48·8 (43)	45·6 (20)	46·8 (34)	+·6
O_1', R	38·4 (33)	38·3 (38)	37·6 (20)	37·2 (32)	+·8
O_1', L	—	37·4 (37)	—	36·3 (32)	—
O_2, R	32·0 (34)	31·5 (38)	31·3 (21)	32·2 (32)	+·2
O_2, L	32·1 (33)	31·6 (37)	31·4 (18)	31·9 (33)	+·5
$P\angle$	84°·0 (33)	82°·2 (41)	83°·3 (20)	81°·2 (33)	+0°·6 ?
$100\ G'H/GB$	70·9 (34)	72·5 (41)	71·7 (17)	70·5 (33)	—
$100\ NB/NH, L$	51·6 (34)	50·9 (42)	51·8 (20)	51·9 (34)	—
$100\ O_2/O_1', R$	83·3 (33)	82·2 (37)	83·4 (20)	86·6 (32)	—
$100\ O_2/O_1', L$	—	84·7 (37)	—	87·8 (32)	—
w_1	109·5 (30)	109·9 (36)	105·2 (16)	104·1 (30)	− 1·4 ?

to render the pooling of the series unjustifiable. Table IV gives the means for the remaining measurements taken by Professor Derry on the 1928—30 series and for similar, but differently defined, measurements taken on the 1924—25 series by Miss Stoessiger. It is clear that entirely fallacious conclusions would have been reached if some of these corresponding pairs had been supposed comparable.

5. *The Nature of the Badari Series.* Certain features of the Badari series can be estimated from the data given in Tables II—IV above. The characters for which pooled means are given (Table II) have constants based on fairly adequate numbers of skulls and the sex ratios (male mean/female mean) for the absolute measurements are found to be unusually small, though of the same order as those

TABLE IV.

Mean Measurements of two Badari Series for corresponding but differently defined Characters.

Prof. Derry's measurements	1928—30 (Derry) ♂	1928—30 (Derry) ♀	Corresponding but differently defined biometric measurements	1921—25 (Stoessiger) ♂	1921—25 (Stoessiger) ♀
Monaco transverse arc ...	296·1 (41)	283·0 (32)	Q'	302·0 (34)	298·8 (20)
Auricular height ...	115·7 (41)	111·8 (33)	OH	111·0 (34)	108·6 (21)
Chord bas.-prosth. ...	98·5 (40)	94·2 (33)	GL	95·0 (33)	92·6 (20)
Interorbital breadth ...	21·7 (40)	21·1 (32)	DC	22·4 (32)	20·6 (21)
100 Chord bas.-prosth./LB	97·4 (40)	97·7 (33)	.	...	...
Bigonial breadth ...	91·6 (40)	85·8 (31)	w_2	88·8 (32)	86·2 (21)
Max. breadth of ramus ...	42·4 (39)	40·2 (33)	...	...	–
Length of corpus ...	79·0 (40)	75·3 (33)	$c_y l$	76·2 (33)	74·1 (19)
Length of ramus... ...	61·1 (38)	55·0 (33)	rl	57·6 (33)	53·2 (18)
Mandibular angle ...	117°·7 (41)	121°·0 (33)	$M\angle$	120°·0 (34)	123°·3 (19)

observed for other series*. Owing to the difficulty of sexing the Badari series, it would be unwise to attach much importance to this result. The mean cephalic and breadth-height indices are about one unit less for the male than for the female series, as is usually found. Other expected relations are that the female orbital index (Table III) and mandibular angle (Table IV) are greater than the male values. It is probable that the differences between the means for the two sexes in the case of all the other indices and of the profile angle are quite insignificant. There is every reason to believe that the male and female series represent the same racial type.

Sexual comparisons in variability can only be made in the case of the 18 characters dealt with in Table II. If coefficients of variation are used for the absolute measurements and standard deviations for the indices, it is found that for 15 of these the male variability is the greater. Dividing the difference of the constants by its probable error gives values over 3 in the case of S (3·1), H' (3·3), S_1 (3·4), S_3 (3·4), L (5·3) and U (5·6), while for all of these it is the male variability which is the greater. A similar clear preponderance of male over female variability has been observed in the case of the long Egyptian E series of the 26th—30th Dynasty skulls, but for other Egyptian and non-Egyptian series it has been far more usual to find a closer approach to sexual equality in this respect. The material suggests that there are racial differences between sexual differences in variability and the Badari type appears to be close to one extreme of the range. It is possible that inaccuracies in sexing have favoured the last conclusion.

* Sex ratios for seven calvarial measurements are given in *Biometrika*, Vol. xxv. (1933), p. 259 in the case of the Kerma, the E Egyptian and the Teita negro series. The Badari values for each character are smaller than any shown there, though the difference between the Badari and the next smallest sex ratio is small in every case, its maximum value being ·009.

A comparison of the variability of the Badari and other series is of particular interest since it is believed that we are dealing with the earliest cranial sample of any length which has been obtained from any part of the world*. It is implied, or distinctly stated, in a good deal of the literature of physical anthropology that present-day populations are racially far more heterogeneous, and hence far more variable, than those which existed at the beginning of the Christian era, say, and that by going back further in time we must approach closer and closer to that ideal and undefined population which may be called a "pure" race. There appears to be no valid reason for attempting to divide the Badari series into sub-samples which would show lesser variabilities than the total sample. It is reasonable to assume that the people represented formed part of an intermarrying group which could not be divided into racial components owing to their complete inter-mixture for several generations. The biometrician accepts such a group as representing a racially homogeneous type, and comparisons may be made with other samples presumed to have been drawn at random from similarly constituted populations. The Badari variabilities for the 16 cranial characters in Table II (omitting the mandibular measurements h_1 and rb') are compared below with those of the long Egyptian E series, using coefficients of variation for the absolute measurements and standard deviations for the indices:

Male.—Badari variabilities greater 11 characters, Egyptian E greater 5; differences exceeding three times their probable errors B (3·2, Badari greater).

Female.—Badari variabilities greater 7 characters, Egyptian E greater 9; differences exceeding three times their probable errors $100\ H'/L$ (3·2), L (4·2), S_3 (4·5), U (4·8), S_1 (5·6), the Egyptian E being the greater in these five cases.

It appears that the male Badari series is slightly more variable than the male Egyptian E, but for the females the position is reversed and the difference is more significant. The Egyptian E has been compared with several other cranial series in a similar way, and it has been found that its variability is slightly less than those of samples from different parts of the world which are presumed to represent racially homogeneous populations. The following comparison is between the Badari constants and those given for the Farringdon Street English series which was obtained from a single London cemetery used in the seventeenth century†, no selection of specimens on which measurements could be taken having been made:

Male.—Badari variabilities greater 4 characters, Farringdon Street greater 12; differences exceeding three times their probable errors GB (3·1, Farringdon Street greater).

* Since the later discovery of Tasian remains, the Badari is generally considered to be the second oldest culture of predynastic Egypt known. If the beginning of dynastic times is dated at 3000 B.C. and the length of the predynastic period is estimated as a minimum of 2000 years, then the Badari people must have lived as long ago as 4000 B.C. and probably earlier. Most authorities apparently agree in making them ante-date all European neolithic cultures by a considerable period. See, for example, Miles Burkitt and V. Gordon Childe: "A Chronological Table of Prehistory." *Antiquity*, Vol. VI. (1932), pp. 185—205.

† Beatrix G. E. Hooke, "A Third Study of the English Skull with special reference to the Farringdon Street Crania." *Biometrika*, Vol. XVIII. (1926), pp. 1—55.

Female.—Badari variabilities greater 2 characters, Farringdon Street greater 14; differences exceeding three times their probable errors LB (3·4), NB (3·6), S_3 (4·3), H' (4·8), S (5·3), 100 H'/L (5·6), S_1 (7·6), U (7·7), L (8·8), the Farringdon Street being the greater in these nine cases.

While there appears to be no marked difference in variability between the two male series, the Farringdon Street female must be considered to be very significantly more variable than the Badari female series. The last conclusion is likely to convey a false impression, however, since it may suggest that the one series is much more variable than the other. To obtain some measure of the order of the difference, the Farringdon Street constant was divided by the Badari constant in the case of each character, and then the average of these ratios was found for the 16 characters. This was also done in the case of the other three comparisons above and the results are: with Egyptian E, male 0·952, female 1·073; with Farringdon Street, male 1·080, female 1·281. Judging in this way, the English female variability is thus of the order 1·3 times the Badari female variability, which is an extraordinarily low ratio, since we are here dealing with series accepted as representing racially homogeneous types which show a divergence in variability approaching the maximum found among such series*. In the last 6000 years there appears to have been little change in the variability of racial populations, and it is safe to conclude, perhaps, that "pure" types much less variable than any found to-day have only existed in that period in the imaginations of some anthropologists.

6. *Racial Comparisons.* Mean cranial measurements based on 30 or more male specimens have been collected for 150 series representing racial types from different parts of the world, excluding many available for ancient Egypt and Europe to prevent these regions being over-represented. It is found that the Badari means are quite close to the inter-racial averages (i.e. the unweighted averages of the serial means), except in the case of: B which is small, though 12 out of 150 means are smaller; B' small, but 7 out of 126 smaller; J small, but 2 out of 141 smaller; U small, but 14 out of 116 smaller; 100 B/L low, but 17 out of 150 lower; 100 B/H' low, but 15 out of 148 lower; NH, L small, but 19 out of 144 smaller; O_2 small, but 4 out of 148 smaller (than Stoessiger's mean 31·5); S_2 large, but 17 out of 98 larger. The Badari type is thus seen to have calvarial and facial breadths which are unusually, but not extremely, small, while its calvarial antero-posterior lengths (apart from the S_2 arc from bregma to lambda) and heights are not at all exceptional. The horizontal circumference (U) is hence

* There appears to be only one cranial series, for which the constants have been published, which is appreciably less variable than the Badari and Egyptian E. This is one of Guanche skulls measured by Prof. E. A. Hooton ("The Ancient Inhabitants of the Canary Islands." *Harvard African Studies*, Vol. VII. 1925). Data for 12 of the 16 cranial characters in Table II above are given. Dividing the Tenerife constants—coefficients of variation for absolute measurements and standard deviations for indices—by the Badari values gives for these a mean male ratio 0·914 and a mean female 1·040. It is not surprising to find that the population of a small island shows peculiarly small variation since inbreeding must have been common. Most European samples—whether derived from single cemeteries or from several cemeteries believed to have been used by closely related groups of people—show variabilites which are close to the Farringdon Street values.

small and the cephalic index low. Capacities could not be found directly owing to the fragility of the skulls, but the average for them must be unusually small. The facial skeleton has unusually small heights. Comparative material for the mandible is scanty, but the Badari bicondylar breadth (w_1) appears to be the smallest male mean recorded, thus agreeing with the conclusion that the breadths of the cranium are unusually small. Comparisons between the Badari means for single characters and those available for other ancient Egyptian series have been made by Miss Stoessiger*, and it has been shown that the Badari type diverges slightly from them in being more prognathous and in having a higher nasal index. In these respects it is more negroid.

The racial relationships of the Badari type can be judged better from coefficients of racial likeness. There are 27 reduced values, of which several have been previously published, in Table V below and the male means given by Miss Stoessiger were used in computing all these constants. New means based on larger numbers of skulls are now available for 12 of the characters used in computing the coefficients, but these are all very close to Miss Stoessiger's means, and by using hers in all cases we have all 31 means based on approximately the same numbers which is an advantage. As would have been anticipated, the closest connections are found with other Predynastic Egyptian series. The dynastic series are further removed, and types more similar to the Badari than some of them are found in Abyssinia to-day and also among the criminal classes in Cairo (Sydney Smith). All other African types with which comparisons have been made are more widely removed from the Badari type. Miss Stoessiger found that there is an extraordinarily close similarity between the last and some Indian types of cranium, and the revised coefficients in Table V confirm her conclusion. A comparison of the reduced coefficients there with several hundred others, published and unpublished, suggests the following conclusions:

(i) The Badari type resembles most closely those of the other Predynastic Egyptian series available. Its connections with these are as close as those normally found between different samples representing the populations of the same country at different times, but in the same era. None are so close that the Badari and any other predynastic series can be supposed to represent precisely the same population†. The Badari resembles the other early predynastic series more closely than it does the late predynastic series. With advancing time the type in the predynastic era was changing by becoming less prognathous and by acquiring slightly lower nasal and higher cephalic indices. A slow progression in the same direction appears to have continued throughout the whole of the dynastic era and down to Roman times in Upper and Middle Egypt at least. The later of the known Egyptian types for this period differ quite markedly from the Badari

* *Loc. cit.* p. 124.

† This seems to be the safest conclusion to draw in spite of the fact that an insignificant coefficient of racial likeness is found between the Badari male means and those of the Early Predynastic series from Abydos, El-Amrah and Hou measured by Thomson and MacIver; only 14 of the 31 measurements used whenever possible in computing the coefficients are available for the latter.

TABLE V.

Male reduced Coefficients of Racial Likeness between the Badari and other Series.

Ancient Egyptian and related Series	C.R.L.*	Negro Series and Hottentots	C.R.L.	Other Series	C.R.L.
Early Predynastic: Abydos, El-Amrah and Hou†	1·72 ± ·69 (14)	Hottentots	16·84 ± ·69 (24)	Dravidians¶ ...	6·70 ± ·69 (18)
Predynastic: Naqada A and Q†	4·02 ± ·38 (31)	Negroes from Egypt ...	18·38 ± ·54 (27)	Sardinians** ...	9·72 ± ·59 (14)
Late predynastic: El-Amrah and Hou†	7·62 ± ·49 (14)	Tanganyika	22·83 ± ·57 (22)	Veddahs¶ ...	10·53 ± ·63 (20)
12th—13th dynasty: Kerma‡	7·77 ± ·33 (31)	Teita	24·54 ± ·47 (31)	Nepalese¶ ...	12·73 ± ·44 (31)
6th—12th dynasty: Denderah†	7·78 ± ·45 (14)	Angoni	32·67 ± ·71 (23)	Galla and Somali‖	14·26 ± ·67 (22)
Modern Abyssinians: Tigre district†	8·45 ± ·41 (28)	Gaboon	44·51 ± ·40 (29)	Hindus¶ ...	14·54 ± ·44 (21)
Modern Egyptians: Cairo§	9·83 ± ·58 (15)	Congo	46·33 ± ·44 (31)	Tamils¶ ...	14·64 ± ·30 (31)
12th—15th dynasty: Hou and Abydos†	10·34 ± ·57 (14)	Kaffirs	54·44 ± ·45 (29)	Bushmen‖ ...	37·96 ± ·53 (24)
18th—20th dynasty: Thebes†	15·15 ± ·43 (29)	Cameroons	61·34 ± ·37 (28)	Tasmanians‖ ...	49·55 ± ·49 (24)

* The numbers in brackets are the numbers of characters on which the coefficients are based.

† Means in *Biometrika*, Vol. XVII. (1925), pp. 14—36.

‡ Means and coefficient given by Collett: *Ibid.* Vol. XXV. (1933), pp. 261 and 270.

§ Means given by Sidney Smith in *Journal of Anatomy*, Vol. LX. (1926), p. 123.

‖ Means or references and coefficients given by Kitson, *Biometrika*, Vol. XXIII. (1931), pp. 285—298.

¶ Means or references given by Woo and Morant, *Ibid.* Vol. XXIV. (1932), pp. 114—118.

** Means in *Ibid.* Vol. XX^{B}. (1928), pp. 370—371.

type in being less prognathous, and in having lower nasal but higher cephalic indices.

(ii) Close connections can be found between these later Egyptian types and some representing European populations in contemporary or later times, but the Badari and other Predynastic Egyptian types bear no close resemblance to any recorded for a European population other than that of modern Sardinians. The connection between the Badari and Sardinian series is not very close, but it is closer than that found between the Predynastic and some late Dynastic Egyptian series.

(iii) Connections of the same order are found between the Badari and all the modern Indian series available, the most intimate being with a Dravidian series. The other Predynastic Egyptian types are rather further removed from the Indian, and between the last and the Dynastic Egyptians no close resemblances are found.

(iv) A wider gap separates all the Egyptian from African negro types, but the resemblances between these and the Badari and other Predynastic types are appreciably closer than those between the negro and late Dynastic Egyptian types.

The interpretation of the observed situation in terms of racial descent must necessarily be hazardous in the present state of our knowledge, and far more material than that available at present may be needed in order to reveal the actual relationships, which are almost certainly complex, between the populations represented. *Meanwhile it is of interest to note that the Badari racial type, which* is believed to be the earliest of which we have any adequate knowledge of a statistical kind, is also peculiarly generalised.

I.

THE RATIO OF THE MEAN DEVIATION TO THE STANDARD DEVIATION AS A TEST OF NORMALITY.

BY R. C. GEARY, M.Sc.

THE problem to be considered in this paper is the following. A sample of n' elements with measures $x_1, x_2, \ldots x_{n'}$, is drawn at random from a universe which is presumed continuous: to determine from the sample whether the universe may be regarded as normal, that is to say whether the probability of drawing therefrom an element, measuring from $x - \frac{\delta x}{2}$ to $x + \frac{\delta x}{2}$, when δx is small, is

$$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}} \delta x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Two cases are considered: (i) that in which the universal mean a may be presumed known, but the standard deviation σ is unknown, and (ii) that in which both the mean and standard deviation are unknown.

The usual method of dealing with this problem, and that which in principle is adopted here, is that of devising a function, or functions, of the sample variables $x_1, x_2, \ldots x_{n'}$, which assume characteristic values for infinite samples (i.e. $n' = \infty$) drawn from a normal universe. We try to find the actual or approximate frequency distributions, for different sizes of sample, on the assumption that the unknown parent universe is normal. If, for the actual sample, the values of such functions are very improbable, we decide that the universe from which the sample was drawn is not normal.

It is quite obvious that in problems of this kind we can only determine "necessary" (and not "sufficient") conditions. We may hope to be in a position to say that the parent universe was *not* A; we can never say on the evidence of the limited sample alone that it probably *was* drawn from a universe B. In the first place, the number of *independent* tests which may be applied is strictly limited: it cannot exceed n', the number in the sample, when the parent universe is completely known. In fact, if n' algebraically independent functions of the n' variables be constructed, all other functions of the variables will be algebraically related to these n' functions, and their frequency distributions will in consequence be absolutely determined by the multiple frequency distribution of the n' functions. If all the n' tests showed that the given sample could be regarded as not improbably having been drawn from a universe B, it could equally plausibly be

regarded as having been drawn from an infinity of other universes differing from B (in functional form, or in the values of the parameters) to a greater or lesser extent according as n' is small or large.

The range of choice of possible tests of normality is extensive. Clearly it is desirable that the test functions should satisfy the following conditions:

(i) They should be as simple as possible, not only for convenience of calculation but because the simpler they are the more likely is it that their frequency distributions, or at least their moments and Thiele semi-invariants, can be determined.

(ii) They should be symmetrical in the sampled variables so as to avoid bias.

(iii) When n' tends towards infinity they should assume values characteristic of the normal universe.

(iv) They should be independent of the unknown parameters a and σ [see (1) above], in the sense of being invariant for the transformation

$$x_i' = \frac{x_i - a}{\sigma} \; (i = 1, 2, \ldots, n').$$

Hitherto* the functions $\sqrt{\beta_1}$ and β_2, with

$$\beta_1 = m_3^2/m_2^3, \quad \beta_2 = m_4/m_2^2,$$

and $n'm_2 = \sum_{i=1}^{n'} (x_i - \bar{x})^2, \quad n'm_3 = \sum_{i=1}^{n'} (x_i - \bar{x})^3, \quad n'm_4 = \sum_{i=1}^{n'} (x_i - \bar{x})^4, \quad n'\bar{x} = \sum_{i=1}^{n} x_i,$

have been used for testing normality. They obviously satisfy conditions (ii) and (iv) above. With regard to (i), they are possibly the simplest functions of the sample moments which satisfy the other conditions. Their numerical computation is not difficult. With regard to condition (iii), $\sqrt{\beta_1}$ and β_2 assume the values zero and 3 respectively when n' tends towards infinity.

From the work of C. C. Craig and R. A. Fisher the lower moments of $\sqrt{\beta_1}$ and β_2 for normal samples may be regarded as completely determined for all values of n'. R. A. Fisher† has given the exact values of the second, fourth and sixth Thiele semi-invariants of $\sqrt{\beta_1}$—the odd semi-invariants are of course zero, since $\sqrt{\beta_1}$ is symmetrically distributed for normal samples. These values show that for quite small values of n' the fourth and sixth semi-invariants approach closely to their normal values zero. $\sqrt{\beta_1}$ cannot, however, be regarded as an efficient test of normality. Infinite samples drawn from all symmetrical universes will give a zero value of β_1. Accordingly it is only a test of symmetry and cannot be expected to distinguish the normal from other symmetrical universes.

I have used the formulae for the B_1 and B_2 of β_2 for normal samples, which E. S. Pearson‡ has derived from R. A. Fisher's results, in the computation of the following table.

* [The P, χ^2 and P, λ_n tests have been applied, and are available. ED.]

† "The Moments of the Distribution for Normal Samples of Measures of Departure from Normality." *Proc. Royal Society of London*, Series A, Vol. cxxx. p. 22.

‡ "Notes on Tests for Normality," *Biometrika*, Vol. XXII. Parts III and IV.

TABLE A.

Values of $\sqrt{B_1}$ and B_2 of β_2 (Normal Samples).

Size of sample n'	$\sqrt{B_1}$	B_2
6	0·71	2·01
11	1·48	6·01
26	1·75	8·03
36	1·69	8·80
51	1·57	8·38
101	1·27	6·75

Even in samples of 101 the value of B_1 is 1·27 and the value of B_2 is 6·75, very far removed from the normal values 0 and 3 respectively. Furthermore it is evident that the approach to normality of the distribution of β_2 only improves very slowly with increasing size of sample. As the prospect of obtaining the exact or even approximate frequency distributions for β_2 for all values of n' seems remote, and as the distribution of β_2 for large samples cannot be presumed normal, it appears desirable to try to devise other tests which do not suffer from these disadvantages.

The value of the mean deviation d for infinite normal samples is

$$d = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} |x - a|\, e^{-\frac{(x-a)^2}{2\sigma^2}}\, dx = \sqrt{\frac{2}{\pi}}\,\sigma,$$

so that

$$\frac{d}{\sigma} = \sqrt{\frac{2}{\pi}} = \cdot 7978845608\ldots.$$

The test which will now be examined is that of estimating d/σ from the sample and attempting to determine the probability, or improbability, that the parent universe was normal from the difference between the estimate and $\sqrt{\frac{2}{\pi}}$.

Moments of w_n.

Suppose that, in the first place, the mean of the parent normal universe is zero. Later on the reduction of the most general case of the universal mean unknown to this case will be discussed. A random sample of n is drawn with measures $y_1, y_2, \ldots, y_n$, and the estimate of d/σ is taken to be

$$w_n = |\bar{y}|/s = \frac{\Sigma\,|y_t|}{ns} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

with

$$ns^2 = \Sigma y_t^2.$$

It will be evident at once that

$$\frac{1}{\sqrt{n}} \leqslant w_n \leqslant 1.$$

Furthermore the frequency distribution of w_n will obviously be independent of σ, the unknown universal standard deviation, which in consequence may be presumed unity, without loss of generality.

The probability of obtaining the given sample is then

$$\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\Sigma y_i^2}\,\delta y_1 \ldots \delta y_n.$$

Change the variables y_i into generalised polar coordinates (ρ; $\phi_0, \phi_1, \ldots \phi_{n-2}$), and for brevity write s_i for $\sin\phi_i$, c_i for $\cos\phi_i$. Then

$$\left.\begin{aligned} y_1 &= \rho s_{n-2} s_{n-3} \ldots s_1 s_0 \\ y_2 &= \rho s_{n-2} s_{n-3} \ldots s_1 c_0 \\ y_3 &= \rho s_{n-2} s_{n-3} \ldots c_1 \\ &\vdots \qquad \vdots \\ y_{n-1} &= \rho s_{n-2} c_{n-3} \\ y_n &= \rho c_{n-2} \\ \text{so that} \quad \Sigma y_i^2 &= \rho^2 = ns^2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and the integral element becomes

$$\left(\frac{1}{\sqrt{2\pi}}\right)^n \rho^{n-1} e^{-\frac{\rho^2}{2}}\,\delta\phi_0\, s_1\delta\phi_1\, s_2^2\delta\phi_2 \ldots s_{n-2}^{n-2}\,\delta\phi_{n-2} \quad \ldots\ldots\ldots\ldots(4),$$

from which it follows that $\rho\,(=\sqrt{n}s)$ and the $n-1$ polar angles are mutually independent. After the transformation, w_n, given by (2), is a function of the polar angles only and therefore independent of s in normal samples. Indicating as before universal mean values by square brackets, it follows that

$$[|\bar{y}|^p] = [w_n^p s^p] = [w_n^p]\,[s^p],$$

or

$$[w_n^p] = [|\bar{y}|^p] \div [s^p],$$

where p is any whole number, so that the moments (fixed origin) of w_n are known when the corresponding moments of $|\bar{y}|$ and s are known. Now, from (4), the frequency distribution of s is

$$Ce^{-\frac{ns^2}{2}} s^{n-1} \quad (0 \leqslant s \leqslant \infty),$$

so that the universal moments of s are as follows:

$$\left.\begin{aligned} &s_1' = [s] && s_2' = [s^2] = 1 \\ &s_3' = [s^3] = \frac{n+1}{n} s_1' && s_4' = [s^4] = \frac{n+2}{n} \\ &s_5' = [s^5] = \frac{(n+1)(n+3)}{n^2} s_1' && s_6' = [s^6] = \frac{(n+2)(n+4)}{n^2} \\ &s_7' = [s^7] = \frac{(n+1)(n+3)(n+5)}{n^3} s_1' && s_8' = [s^8] = \frac{(n+2)(n+4)(n+6)}{n^3} \\ &\text{etc.} \end{aligned}\right\} \ldots(5),$$

and

$$s_1' = [s] = C\int_0^\infty e^{-\frac{ns^2}{2}} s^n\, ds,$$

with $$1 = C\int_0^\infty e^{-\frac{ns^2}{2}} s^{n-1}\,ds,$$

or (for n even) $$s_1' = \frac{n!}{\{(\tfrac{1}{2}n)!\}^2}\frac{n^{\frac{1}{2}}}{2^n}\sqrt{\frac{\pi}{2}}.$$

But by Stirling's Theorem

$$n! = n^{n+\frac{1}{2}}\sqrt{2\pi}\,e^{-n+\frac{1}{12n}-\frac{1}{360n^3}+\frac{1}{1260n^5}-\frac{1}{1680n^7}+\dots},$$

whence $$s_1' = e^{-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{17}{112n^7}-\dots} \quad \text{......................(6).}$$

The universal moments of $|\bar{y}| = \frac{1}{n}\Sigma|y_i|$ may be determined very easily since

$$\begin{aligned}[|y|^p] &= \sqrt{\frac{2}{\pi}}\int_0^\infty y^p e^{-\frac{y^2}{2}}\,dy \\ &= \begin{cases}(p-1)(p-3)\dots 3\cdot 1 \text{ when } p \text{ is even,} \\ (p-1)(p-3)\dots 4\cdot 2\sqrt{\dfrac{2}{\pi}} \text{ when } p \text{ is odd,}\end{cases}\end{aligned}$$

and since the variables y_i are mutually independent. The first eight moments, m_k', of $|\bar{y}|$ are as follows:

$$\left.\begin{aligned}
m_1' &= \frac{1}{n}[\Sigma|y_i|] = \frac{1}{n}\Sigma[|y_i|] = \sqrt{\frac{2}{\pi}} = a \\
m_2' &= \frac{1}{n^2}\{n_1 + a^2 n_2\} \\
m_3' &= \frac{a}{n^3}\{2n_1 + 3n_2 + a^2 n_3\} \\
m_4' &= \frac{1}{n^4}\{3n_1 + (8a^2+3)n_2 + 6a^2 n_3 + a^4 n_4\} \\
m_5' &= \frac{a}{n^5}\{8n_1 + 35n_2 + (20a^2+15)n_3 + 10a^2 n_4 + a^4 n_5\} \\
m_6' &= \frac{1}{n^6}\{15n_1 + (88a^2+45)n_2 + (165a^2+15)n_3 + (40a^4+45a^2)n_4 \\
&\qquad + 15a^4 n_5 + a^6 n_6\} \\
m_7' &= \frac{a}{n^7}\{48n_1 + 483n_2 + 7(64a^2+75)n_3 + 105(5a^2+1)n^4 \\
&\qquad + 35(2a^4+3a^2)n_5 + 21a^4 n_6 + a^6 n_7\} \\
m_8' &= \frac{1}{n^8}\{105n_1 + (1280a^2+735)n_2 + (4562a^2+630)n_3 \\
&\qquad + (1568a^4+2940a^2+105)n_4 + (1330a^4+420a^2)n_5 \\
&\qquad + (112a^6+210a^4)n_6 + 28a^6 n_7 + a^8 n_8\} \\
\text{where} \quad & n_1 = n, \quad n_2 = n(n-1), \quad n_3 = n(n-1)(n-2), \text{ etc.}
\end{aligned}\right\}\dots(7).$$

The moments, μ_k', of w_n may be written down at once from (5), (6) and (7) since

$$\mu_k' = m_k'/s_k'.$$

Finally the Thiele semi-invariants λ_k of w_n can be found from the recurrence formulae

$$\begin{aligned}
\mu_1' &= \lambda_1,\\
\mu_2' &= \lambda_1\mu_1' + \lambda_2,\\
\mu_3' &= \lambda_1\mu_2' + 2\lambda_2\mu_1' + \lambda_3,\\
\mu_4' &= \lambda_1\mu_3' + 3\lambda_2\mu_2' + 3\lambda_3\mu_1' + \lambda_4,\\
&\text{etc.}
\end{aligned}$$

On account of the intervention of the exponential in the formula for s_1', it is not possible to give simple rational expressions for the λ_k such as R. A. Fisher has given for the semi-invariants of his functions $k_3/k_2^{\frac{3}{2}}$ and k_4/k_2^2 (very similar to $\sqrt{\beta_1}$ and β_2). The calculation of the semi-invariants of w_n for any particular values of n, step by step as indicated above, is a lengthy process but presents no particular difficulty. As the value of λ_2 (equal to the square of the standard deviation) is quite small even for low values of n (see Table B below), and as λ_k appears to be *of the same order of magnitude as $\lambda_2^{k/2}$, it is necessary to work to many places* of decimals in order to obtain sufficiently accurate values for the higher semi-invariants. The values of the first four semi-invariants and of $\sqrt{\beta_1}$ and β_2 for certain sizes of samples are given in the following table.

TABLE B.

Semi-invariants, etc., of w_n.

Size of sample n $(=n'-1)$	λ_1	λ_2	λ_3	λ_4	$\sqrt{B_1}=\lambda_3/\lambda_2^{\frac{3}{2}}$	$B_2=\lambda_4/\lambda_2^2+3$
5	·838525	·00617082	− ·00027074	·000006184	−0·56	3·16
10	·818049	·00375295	− ·00010696	·000003557	−0·47	3·25
25	·805901	·00167809	− ·00002250	·000000461	−0·33	3·16
50	·801884	·00086990	− ·00000816	·000000072	−0·24	3·10
100	·799882	·000442780	− ·00000161	·000000010	−0·17	3·05

Even for small samples, the values of $\sqrt{B_1}$ and B_2 are quite near the normal values. *The values of $\lambda_p/\lambda_2^{p/2}$ for $p=3$, 4, 5 and 6 for samples of 10 are as* follows, correct to two decimal places:

$$\lambda_3/\lambda_2^{\frac{3}{2}} = -0{\cdot}47, \qquad \lambda_4/\lambda_2^2 = +0{\cdot}25,$$
$$\lambda_5/\lambda_2^{\frac{5}{2}} = -0{\cdot}10, \qquad \lambda_6/\lambda_2^3 = -0{\cdot}29.$$

From this investigation it appears very likely that, for quite small samples drawn at random from a normal universe with mean zero, the distribution of w_n is fairly close to normal.

The General Case.

In the foregoing analysis it has been assumed that the universal mean of the parent normal universe is known to be zero. This case has some practical importance. It occurs when the sample variables are themselves differences of pairs of variables drawn from the same universe. In the general case if the n' random and independent observations were $x_1, x_2, \ldots x_{n'}$ we might take $y_1 = x_1 - x_2$, $y_2 = x_3 - x_4$, etc. (if n' were even), so that if the x's are normally distributed so are the y's, but with universal mean zero, and the w_n of the y's might be used to test normality. It is evident, however, that this procedure would not sufficiently utilise the information to be derived from the sample of x's.

The simplest transformation is of course

$$y_i = x_i - \bar{x} \quad (i = 1, 2, \ldots n'),$$

and in fact it is implicit in the tests $\sqrt{\beta_1}$ and β_2. It has the great advantage of being symmetrical, but cannot be used in the calculation of w_n because the y's, while normally distributed with mean zero, are not mutually independent.

The properties of an interesting transformation will now be examined*. Suppose that the n' x's are arranged in random order, for example in the order in which they were drawn. Let

$$\left.\begin{aligned}
y_1 &= \left(\frac{x_1 + x_2}{2} - x_2\right)\sqrt{\frac{2}{1}} = \frac{x_1 - x_2}{\sqrt{2}} \\
y_2 &= \left(\frac{x_1 + x_2 + x_3}{3} - x_3\right)\sqrt{\frac{3}{2}} = \frac{x_1 + x_2 - 2x_3}{\sqrt{6}} \\
&\;\;\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
y_{n'-1} &= \left(\frac{x_1 + x_2 + \ldots + x_{n'}}{n'} - x_{n'}\right)\sqrt{\frac{n'}{n'-1}} = \frac{x_1 + x_2 + \ldots + x_{n'-1} - (n'-1)\,x_{n'}}{\sqrt{n'(n'-1)}} \\
y_{n'} &= \frac{x_1 + x_2 + \ldots + x_{n'}}{\sqrt{n'}}
\end{aligned}\right\} \quad \ldots\ldots(8).$$

The reciprocal is

$$\left.\begin{aligned}
x_1 &= \frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{6}} + \ldots + \frac{y_{n'-1}}{\sqrt{n'(n'-1)}} + \frac{y_{n'}}{\sqrt{n'}} \\
x_2 &= -\frac{y_1}{\sqrt{2}} + \frac{y_2}{\sqrt{6}} + \ldots + \frac{y_{n'-1}}{\sqrt{n'(n'-1)}} + \frac{y_{n'}}{\sqrt{n'}} \\
&\;\;\vdots \qquad\qquad\qquad \vdots \\
x_{n'-1} &= \qquad\qquad -y_{n'-2}\sqrt{\frac{n'-2}{n'-1}} + \frac{y_{n'-1}}{\sqrt{n'(n'-1)}} + \frac{y_{n'}}{\sqrt{n'}} \\
x_{n'} &= \qquad\qquad\qquad\qquad -y_{n'-1}\sqrt{\frac{n'-1}{n'}} + \frac{y_{n'}}{\sqrt{n'}}
\end{aligned}\right\} \quad \ldots\ldots(8').$$

* This transformation has been given by William Burnside in his *Theory of Probability* (Cambridge University Press, 1928, p. 97). Dr Karl Pearson kindly informs me that it is due to Helmert (1875). Its geometrical properties indicate its fundamental importance. For example, when $n'=4$, the four

The structure of the transformation is evident: y_1 represents the difference (to a numerical factor) between the mean of the first two observations and the second, y_2 the difference (to a numerical factor) between the mean of the first three observations and the third, etc. The transformation is completed by setting $y_{n'}$ equal to $\sqrt{n'}\,\bar{x}$. Quite apart from its mathematical implications, which will presently be examined, the method of analysis of the primary data is therefore a very natural one.

The new variables $y_1, y_2, \ldots y_{n'}$ have the following properties:

(i) They are normally distributed with the same standard deviation as the x's.

(ii) The universal mean of each of the variables $y_1, y_2, \ldots y_{n'-1}$ is zero.

(iii) Each pair of the first $n'-1$ variables are uncorrelated and accordingly, since they are normally distributed, they are independent.

(iv) The transformation is orthogonal, i.e. $\Sigma x_i^2 = \Sigma y_i^2$, or

$$\sum_{i=1}^{n'-1} y_i^2 = \sum_{i=1}^{n'} x_i^2 - n'\bar{x}^2 = ns^2$$

with $n = n'-1$.

The n variables y_i satisfy all the conditions specified in the previous section and accordingly we take

$$w_n = \frac{\Sigma\,|y_i|}{ns}$$

as the test of normality.

The advantages of w_n, regarded as a function of the original variables x, are as follows: like β_2 it assumes a characteristic value for infinite normal random samples; the values of its semi-invariants indicate that its distribution is far closer to the normal, even for moderate samples, than that of β_2; and as will appear in the following section its frequency distribution can be determined for all normal samples. Its principal disadvantage is that it is not symmetrical in the original variables, which means that in general it will assume different values for the $n'!$ permutations of the x_i. Consequently there is a risk not present in the use of β_2 that an abnormal value of w_n may indicate an unrepresentative permutation of the sample and not a non-normal parent universe. Accordingly it would seem advisable in all cases to randomise the sample a few times and to calculate w_n for each permutation. The mean or median value of the different w_n's might then be taken as the representative value for the purpose of determining the probability of normality.

Another course meriting practical and algebraical investigation is that of taking

$$w_n' = \sum_{i=1}^{n'} |x_i - \bar{x}| \Big/ s\sqrt{n'(n'-1)} = \frac{1}{\sqrt{n'}}\,\Sigma\,|x_i - \bar{x}| \Big/ \sqrt{\Sigma\,(x_i - \bar{x})^2}$$

planes $x_1=0$, $x_2=0$, $x_3=0$, $x_4=0$ in the y-space for $y_4=$constant are the sides of the regular tetrahedron with centre of gravity at the origin, one side parallel to the plane $y_3=0$, one vertex on the y_3-axis and one edge parallel to the y_1-axis. This transformation is probably the most symmetrical orthogonal transformation of variables, one of which is $y_{n'} = \frac{1}{\sqrt{n'}}\,\Sigma x_i$, which could be devised.

instead of w_n. It may be shown that

$$[w_n'] = [w_n],$$

and, of course:

$$\operatorname*{Lim}_{n\to\infty} w_n' = \operatorname*{Lim}_{n\to\infty} w_n,$$

and w_n', a symmetrical function of the x_i, has a unique value for each sample. It is evident that for moderately sized samples the value of w_n' will not differ much from the values of w_n for the different permutations. From work at present in progress it appears that in moderately sized samples the distribution of w_n' is very similar to that of w_n. For $n' = 51$ the values of the first three moments and of B_1 for these test functions are compared in the following table:

	Mean λ_1	Standard Deviation $\sqrt{\lambda_2}$	Third Moment λ_3	$\sqrt{B_1} = \lambda_3/\lambda_2^{\frac{3}{2}}$
$w'_{(50)}$	·8010	·0292	$-$·0000585	$-$0·24
$w_{(50)}$	·8010	·0295	$-$·0000610	$-$0·24

No significance attaches to the equality of the means: they have been made so by choice of constants. To the degree of accuracy indicated, it is very likely that for samples of more than 40 the probability points for w_n shown on Table F below may be taken as applying to w_n'.

When transformation (8) has been applied to the variables x_i the distribution of

$$u_n = \bar{y}/s$$

is given by

$$C \sin^{n-3}\theta\, \delta\theta_{n-2} \qquad (\pi \geqslant \theta_{n-2} \geqslant 0)$$

with

$$u_n = \cos\theta_{n-2}.$$

This is not an efficient test of normality or even for asymmetry in the parent universe because the value of $\bar{y}$, and hence of u_n, will be small for samples drawn from any universe. u_n might, however, be used to test the "representativeness" of the permutation of the n' variables x_i: if the value of u_n is not abnormal, then the permutation is probably representative.

If the parent universe were not normal, the efficiency of the w_n test would be jeopardised if as a result of transformation the distributions of the transformed variables y_i were brought appreciably closer to normality. It is fairly evident that in general this will not occur. If a be the universal mean of the x_i, the value of the kth transformed variable y_k will tend towards $a - x_{k+1}$ with increasing k. It may be shown that if the pth semi-invariant of the variables x_i is λ_p, the pth semi-invariant of the transformed variable y_k is

$$\left\{k\left(\frac{1}{\sqrt{k(k+1)}}\right)^p + \left(-\sqrt{\frac{k}{k+1}}\right)^p\right\}\lambda_p,$$

from which it follows that if the x_i's are symmetrically distributed, so are the y_k's. The character of the distribution has not been modified fundamentally by the transformation.

The Frequency Distribution of w_n. Theoretical Aspects.

The probability of obtaining a given value of

$$w_n = |\bar{y}|/s$$

is equal to 2^n times the probability of obtaining that value with positive values of the variables y_i. The frequency distribution of w_n can therefore be found by confining attention to the region

$$y_i \geqslant 0 \qquad (i = 1, 2, \ldots n)$$

which on transformation into generalised polar coordinates corresponds to the region

$$\frac{\pi}{2} \geqslant \phi_i \geqslant 0 \qquad (i = 0, 1, 2, \ldots n-2).$$

Now
$$\sqrt{n}\, w_n = c_{n-2} + s_{n-2}(c_{n-3} + s_{n-3}(c_{n-4} + \ldots$$
$$= c_{n-2} + \sqrt{n-1}\, w_{n-1} s_{n-2},$$

where, as before, c_i and s_i represent $\cos\phi_i$ and $\sin\phi_i$ respectively.

For a given value of w_n, since $w_{n-1} \leqslant 1$,

$$\sqrt{n}\, w_n \leqslant c_{n-2} + \sqrt{n-1}\, s_{n-2},$$

or, on dividing across by $\sqrt{n}$ and setting

$$\cos\alpha = \frac{1}{\sqrt{n}}, \quad \cos\beta = w_n,$$

we find
$$\cos(\phi_{n-2} - \alpha) \geqslant \cos\beta,$$

whence the limiting values of ϕ_{n-2} are given by

$$\alpha + \beta \geqslant \phi_{n-2} \geqslant \alpha - \beta.$$

It may be shown that $\alpha + \beta \geqslant \frac{\pi}{2}$, so that the condition $\alpha + \beta \geqslant \phi_{n-2}$ may be ignored.

The condition $w_{n-1} \geqslant \frac{1}{\sqrt{n-1}}$, on a similar analysis, yields the following limiting values of ϕ_{n-2},

$$\phi_{n-2} \geqslant \frac{\pi}{4} + \gamma; \quad \phi_{n-2} \leqslant \frac{\pi}{4} - \gamma,$$

with $\cos\gamma = \sqrt{\frac{n}{2}}\, w_n$. These conditions only apply when

$$\sqrt{\frac{n}{2}}\, w_n \leqslant 1,$$

or in practice for very small samples.

Since the polar angles ϕ_i are all independent for normal samples, the joint probability of w_{n-1} and ϕ_{n-2} is given by

$$C_{n-2}\, s_{n-2}^{n-2}\, \delta\phi_{n-2} f_{n-1}(w_{n-1})\, \delta w_{n-1},$$

where $f_{n-1}(w_{n-1})$ represents the frequency distribution of w_{n-1} and where the constant C_{n-2} is determined by

$$C_{n-2}\int_0^{\pi/2} s_{n-2}^{n-3}\,d\phi_{n-2} = 1.$$

Now, when ϕ_{n-2} is fixed,

$$\sqrt{n}\,\delta w_n = \sqrt{n-1}\,s_{n-2}\,\delta w_{n-1},$$

and, on substituting for δw_{n-1}, the integral element becomes

$$C_{n-2}\sqrt{\frac{n}{n-1}}\,s_{n-2}^{n-3}\,\delta\phi_{n-2}\,f_{n-1}(w_{n-1})\,\delta w_n.$$

The frequency distribution of w_n is accordingly found from that of w_{n-1} by the integral iteration formula

$$f_n(w_n) = C_{n-2}\sqrt{\frac{n}{n-1}}\int_0^{\cos(\alpha-\beta)} s_{n-2}^{n-4}\,f_{n-1}(w_{n-1})\,dc_{n-2} \quad\text{...........(9)},$$

with
$$w_{n-1} = \frac{\sqrt{n}\,w_n - c_{n-2}}{\sqrt{n-1}\,s_{n-2}}.$$

When $w_n \leqslant \sqrt{\dfrac{2}{n}}$ the value of the function $f_{n-1}(w_{n-1})$ is zero between the limits $\cos\left(\dfrac{\pi}{4}+\gamma\right)$ and $\cos\left(\dfrac{\pi}{4}-\gamma\right)$.

A limiting value of w_{n-1} regarded as a function of ϕ_{n-2} (with w_n fixed) is found by setting

$$\frac{dw_{n-1}}{d\phi_{n-2}} = 0,$$

which gives
$$c_{n-2} = \frac{1}{\sqrt{n}\,w_n}.$$

This is found to yield a minimum value of w_{n-1}; whence, on substitution and reduction

$$(n-1)\,w_{n-1}^2 \geqslant n w_n^2 - 1.$$

If $w_n^2 \geqslant p/n$, where p is an integer less than n, this condition gives

$$w^2_{n-1} \geqslant (p-1)/(n-1).$$

It will presently be shown that for $n=3$ the frequency distribution of w_3 is represented by different functions in the intervals

$$1 \geqslant w_3 \geqslant \sqrt{\tfrac{2}{3}}$$

and
$$\sqrt{\tfrac{2}{3}} \geqslant w_3 \geqslant \sqrt{\tfrac{1}{3}}$$

respectively. From this property and from the foregoing iteration formula it may be deduced that the frequency distribution of w_n has a different functional form in each of the intervals

$$\sqrt{\frac{p}{n}} \geqslant w_n \geqslant \sqrt{\frac{p-1}{n}} \qquad (p = 2, 3, \ldots, n).$$

Form of $f_n(w_n)$ near $w_n = 1/\sqrt{n}$.

The function w_n assumes its minimum value $1/\sqrt{n}$ when $n-1$ of the variables y_i are zero, and one is not zero. Suppose that y_1 is the non-zero variable and that the other $n-1$ variables are so small that their squares are negligible. Restricting the variables to their positive values, the joint probability of $y_1, y_2, \ldots y_n$ may be written

$$\left(\sqrt{\frac{2}{\pi}}\right)^n e^{-\frac{y_1^2}{2}} \delta y_1 \delta y_2 \ldots \delta y_n,$$

as it may be assumed without loss of generality that the standard deviation of the variables is unity. Now

$$w_n = \Sigma y_i / \sqrt{n}\, y_1,$$

whence $\qquad y_n = v_n y_1 - y_2 - \ldots - y_{n-1}$ with $v_n = \sqrt{n}\, w_n - 1$.

Change the variable y_n in the integral element so that

$$\delta y_n = y_1 \delta v_n$$

and integrate for positive values of the variables. The element of probability for v_n in the neighbourhood of $v_n = 0$ $\left(\text{i.e. } w_n = \frac{1}{\sqrt{n}}\right)$ is found to be

$$\left(\sqrt{\frac{2}{\pi}}\right)^n \delta v_n \int_0^\infty e^{-\frac{y_1^2}{2}} y_1 dy_1 \int_0^{v_n y_1} dy_2 \ldots \int_0^{v_n y_1 - y_2 - \ldots - y_{n-3}} dy_{n-2} \int_0^{v_n y_1 - y_2 - \ldots - y_{n-2}} dy_{n-1}$$

$$= \frac{2^{n-1}}{\pi^{n/2}} \frac{(\frac{1}{2}n-1)!}{(n-2)!} (v_n)^{n-2} \delta v_n.$$

The last expression represents the probability of v_n near $v_n = 0$ when a particular variable is presumed not zero. Since any of the n variables may be presumed not zero to give the value $\frac{1}{\sqrt{n}}$ of w_n the probability of v_n is n times the foregoing expression. Changing to the original variable w_n the probability of w_n in the neighbourhood of $w_n = \frac{1}{\sqrt{n}}$ is finally

$$\frac{2^{n-1}}{\pi^{n/2}} \frac{n^{\frac{n+1}{2}} (\frac{1}{2}n-1)!}{(n-2)!} \left(w_n - \frac{1}{\sqrt{n}}\right)^{n-2} \delta w_n.$$

Form of $f_n(w_n)$ near $w_n = 1$.

Another integral-iteration formula for $f_n(w_n)$ will now be derived which, while more complicated in manner of derivation than (9), seems more amenable to algebraical treatment. As before, let y_i $(i = 1, 2, \ldots n)$ be the normal variables with mean zero and standard deviation unity. Our attention may be confined to the positive region of the y_i space. In succession transform the variables y_i orthogonally using transformation (8) and the variables z_i into generalised polar coordinates r; $\theta_0, \theta_1, \ldots \theta_{n-2}$, so that

$$z_n = \frac{\Sigma y_i}{\sqrt{n}} = r \cos \theta_{n-2}.$$

Then since the $y_i \geqslant 0$ and $ns^2 = \Sigma y_i^2 = \Sigma z_i^2 = r^2$,

$$w_n = \frac{\Sigma y_i}{ns} = \cos\theta_{n-2},$$

and the integral element of the probability of positive values of the y_i, namely,

$$\left(\sqrt{\frac{2}{\pi}}\right)^n e^{-\frac{1}{2}\Sigma y_i^2}\,\delta y_1 \ldots \delta y_n,$$

becomes
$$\left(\sqrt{\frac{2}{\pi}}\right)^n e^{-\frac{r^2}{2}} r^{n-1}\,dr\,\delta\theta_0\, s_1\delta\theta_1 \ldots s_{n-3}^{n-3}\delta\theta_{n-3}\, s_{n-2}^{n-2}\,\delta\theta_{n-2}.$$

The segregation of the factor

$$s_{n-2}^{n-2}\,\delta\theta_{n-2},$$

immediately expressible in terms of w_n, has thus been effected, and the probability of θ_{n-2} may be formally set down as

$$F_{n-2}(ct_{n-2})\, s_{n-2}^{n-2}\,\delta\theta_{n-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

(ct_{n-2} indicates $\cot\theta_{n-2}$), the function $F_{n-2}(ct_{n-2})$ being given by

$$F_{n-2}(ct_{n-2}) = K_n \int_D d\theta_0\, s_1\, d\theta_1 \ldots s_{n-3}^{n-3}\, d\theta_{n-3} \quad \ldots\ldots\ldots\ldots(11),$$

with
$$K_n = \left(\sqrt{\frac{2}{\pi}}\right)^n \int_0^\infty e^{-\frac{r^2}{2}} r^{n-1}\,dr = \frac{2^{n-1}}{\pi^{\frac{n}{2}}}\left(\tfrac{1}{2}n - 1\right)!$$

and the domain of integration D, corresponding to

$$y_i \geqslant 0 \qquad (i = 1, 2, \ldots n),$$

is given by

$$s_{n-2}\left(s_{n-3}\left(\ldots s_1\left(-\frac{s_0}{\sqrt{2}} + \frac{c_0}{\sqrt{6}}\right) + \frac{c_1}{\sqrt{12}}\right) + \frac{c_2}{\sqrt{20}}\right) + \ldots + \frac{c_{n-3}}{\sqrt{n(n-1)}}\right) + \frac{c_{n-2}}{\sqrt{n}} \geqslant 0,$$

$$s_{n-2}\left(s_{n-3}\left(\ldots s_1\left(-\frac{s_0}{\sqrt{2}} + \frac{c_0}{\sqrt{6}}\right) + \frac{c_1}{\sqrt{12}}\right) + \frac{c_2}{\sqrt{20}}\right) + \ldots + \frac{c_{n-3}}{\sqrt{n(n-1)}}\right) + \frac{c_{n-2}}{\sqrt{n}} \geqslant 0,$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$s_{n-2}\left(-\sqrt{\frac{n-2}{n-1}}\, s_{n-3}c_{n-4} + \frac{c_{n-3}}{\sqrt{n(n-1)}}\right) + \frac{c_{n-2}}{\sqrt{n}} \geqslant 0$$

$$-\sqrt{\frac{n-1}{n}}\, s_{n-2}c_{n-3} + \frac{c_{n-2}}{\sqrt{n}} \geqslant 0,$$

from (8′). Dividing across by the positive s_{n-2}, it will be observed that all the terms except the last of each of the inequalities are free of the angle θ_{n-2} (and hence of w_n) and the last term becomes $\frac{ct_{n-2}}{\sqrt{n}}$. This accounts for the form of (10). Furthermore it may easily be shown that for all values of the angles $\theta_0, \theta_1, \ldots \theta_{n-3}$ the *minimum* value of the left-hand terms of the inequalities, for θ_{n-2} fixed, is

$$-\sqrt{\frac{n-1}{n}} + \frac{ct_{n-2}}{\sqrt{n}},$$

so that if ct_{n-2} satisfies the inequality

$$ct_{n-2} \geqslant \sqrt{n-1} \quad \text{or} \quad c_{n-2} = w_n \geqslant \sqrt{\frac{n-1}{n}},$$

the domain of integration D becomes

$$2\pi \geqslant \theta_0 \geqslant 0,$$
$$\pi \geqslant \theta_i \geqslant 0 \qquad (i = 1, 2, \dots (n-3)),$$

and the function $F_{n-2}(ct_{n-2})$ becomes a constant. It follows that *the probability of* $w_n = \cos\theta_{n-2}$ *when* $1 \geqslant w_n \geqslant \sqrt{\frac{n-1}{n}}$ *is given by*

$$A_{n-2} \sin^{n-2}\theta_{n-2}\,\delta\theta_{n-2},$$

the constant being given by

$$A_{n-2}\int_0^\pi \sin^{n-2}\theta\, d\theta = 2^n.$$

Adverting to the general case, it will be observed that if, after dividing all the inequalities except the last by the positive quantity $s_{n-2}s_{n-3}$, a new angle θ'_{n-3} given by

$$\frac{ct'_{n-3}}{\sqrt{n-1}} = \frac{ct_{n-3}}{\sqrt{n(n-1)}} + \frac{ct_{n-2}}{s_{n-3}\sqrt{n}} \quad \text{..........................(12)}$$

is introduced, the form of the resulting $n-1$ inequalities is identical with that of the original n inequalities and hence it follows that the function $F_{n-2}(ct_{n-2})$, given by (11), may be written as

$$F_{n-2}(ct_{n-2}) = \frac{K_n}{K_{n-1}}\int_{-1}^{\frac{ct_{n-2}}{\sqrt{n-1}}} F_{n-3}(ct'_{n-3})\, s_{n-3}^{n-\frac{1}{3}}\, dc_{n-3} \quad \text{............(13)},$$

ct'_{n-3} being given by (12). This is the iteration formula required. It will now be utilised to determine the form of $F_{n-2}(ct_{n-2})$, and hence the frequency of w_n when $\sqrt{\frac{n-1}{n}} \geqslant w_n \geqslant \sqrt{\frac{n-2}{n}}$; to prove in fact that the derivative

$$F'_{n-2}(ct_{n-2}) = H_{n-2}\left(1 - \frac{ct^2_{n-2}}{n-1}\right)^{\frac{n-4}{2}},$$

H_{n-2} being a constant. This will be proved by induction. Let it be assumed that the result holds for $n-1$, i.e. that for

$$\sqrt{\frac{n-2}{n-1}} \geqslant c'_{n-3} \geqslant \sqrt{\frac{n-3}{n-1}},$$

$$F'_{n-3}(ct'_{n-3}) = H_{n-3}\left(1 - \frac{ct'^2_{n-3}}{n-2}\right)^{\frac{n-5}{2}}.$$

Now it may be proved that if $w_n = c_{n-2} \geqslant \sqrt{\frac{n-2}{n}}$, then it follows that

$$ct'_{n-3} \geqslant \sqrt{\frac{n-3}{2}},$$

which is equivalent to $c'_{n-3} \geqslant \sqrt{\dfrac{n-3}{n-1}}$. Also the function $F_{n-3}(ct'_{n-3})$ is equal to a constant A_{n-3} when $c'_{n-3} \geqslant \sqrt{\dfrac{n-2}{n-1}}$, i.e. when $ct'_{n-3} \geqslant \sqrt{n-2}$. The last inequality yields the following limits for c_{n-3},

$$c_{n-3} \leqslant \cos(\alpha+\beta),$$

$$c_{n-3} \geqslant \cos(\alpha-\beta),$$

with $\cos\alpha = -1/(n-1)$ and $\cos\beta = ct_{n-2}/\sqrt{n-1}$.

Hence, from (13),

$$\frac{K_{n-1}}{K_n} F_{n-2}(ct_{n-2}) = A_{n-3}\int_{-1}^{\frac{ct_{n-2}}{\sqrt{n-1}}} s_{n-3}^{n-4}\, dc_{n-3} + \int_{\cos(\alpha+\beta)}^{\cos(\alpha-\beta)} \{F_{n-3}(ct'_{n-3}) - A_{n-3}\}\, s_{n-3}^{n-4}\, dc_{n-3}.$$

Differentiating across with respect to ct_{n-2} and noting that the integral element of the second integral vanishes at the limits of integration, we find

$$\frac{K_{n-1}}{K_n} F'_{n-2}(ct_{n-2}) = \frac{A_{n-3}}{\sqrt{n-1}}\left(1-\frac{ct^2_{n-2}}{n-1}\right)^{\frac{n-4}{2}} + K_{n-3}\int_{\cos(\alpha+\beta)}^{\cos(\alpha-\beta)} \left(1-\frac{ct'^2_{n-3}}{n-2}\right)^{\frac{n-5}{2}} s_{n-3}^{n-3}\, dc_{n-3}.$$

On substitution for ct'_{n-3} in the latter integral, upon reduction it is found to be of the form (to a constant factor)

$$\int_{x_0}^{x_1} \{(x-x_0)(x-x_1)\}^{\frac{n-5}{2}}\, dx,$$

which (to a constant factor) is equal to

$$(x_1-x_0)^{n-4}.$$

In this case

$$x_1 - x_0 = \cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$$

$$= 2\sqrt{\frac{n(n-2)}{(n-1)^2}}\left(1-\frac{ct^2_{n-2}}{n-1}\right)^{\frac{1}{2}}.$$

It follows that (to a constant factor) $F'_{n-2}(ct_{n-2})$ is equal to

$$\left(1-\frac{ct^2_{n-2}}{n-1}\right)^{\frac{n-4}{2}}.$$

All the constants have been determined, and the final result is as follows: *the probability of $w_n = \cos\theta$ when $\sqrt{\dfrac{n-1}{n}} \geqslant w_n \geqslant \sqrt{\dfrac{n-2}{n}}$ is*

$$\left\{\frac{2^{n-1}}{\pi}\frac{n(n-2)}{\sqrt{n-1}}\int_0^{ct_{n-2}}\left(1-\frac{x^2}{n-1}\right)^{\frac{n-4}{2}} dx - \frac{n-2}{2}A_{n-2}\right\}\sin^{n-3}\theta\,\delta\theta,$$

with

$$A_{n-2}\int_0^{\pi}\sin^{n-2}\theta\, d\theta = 2^n.$$

Frequency Distributions for $n = 2, 3, 4$ and 5.

The foregoing general propositions completely determine the probabilities of w_n for $n = 2$ and $n = 3$ as follows:

$\underline{n = 2.}$

$$1 \geqslant w_2 \geqslant \frac{1}{\sqrt{2}} : \frac{4}{\pi} \frac{\delta w_2}{\sqrt{1 - w_2^2}}.$$

$\underline{n = 3.}$

$$1 \geqslant w_3 \geqslant \sqrt{\tfrac{2}{3}} : 4\delta w_3,$$

$$\sqrt{\tfrac{2}{3}} \geqslant w_3 \geqslant \sqrt{\tfrac{1}{3}} : 4\left(1 - \frac{3}{\pi} \cos^{-1} \frac{w_3}{\sqrt{2 - 2w_3^2}}\right) \delta w_3.$$

$\underline{n = 4.}$

The probabilities of w_4 when $w_4 \geqslant \dfrac{1}{\sqrt{2}}$ are as follows:

$$1 \geqslant w_4 \geqslant \sqrt{\tfrac{3}{4}} : \frac{32}{\pi} \sin \theta \delta w_4,$$

$$\sqrt{\tfrac{3}{4}} \geqslant w_4 \geqslant \sqrt{\tfrac{2}{4}} : \frac{32}{\pi} \left(\frac{2}{\sqrt{3}} \cos \theta - \sin \theta\right) \delta w_4,$$

with $w_4 = \sin \theta$. The form of the probability function has also been determined for the third interval, namely $\sqrt{\tfrac{2}{4}} \geqslant w_n \geqslant \sqrt{\tfrac{1}{4}}$, but perhaps the details need not be given. The first derivative of $F_2(ct_2)$ was found to be

$$F_2'(ct_2) = \frac{64}{\sqrt{3}\pi} \left(1 - \frac{3}{\pi} \cos^{-1} \sqrt{\frac{2ct_2^2}{3 - ct_2^2}}\right).$$

(Compare with the frequency distribution of w_3 in the second interval.) It was found possible to integrate this expression so as to give the following expression for the probability of w_4 in the interval

$$\sqrt{\tfrac{2}{4}} \geqslant w_4 \geqslant \sqrt{\tfrac{1}{4}} : \frac{32}{\pi} \left(\frac{2}{\sqrt{3}} \cos \theta - \sin \theta - 2\sqrt{3} \cos \theta \cos^{-1} \sqrt{\frac{2 \cot^2 \theta}{3 - \cot^2 \theta}} \right.$$
$$\left. + 6 \cos^{-1} \sqrt{\frac{2}{3 - \cot^2 \theta}}\right) \delta w_4,$$

with $\cos \theta = w_4$.

$\underline{n = 5.}$

The probabilities of w_5 in the first two intervals are as follows:

$$1 \geqslant w_5 \geqslant \sqrt{\tfrac{4}{5}} : 24(1 - w_5^2) \delta w_5,$$

$$\sqrt{\tfrac{4}{5}} \geqslant w_5 \geqslant \sqrt{\tfrac{3}{5}} : \frac{12}{\pi} \left(2\pi - 10 \cos^{-1} \frac{\cot \theta}{2} + 5 \cot \theta \sqrt{1 - \frac{\cot^2 \theta}{4}}\right) \sin^2 \theta \delta w_5,$$

with $\cos \theta = w_5$.

It seems not unlikely that further investigation would have given more symmetrical expression to the iteration and other formulae in this section, and that exact values for the frequencies for some values of $n > 4$ and some intervals

of $w_n \leqslant \sqrt{\frac{n-2}{n}}$ might have been obtained. The complicated expressions obtained even for $n=4$ and $n=5$ suggest however that such formulae would be exceedingly cumbersome. The present paper has the practical objective of deriving the actual numerical probabilities of w_n for samples of different sizes, and it will be shown that the results so far obtained are adequate for this purpose

Computation of Frequency Tables for w_n.

In Table C will be found the ordinates, and in Table D the cumulative probabilities of w_n for $n=2$ to $n=10$ inclusive for certain values of w_n. These values were more or less arbitrarily selected, the pivotal point $w_n = \cdot 80$ as very close to the mean $\sqrt{\frac{2}{\pi}}$ for infinite samples, and the interval of $\cdot 03$ as about one-half of the standard deviation for $n=10$ and therefore less than half the standard deviation for $n<10$. The values of the ordinates for $n=2$, 3 and 4 and for $w_n > \sqrt{\frac{n-2}{n}}$ are the actual values calculated from the formulae given in the last section. The remaining ordinates were computed by approximate integration from the integral iteration formula (9). For instance, in the calculation of the ordinates for the different values of w_5, the integral element on the right-hand side was regarded as a function of $\cos\phi_3$ (i.e. c_3) and its values found at intervals of $\cdot 05$ starting with $\cos\phi_3 = 0$. The values of $f_4(w_4)$ were found by osculatory interpolation* (used in life table construction) from the previously ascertained values of $f_4(\cdot 86)$, $f_4(\cdot 83)$, etc.; the values of $f_4(w_4)$ for $w_4 > \sqrt{\frac{1}{2}}$ were the actual values which were easily calculated.

Finally the values of the ordinate for each selected value of w_5 (i.e. $\cdot 47$, $\cdot 50$, etc.) were found by approximate integration by dividing the values of the integral element into contiguous groups of 3, 4 or 5 and applying the familiar formulae to each group. Special methods were used near the upper limits of $\cos\phi_3$ and near the limits of the regions where $f_4(w_4)$ is zero, as at these limits the values of $\cos\phi_3$ were not multiples of $\cdot 05$. As a check on the work the value of the integral

$$\int_{\sqrt{\frac{3}{5}}}^{1} f_5(w_5)\,dw_5$$

was computed by approximate integration and found to equal $0\cdot 999$, which was sufficiently close to unity.

Without adjusting the ordinates to allow for this slight discrepancy the calculation of the ordinates for the frequency of w_6 was proceeded with in an analogous manner, and so on to $n=10$. At each stage the proximity of the total probability to unity was checked: the actual totals ranged from $0\cdot 995$ to $1\cdot 001$ (without any adjustment). The work, which involves about 1000 interpolations, was arduous, but systematic.

* See Supplement to the *Seventy-fifth Annual Report of the Registrar-General of Births, Deaths and Marriages in England and Wales*, Part I—Life Tables, p. 50.

TABLE C.

Ordinates of the Frequency Distribution of w_n.

n' / n / w_n	3 / 2	4 / 3	5 / 4	6 / 5	7 / 6	8 / 7	9 / 8	10 / 9	11 / 10
1·00	∞	4	0	0	0	0	0	0	0
·98	6·3983	4	2·0270	·9504	·4275	·1882	·0814	·0348	·0147
·95	4·0776	4	3·1806	2·3400	1·6531	1·1408	·7740	·5190	·3450
·92	3·2487	4	3·9920	3·6864	3·2698	2·8246	2·3794	1·9780	1·6315
·89	2·7924	4	4·6444	4·9340	4·7941	4·6066	4·3849	4·1346	3·88
·86	2·4951	4	4·9178	5·1070	5·4093	5·6479	5·79	5·86	5·92
·83	2·2828	4	4·0809	4·6789	5·2092	5·63	5·92	6·29	6·62
·80	2·1221	2·7019	3·2978	3·9074	4·34	4·74	5·13	5·47	5·82
·77	1·9955	1·9052	2·5574	2·91	3·21	3·52	3·75	3·98	4·20
·74	1·8930	1·4049	1·8525	1·99	2·17	2·30	2·40	2·50	2·57
·71	1·8081	1·0316	1·1779	1·28	1·32	1·36	1·38	1·37	1·37
·68	—	·7320	·7135	·777	·767	·740	·707	·678	·643
·65	—	·4810	·4362	·403	·396	·370	·336	·302	·270
·62	—	·2647	·2517	·224	·201	·168	·144	·123	·102
·59	—	·0745	·1296	·112	·0930	·0728	·0568	·0446	·0349
·56	—	—	·0532	·0494	·0370	·0294	·0216	·0154	·0104
·53	—	—	·0124	·0181	·0122	·0091	·0069	·0036	·0024
·50	—	—	0	·0040	·0035	·0023	,0016	·0007	·0004
·47	—	—	—	·0004	·0007	·0005	·0003	·0002.	·0001
·44	—	—	—	—	·0000	·0001	·0000	·0000	·0000
(Lower limit of w_n)	(·7071)	(·5773)	(·5000)	(·4472)	(·4082)	(·3780)	(·3536)	(·3333)	(·3162)

In Table D the cumulative probabilities shown for $n=2$ and $n=3$ and for $w_n \geqslant \sqrt{\dfrac{n-1}{n}}$ are the actual values calculated from the formulae in the last section. The remainder of the table was computed by approximate integration, the formula used being based upon that for osculatory interpolation, i.e., for unit interval,

$$\int_0^1 u_{1+x}\,dx = \tfrac{1}{24}\left(-u_0 + 13\left(u_1 + u_2\right) - u_3\right),$$

a formula of some interest in itself. The slight discrepancies in the total probabilities were distributed proportionately over the calculated frequencies.

As a final check, it was calculated from Table D that the mean value of w_{10}, depending for its accuracy on all the interpolations from $n=5$ to $n=10$, was 0·8177 compared with the actual value of 0·8180 given in Table B.

From the values of $\sqrt{B_1}$ and B_2 given in the latter table it might have been anticipated that the Second Approximation to the Law of Error, namely

$$\frac{1}{\sigma\sqrt{2\pi}}\left\{1 - \frac{\sqrt{B_1}}{6}\left(\frac{3w}{\sigma} - \frac{w^3}{\sigma^3}\right)\right\} e^{-\frac{w^2}{2\sigma^2}}\,\delta w$$

TABLE D.

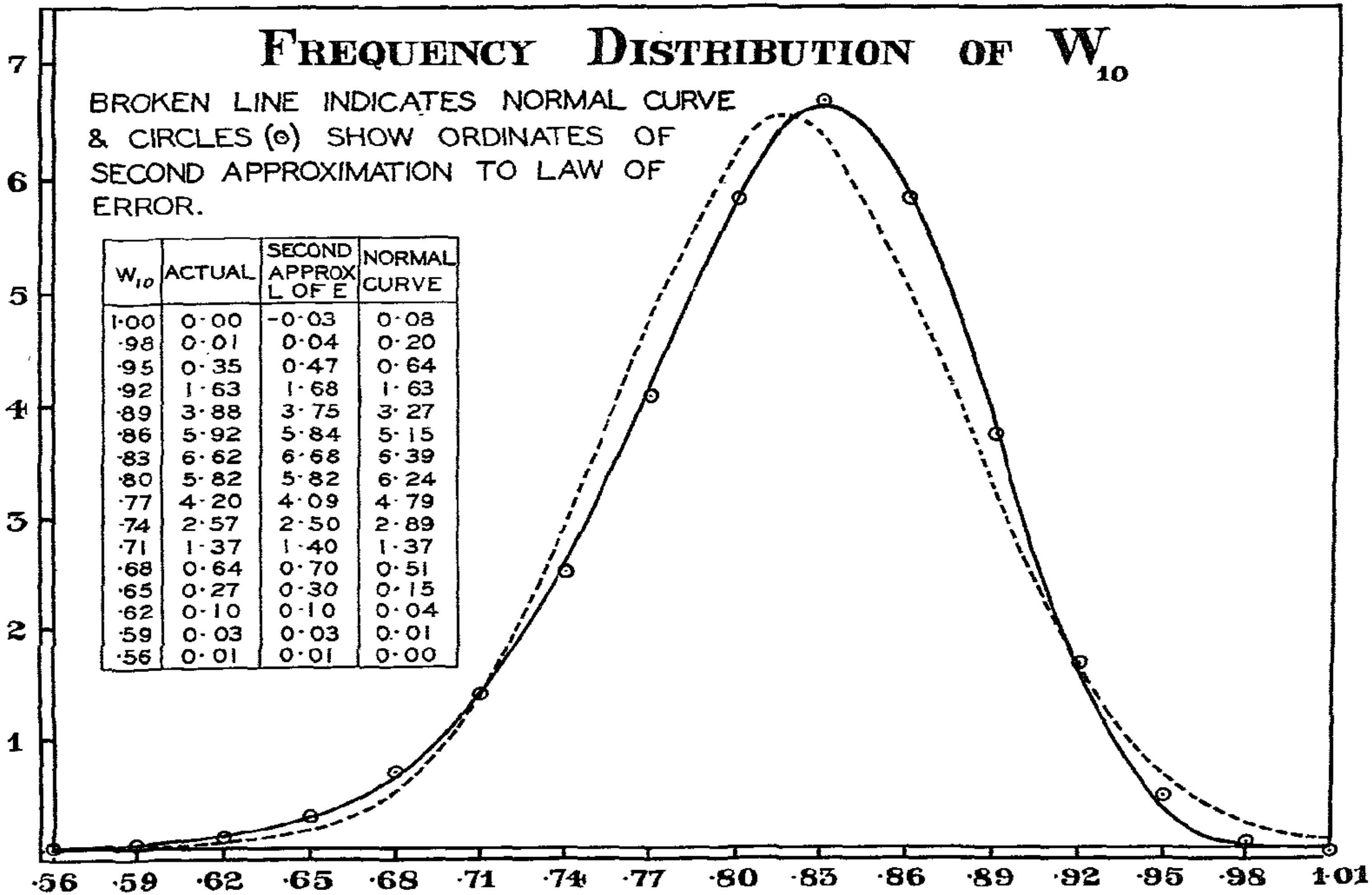
Cumulative Probability of w_n from $w_n = 1$.

n' \ n \ w_n	3 2	4 3	5 4	6 5	7 6	8 7	9 8	10 9	11 10
1·00	0	0	0	0	0	0	0	0	0
·98	·2551	·0800	·0271	·0095	·0034	·0013	·0005	·0002	·0001
·95	·4043	·2000	·1066	·0590	·0335	·0103	·0112	·0060	·0030
·92	·5128	·3200	·2147	·1495	·1071	·0777	·0578	·0420	·0314
·89	·6028	·4400	·3445	·2803	·2205	·1900	·1580	·1336	·1133
·86	·6810	·5600	·4915	·4330	·3850	·3460	·3141	·2857	·2623
·83	·7534	·6800	·6203	·5810	·5464	·5175	·4928	·4712	·4539
·80	·8103	·7856	·7369	·7106	·6900	·6745	·6608	·6500	·6433
·77	·8810	·8535	·8246	·8130	·8043	·7988	·7950	·7920	·7908
·74	·9393	·9027	·8907	·8860	·8846	·8858	·8872	·8894	·8920
·71	·9948	·9300	·9301	·9346	·9364	·9399	·9431	·9466	·9500
·68	—	·9653	·9639	·9651	·9672	·9707	·9736	·9765	·9791
·65	—	·9834	·9807	·9824	·9842	·9868	·9887	·9905	·9921
·62	—	·9945	·9908	·9915	·9929	·9945	·9950	·9966	·9973
·59	—	·9993	·9964	·9964	·9971	·9979	·9984	·9989	·9992
·56	—	—	·9990	·9987	·9990	·9993	·9995	·9997	·9998
·53	—	—	·9999	·9997	·9997	·9998	·9999	·9999	1·0000
·50	—	—	1·0000	·9999	·9999	1·0000	1·0000	1·0000	1·0000
·47	—	—	—	1·0000	1·0000	1·0000	1·0000	1·0000	1·0000
(Lower Limit of w_n)	(·7071)	(·5773)	(·5000)	(·4472)	(·4082)	(·3780)	(·3536)	(·3333)	(·3162)

(σ here indicating the standard deviation of w), would furnish a close approximation to the actual distribution. It is extremely fortunate and somewhat unexpected that, as shown in the diagram, the correspondence is very satisfactory for samples of only 10. On this diagram the normal curve, with the actual mean and standard deviation of w_{10}, is shown for the purposes of contrast and comparison. Circles (⊙) only mark the curve for the Second Approximation, as the ordinates of the latter curve are too close to those of the actual distribution to permit of accurate drawing.

The correspondence between the Second Approximation and the actual distribution is brought out concisely in Table E, p. 330, which shows the 1 % and 5 % probability points for the respective distributions. [By definition, the upper 1 % probability point is the value of w_{10} above which lies 1 % of the total frequency.]

There is scarcely any significant difference between the probability points as calculated by these two systems. While Table B shows that the value of B_2 does not decrease continuously with increasing size of sample, it does so after a certain value of n and never differs much from the normal value of 3. There can be no



FREQUENCY DISTRIBUTION OF W_{10}

BROKEN LINE INDICATES NORMAL CURVE & CIRCLES (⊙) SHOW ORDINATES OF SECOND APPROXIMATION TO LAW OF ERROR.

W_{10}	ACTUAL	SECOND APPROX L OF E	NORMAL CURVE
1·00	0·00	−0·03	0·08
·98	0·01	0·04	0·20
·95	0·35	0·47	0·64
·92	1·63	1·68	1·63
·89	3·88	3·75	3·27
·86	5·92	5·84	5·15
·83	6·62	6·68	5·39
·80	5·82	5·82	6·24
·77	4·20	4·09	4·79
·74	2·57	2·50	2·89
·71	1·37	1·40	1·37
·68	0·64	0·70	0·51
·65	0·27	0·30	0·15
·62	0·10	0·10	0·04
·59	0·03	0·03	0·01
·56	0·01	0·01	0·00

TABLE E.

	1 % points		5 % points	
	Upper	Lower	Upper	Lower
Actual distribution	·941	·656	·911	·710
Second Approximation to Law of Error	·942	·656	·912	·708

reasonable doubt that the Second Approximation will yield sufficiently accurate 1 % and 5 % probability points for samples of 11 ($=n'$) and over. On this assumption the following table has been calculated:

TABLE F.

The 1 % and 5 % probability points of w_n.

Size of Sample n'	n	Upper limit of w_n	Probability points of w_n				Lower limit of w_n	Mean of w_n	Standard Deviation of w_n
			Upper 1 %	Upper 5 %	Lower 5 %	Lower 1 %			
6	5	1	·980	·954	·696	·626	·4472	·8385	·0780
11	10	1	·941	·911	·710	·656	·3162	·8180	·0613
16	15	1	·916	·891	·720	·677	·2581	·8113	·0516
21	20	1	·902	·879	·728	·691	·2236	·8079	·0451
26	25	1	·892	·870	·731	·701	·2000	·8059	·0440
31	30	1	·884	·864	·739	·709	·1826	·8046	·0376
36	35	1	·878	·859	·743	·715	·1690	·8036	·0350
41	40	1	·873	·855	·746	·720	·1581	·8029	·0328
46	45	1	·869	·851	·749	·725	·1491	·8023	·0310
51	50	1	·865	·849	·751	·728	·1414	·8019	·0295
76	75	1	·853	·839	·759	·741	·1155	·8005	·0242
101	100	1	·846	·834	·764	·748	·1000	·7999	·0210
501	500	1	·820	·814	·783	·776	·0447	·7983	·0095
1001	1000	1	·813	·809	·787	·782	·0316	·7981	·0067

Conclusion.

It is a simple matter to devise theoretical universes which random samples will identify as probably non-normal by the w_n test and to determine roughly from Table F the sizes of the samples required. For example, the value of the ratio of mean deviation to standard deviation (in infinite random samples) for the rectangular distribution

$$\tfrac{1}{2}\,\delta x \qquad (+1 \geqslant x \geqslant -1)$$

is $\frac{\sqrt{3}}{2} = {}$·866, so that, with no presumption of knowledge about the nature of the distribution, samples of about 75 might be required in order that in the majority

of cases the values of w_n found would lie above the upper 1% probability point. The values of the ratios (in infinite samples) for the distributions

$$\frac{2}{\pi}\frac{\delta x}{(1+x^2)^2} \text{ and } \tfrac{1}{2}e^{-|x|}\delta x \qquad (+\infty \geqslant x \geqslant -\infty)$$

which bear superficial resemblances to the normal, are $\frac{2}{\pi} = \cdot 637$ and $\frac{1}{\sqrt{2}} = \cdot 707$ respectively, and samples of about 25 and 50 respectively would usually be required to indicate their non-normality. It is important to observe, however, that the test will often react for far smaller samples. These three universes are symmetrical. In cases where asymmetry is suspected a count of + signs and − signs of the transformed variables y_i in moderate sized samples (say of 50 to 100) may reveal such asymmetry. The number of signs should of course be distributed in the "point binomial"

$$(\tfrac{1}{2}+\tfrac{1}{2})^n$$

for normal samples.

The efficiency of w_n for testing normality with the types of samples actually occurring is now a matter for practical investigation. The indications are that *in practice* it will not often react for samples of less than, say, 50. In using it we are manifestly liable to what J. Neyman and E. S. Pearson* have called "Errors of Type II," i.e. of accepting an hypothesis H_0 (in this case that the universe is normal) when some alternative is true. The question arises: if w_n fails to identify non-normality for a sample known to be derived from a non-normal universe will any other *general* method prove effective? In other words, an "existence theorem" for tests of normality seems to be required: a theorem which, while not necessarily indicating the form of the test or tests, will at least establish the "maximum sensitivity" (with a suitable definition of "sensitivity") however approximately and with whatever simplification in the conditions of the problem. This may be quite small for small samples; it is manifestly nil for samples of 2. As shown by R. A. Fisher† for $\sqrt{\beta_1}$ in normal samples of 3 and in the present paper for w_3, the distributions are wholly or partially U-shaped, and in consequence they reveal no region of improbability. On the other hand w_n must identify non-normality in indefinitely large samples, unless the universe has the normal value of $\sqrt{\frac{2}{\pi}}$ for the ratio of mean deviation to standard deviation. The only important non-normal distribution which assumes the normal value for the ratio appears to be the Second Approximation to the Law of Error which also gives the normal value of 3 for β_2 in infinite samples.

Summary.

The values of the moments and of $\sqrt{B_1}$ and B_2 of the distribution of β_2 for normal samples indicate that impracticably large samples would be required in order that the distribution would be presumed normal. As the frequency distribution of β_2 is unknown and as $\sqrt{\beta_1}$ is to be regarded as a test of asymmetry rather

* "The Testing of Statistical Hypotheses in Relation to Probabilities *a priori*." *Proceedings of the Cambridge Philosophical Society*, Vol. XXIX. Part 4, p. 493.

† *Op. cit.*, p. 20.

than of normality, we are constrained to seek new measures of normality. The measure considered in this paper is the ratio of the mean deviation to the standard deviation calculated from the sample, which ratio in infinite random samples equals $\sqrt{\frac{2}{\pi}}$.

Two cases are considered: (i) that in which the experimental technique is such that the universal mean may be presumed known and hence, by a change of origin, presumed zero; (ii) the general case of both universal mean and standard deviation unknown. In the first case, if the sample measures are $y_1, y_2, \dots y_n$, we take

$$w_n = \frac{1}{\sqrt{n}} \frac{\Sigma |y_i|}{\sqrt{\Sigma y_i^2}}.$$

In the second case, if the sample measures are $x_1, x_2, \dots x_{n'}$, arranged in random order, we take

$$w_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{|x_1 + \dots + x_i - i x_{i+1}|}{\sqrt{i(i+1)}} \Big/ \sqrt{\sum_{i=1}^{n'} (x_i - \bar{x})^2},$$
$$n = n' - 1.$$

In this case w_n has not an unique value for each sample and it is suggested tentatively (pending further work) that for moderately sized samples the function

$$w_n' = \frac{1}{\sqrt{n'}} \sum_{i=1}^{n'} |x_i - \bar{x}| \Big/ \sqrt{\sum_{i=1}^{n'} (x_i - \bar{x})^2}$$

be used instead of w_n. The frequency distribution of w_n may be regarded as completely determined for all normal samples. From Table F may be determined the approximate improbability of the value of w_n found from a given sample occurring if the sample were drawn at random from a normal universe.

II.

A COMPARISON OF β_2 AND MR GEARY'S w_n CRITERIA.

By E. S. PEARSON, D.Sc.

1. *The Choice of alternative Tests.*

In the development of statistical sampling theory it has often happened that more than one test of a given hypothesis is available. Generally on theoretical grounds it is possible to specify which of these tests is the most efficient; but it may happen that owing to mathematical difficulties in putting the ideal test into working form or to practical difficulties arising from the extent of computation involved, the statistician will choose to employ a second best but simpler test. In the case of testing the hypothesis that a sample has been drawn from a normally distributed population, it seems likely that for large samples and when only small departures from normality are in question, the most efficient criteria will be based on the moment coefficients of the sample, e.g. on the values of $\sqrt{\beta_1}$ and β_2*. While, however, the sampling moments of these quantities are known*, it has not so far been possible to determine the actual sampling distributions from which the probability integral could be obtained exactly.

In earlier papers in this Journal† I suggested that empirical curves having the correct first four moment coefficients might be used as approximations to the unknown true distributions, and gave tables of 5 °/。 and 1 °/。 levels of probability for $\sqrt{\beta_1}$ and β_2, which have since been republished in *Tables for Statisticians and Biometricians*, Part II (Table XXXVII *bis*). The sampling distribution of $\sqrt{\beta_1}$ being symmetrical and not very leptokurtic presents relatively little difficulty, and a recent investigation by K. Williams‡ suggests that the tables previously given cannot be far in error. A test on $\sqrt{\beta_1}$ is however only sensitive to lack of symmetry, and the adequacy of the tables for the 5 °/。 and 1 °/。 limits for β_2 is at present more doubtful, since the moment coefficients indicate a very skew and leptokurtic distribution which it is difficult to approximate to with confidence from a knowledge of four moment coefficients only.

R. C. Geary in his interesting paper above has therefore suggested an alternative criterion, which is based on the ratio of the mean deviation to the standard

* R. A. Fisher, *Proc. Roy. Soc.* Series A, Vol. 130 (1930), pp. 16—28.

† *Biometrika*, Vol. XXII. (1930), pp. 239—249, and pp. 423—424.

‡ *Biometrika*, Vol. XXVII. (1935), pp. 269—271. Mr Williams has extended my tables for $\sqrt{\beta_1}$ down to $n' = 20$.

deviation. This criterion which, as shown below, is sensitive to changes in the population β_2 has the advantage that

(1) it yields a more manageable sampling distribution, for which he has been able to provide tables of percentage limits;

(2) it involves less trouble in calculation, since the mean deviation is more readily obtained than the fourth moment.

In assessing the value of these alternative tests it is necessary to bear in mind two risks of error in judgment:

Type I, that of rejecting the hypothesis tested when it is true, i.e. of deciding the population is not normal when it is;

Type II, that of failing to detect departure from normality when it really exists in the population.

In the case of β_2, no attempt was made in my previous paper to calculate probability levels for $n < 100$ because of the very high values of $B_1(\beta_2)$ and $B_2(\beta_2)$, and before further investigation is completed, the limits given for samples of less than 200 or 300 must be regarded as only approximate. Mr Geary has however been able to obtain precise levels for his ratio w_n and reports that investigations at present in progress give reasons for believing that for $n' \geqslant 40$ the more generally useful criterion w_n' will have the same sampling distribution as w_n. His test therefore as far as it concerns kurtosis rather than asymmetry does provide a reliable control of the error of Type I. Its efficiency in dealing with the other type of error can be most readily judged on the basis of some numerical investigations. Such investigations have been carried out after consultation with Mr Geary*; the work has been rather lengthy as it seemed desirable to compute values for $\sqrt{\beta_1}$ and β_2 as well as for w_n', and in some cases w_n.

2. *The Experimental Sampling.*

The problem to be dealt with is that of comparing the relative efficiency of w_n, w_n', $\sqrt{\beta_1}$ and β_2 in controlling errors of Type II, i.e. in detecting departure from normality in the sampled population. Samples were drawn with the help of Tippett's Random Sampling Numbers† from the seven different populations, which are shown in Table I. Further details of the experiment are given in an Appendix to this paper.

The populations II, III, IV, VI and VII have been used in a number of previous sampling experiments‡; I and V, the rectangular and double exponential populations are referred to by Geary on p. 330 of his paper. Population IV was derived from the equation

$$y = y_0(1 + x^2)^{-3} \quad \text{.................................(1).}$$

* My apologies are due to Mr Geary for the delay in the publication of his paper which has resulted, but I hope that he will regard the work as worth while. The heavy computation has been undertaken by students in the Department of Applied Statistics, University College, London, and in particular by Miss C. M. Thompson.

† *Tracts for Computers*, No. xv.

‡ For an account of these distributions see also J. M. le Roux, *Biometrika*, Vol. xxiii. (1931), pp. 158—161.

When the distribution law is very leptokurtic there is inevitably a certain difference between the moments of the probability distribution represented by the Random Number Scale (adjusted to a total of 10,000) used in sampling and the moments of the theoretical law. It is the former that have been entered in Table I. For example, for the double exponential curve $\beta_2 = 6{\cdot}0$ and (Mean deviation)/(Standard deviation) $= w_n = {\cdot}707$; but for the sampling scale used $\beta_2 = 5{\cdot}88$ and $w_n = {\cdot}700$.

TABLE I.

Populations sampled.

No.	Type	$\sqrt{\beta_1}$	β_2	$\frac{\text{Mean deviation}}{\text{Standard deviation}}$
I	Rectangular ...	0·0	1·8	866
II	Pearson Type II	0·0	2·5	·818
III	" " VII	0·0	4·1	·768
IV	" " VII	0·0	7·1	·734
V	Double Exponential	0·0	5·9	·700
VI	Pearson Type III	0·7	3·75	·789
VII	" " I	1·0	3·8	·797

Again the law of equation (1), p. 334, above has $\beta_2 = 9{\cdot}0$, while for the sampling scale $\beta_2 = 7{\cdot}05$; the difference arises owing to the practical difficulty of representing adequately for sampling purposes a curve of this form. These points are referred to again in the Appendix.

The sampling may be divided into three groups.

Series A.

Six samples of $n' = 50$* were drawn from each of the populations II, III, VI and VII. If the 50 values of the variable x are denoted by $x_1, x_2, \ldots x_{50}$, then Geary's criterion w_n may be calculated from the $n = n' - 1 = 49$ values of y obtained by the *transformation* of his equation (8). By random re-arrangements of the 50 x's, four series of y's were obtained, and hence four different values of w_n, for each of the six samples from the four populations. Samples of 50 were also obtained from a Normal population and treated in the same manner. Each series of 50 values of x gave a single value of

$$w_n' = \frac{\sum_{i=1}^{n'} |x_i - \bar{x}|}{\sqrt{n' \sum_{i=1}^{n'} (x_i - \bar{x})^2}} = \frac{\text{Mean deviation}}{\text{Standard deviation}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where $\bar{x}$ is the mean value of x in the sample. Table II contains all these values of w_n and w_n', which have also been plotted in Fig. 1. On this diagram the lower

* n' will be used for the sample size to conform with Geary's notation.

TABLE II.

Samples of 50; Values of w_n and w_n'.

Sample	Normal Population w_n	Normal Population w_n'	Population II w_n	Population II w_n'	Population III w_n	Population III w_n'	Population VI w_n	Population VI w_n'	Population VII w_n	Population VII w_n'
1 *a*	·791	·802	·856	·801	·760	·769	·822	·822	·764	·778
1 *b*	·820		·831		·755		·813		·791	
1 *c*	·813		·850		·780		·833		·782	
1 *d*	·800		·842		·761		·809		·775	
2 *a*	·799	·822	·859	·854	·791	·774	·792	·792	·832	·805
2 *b*	·780		·859		·797		·815		·817	
2 *c*	·838		·864		·762		·800		·737	
2 *d*	·834		·861		·756		·817		·830	
3 *a*	·774	·749	·810	·814	·827	·811	·830	·796	·830	·817
3 *b*	·754		·820		·799		·798		·826	
3 *c*	·759		·788		·788		·783		·807	
3 *d*	·758		·816		·818		·794		·833	
4 *a*	·817	·825	·808	·790	·786	·789	·821	·814	·800	·788
4 *b*	·827		·809		·809		·818		·791	
4 *c*	·833		·807		·778		·784		·817	
4 *d*	·830		·830		·785		·795		·774	
5 *a*	·824	·826	·782	·816	·786	·782	·789	·787	·833	·812
5 *b*	·812		·788		·767		·808		·800	
5 *c*	·796		·824		·783		·792		·812	
5 *d*	·817		·795		·787		·808		·767	
6 *a*	·801	·807	·835	·841	·758	·725	·811	·847	·828	·806
6 *b*	·800		·830		·748		·847		·823	
6 *c*	·804		·834		·720		·859		·776	
6 *d*	·827		·848		·754		·868		·800	
Mean	·8045	·805	·8272	·830	·7773	·775	·8127	·810	·8019	·801

N.B. *a*, *b*, *c* and *d* are four random arrangements of the same 50 observations.

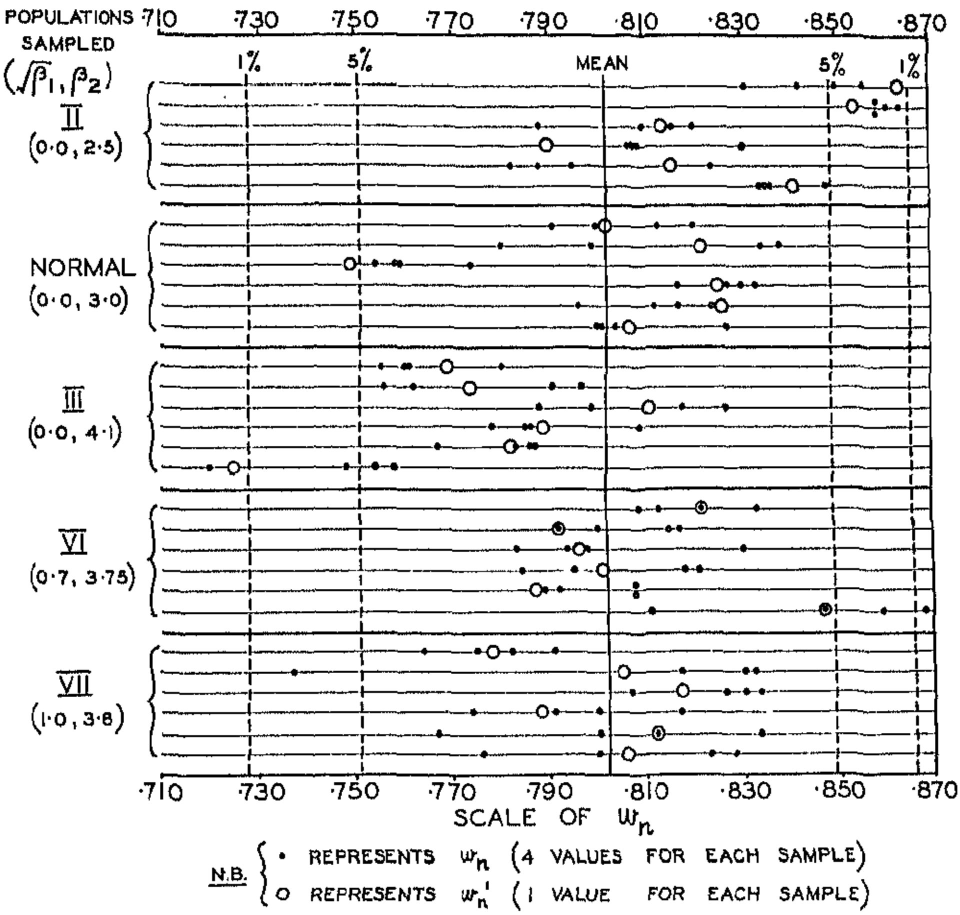
and upper 5 % and 1 % probability limits and the mean level have been drawn a follows:

Lower limits: 5 % ·751, 1 % ·728.

Upper limits: 5 % ·849, 1 % ·866.

Mean level ·802.

FIG. 1. VALUES OF w_n & w_n' IN SAMPLES OF 50.



The figures are taken from Geary's Table F. The diagram shows clearly that:

(*a*) The four values of w_n vary very considerably about w_n'.

(*b*) w_n' cannot be taken as representative of w_n; it sometimes lies outside the range of the four values.

(*c*) When the population is not Normal very few of the sample points fall beyond even the 5 °/₀ limits, so that departures from normality of this order could rarely be detected with samples as small as 50. As far as can be judged from this small experience the chance of detection appears on the whole to be no greater using w_n than w_n'.

The labour involved in the calculation of even a single value of w_n is very considerable, and since for n' as large as 50 there is every reason to believe that w_n' follows closely the same sampling distribution as w_n, it appears that this simpler criterion can be used; in the later experiments therefore w_n' alone has been calculated.

In the case of populations VI and VII the ratio of mean deviation to standard deviation was ·789 and ·797 respectively, differing only very slightly from the Normal value of $\sqrt{2/\pi} = \cdot 798$. It was therefore not to be expected that the w_n' test would be efficient in these cases. In the case of populations II and III the ratios are ·818 and ·768 respectively, but even here owing to the small size of sample, only 2 out of 6 and 1 out of 6, respectively, of the sample values of w_n' are seen to fall beyond the 5 °/₀ limits.

The values of $\sqrt{\beta_1}$ and β_2 were also calculated for each of the six samples; for these criteria of course no change occurs on rearranging the x's. The results are shown in Table III and plotted in Fig. 2, where 5 °/₀ and 1 °/₀ limits for $\sqrt{\beta_1}$ appropriate for samples from a Normal population have been inserted*. The $\sqrt{\beta_1}$ test

TABLE III.

Values of $\sqrt{\beta_1}$ and β_2 in Samples of 50.

Sample No.	Population II		Population III		Population VI		Population VII	
	$\sqrt{\beta_1}$	β_2	$\sqrt{\beta_1}$	β_2	$\sqrt{\beta_1}$	β_2	$\sqrt{\beta_1}$	β_2
1	− ·220	1·962	·484	3·753	·124	2·513	1·086	3·666
2	·201	2·122	·718	4·181	·903	3·289	1·131	3·627
3	·215	2·309	− ·511	2·517	·856	2·984	·074	2·915
4	·203	2·282	− ·204	2·931	·175	2·580	·906	2·931
5	·282	2·332	·434	3·054	·743	3·203	1·022	3·601
6	·161	2·308	·900	5·645	·409	2·301	·994	3·427
Mean	+ ·140	2·219	+ ·288	3·680	+ ·535	2·812	+ ·969	3·362

is reasonably efficient in detecting the skewness of populations VI and VII; for the other two cases (II and III) where the β_2 test would be appropriate, no limits are available. Since however for $n' = 50$, $\sigma_{\beta_2} = \cdot 60$ approximately, it is doubtful whether significant departures from normality could be established in the case of any single sample even if the true sampling distribution were known. It is of interest to note that in all but 2 of the 24 samples, β_2 is less than the corresponding population value. This bias which appears also in the other experiments will be referred to again below.

Series B.

A second experiment consisted in drawing six samples of 300 from each of the two symmetrical Non-normal populations II and III. In this case the 5 °/₀ and 1 °/₀ limits for β_2 previously calculated are probably fairly accurate, so that a satisfactory comparison of the β_2 and w_n' tests is possible. The results are shown in Table IV, p. 340, and Fig. 3.

* These are obtained from my table already referred to, *Biometrika*, Vol. xxii. (1930—1931), p. 248.

FIG. 2. VALUES IN SAMPLES OF 50 OF:-

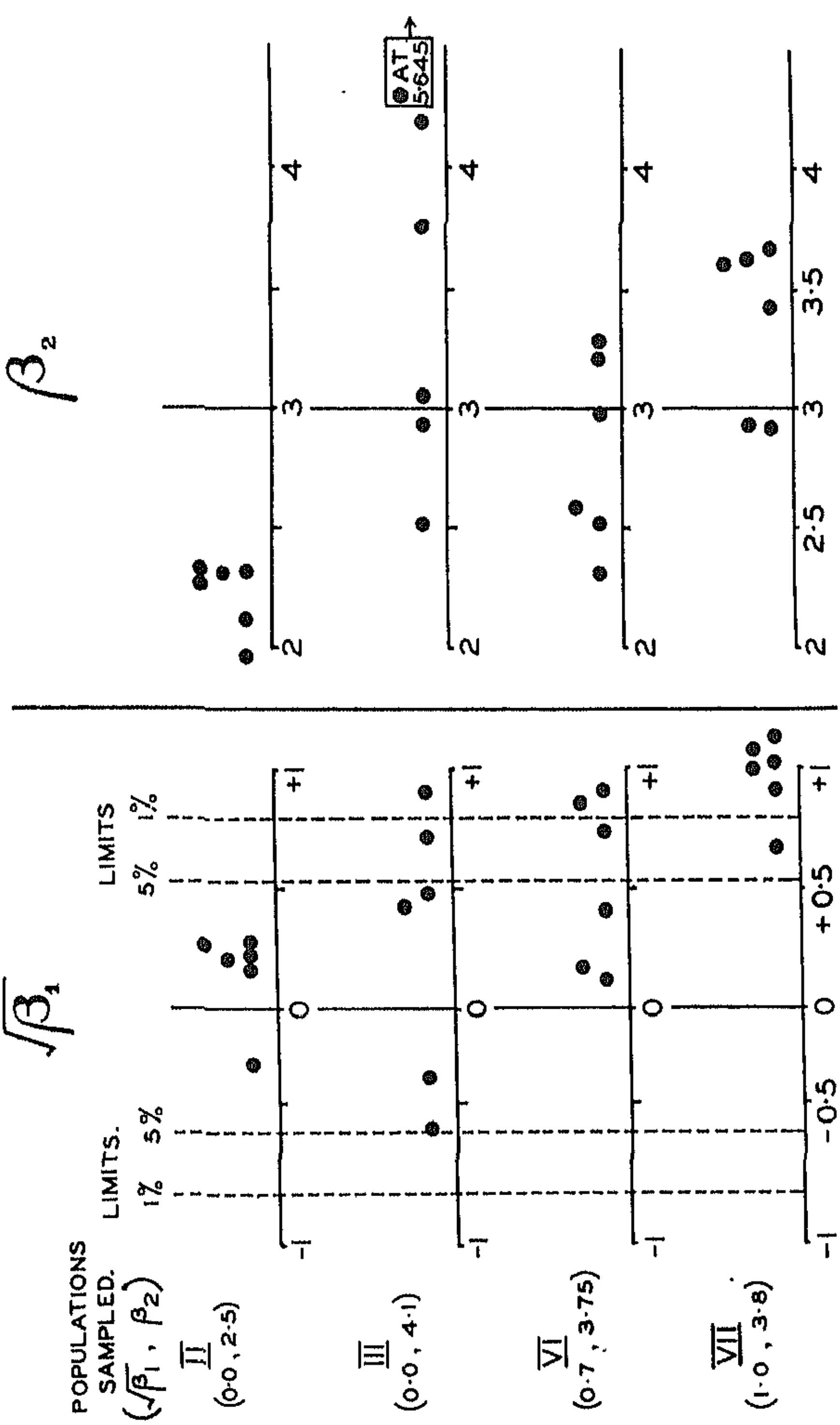
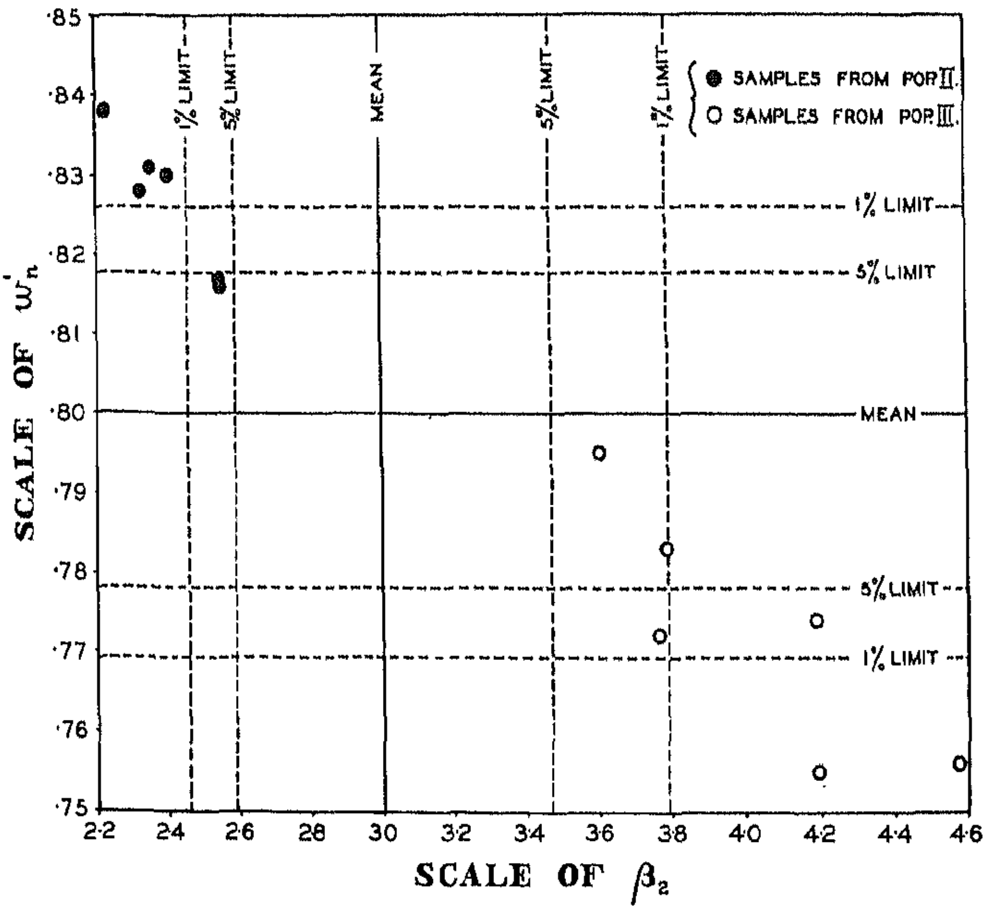


TABLE IV.

Samples of 300; Values of Criteria.

Sample No.	Population II (0·0, 2·5)			Population III (0·0, 4·1)		
	$\sqrt{\beta_1}$	β_2	w_n'	$\sqrt{\beta_1}$	β_2	w_n'
1	− ·021	2·209	·838	+ ·490	4·189	·774
2	− ·006	2·328	·828	− ·300	3·788	·783
3	+ ·074	2·548	·810	− ·020	4·196	·755
4	− ·067	2·352	·831	− ·108	3·762	·772
5	+ ·013	2·401	·830	+ ·249	3·602	·795
6	− ·022	2·545	·817	− ·456	4·573	·756
Mean	− ·005	2·413	·827	− ·023	4·018	·772

FIG. 3. COMPARISON OF β_2 & w_n' IN SAMPLES OF 300.



The correlation between the two criteria is extremely high*, so that a decision based on a knowledge of w_n' is not likely to differ much from that based on β_2 and vice versa. There is certainly a suggestion as indicated in the following table that β_2 is somewhat more efficient in detecting lack of normality than w_n', but with only six samples available the evidence is far from conclusive. The mean values of β_2 are slightly less and the mean values of w_n' slightly more than the population values given in Table I, but the bias is far less than in the samples of 50 (Series A) and 76 (Series C).

TABLE V.

Samples of 300; *Comparison of Criteria.*

	Population II		Population III	
	Using β_2	Using w_n'	Using β_2	Using w_n'
Number of sample points:				
Within 5 °/。 limits	0	2	0	2
Between 5 °/。 and 1 °/。 limits	2	0	3	2
Beyond 1 °/。 limits	4	4	3	2

$\sqrt{\beta_1}$ varies of course about zero, but the standard error increases as the population becomes more leptokurtic† and the values for the 1st and 6th samples for population III (+ ·490 and − ·456) fall far outside the 1°/。 limits (± ·329) for a Normal population. This does not however mean that the $\sqrt{\beta_1}$ test should be used in this case, since the modal value of $\sqrt{\beta_1}$ is zero.

Series C.

In this case ten samples of 76 were drawn from each of the populations I, IV and V; the populations differ more from the Normal than in the preceding case but the samples are considerably smaller. Values of $\sqrt{\beta_1}$, β_2 and w_n' are shown in Table VI, p. 343, and the last two are plotted in the form of a correlation diagram in Fig. 4. In this case no reliable 5°/。 and 1°/。 limits for β_2 are available. As a rough indication of their position, however, the procedure adopted in my earlier paper has been followed, i.e. the first four moment coefficients appropriate for $n' = 76$ have been calculated‡ and the probability levels obtained by quadrature of the Pearson Type IV curve having these moments. The results were as follows:

Lower limits: 1°/。 at 2·14, 5°/。 at 2·27.

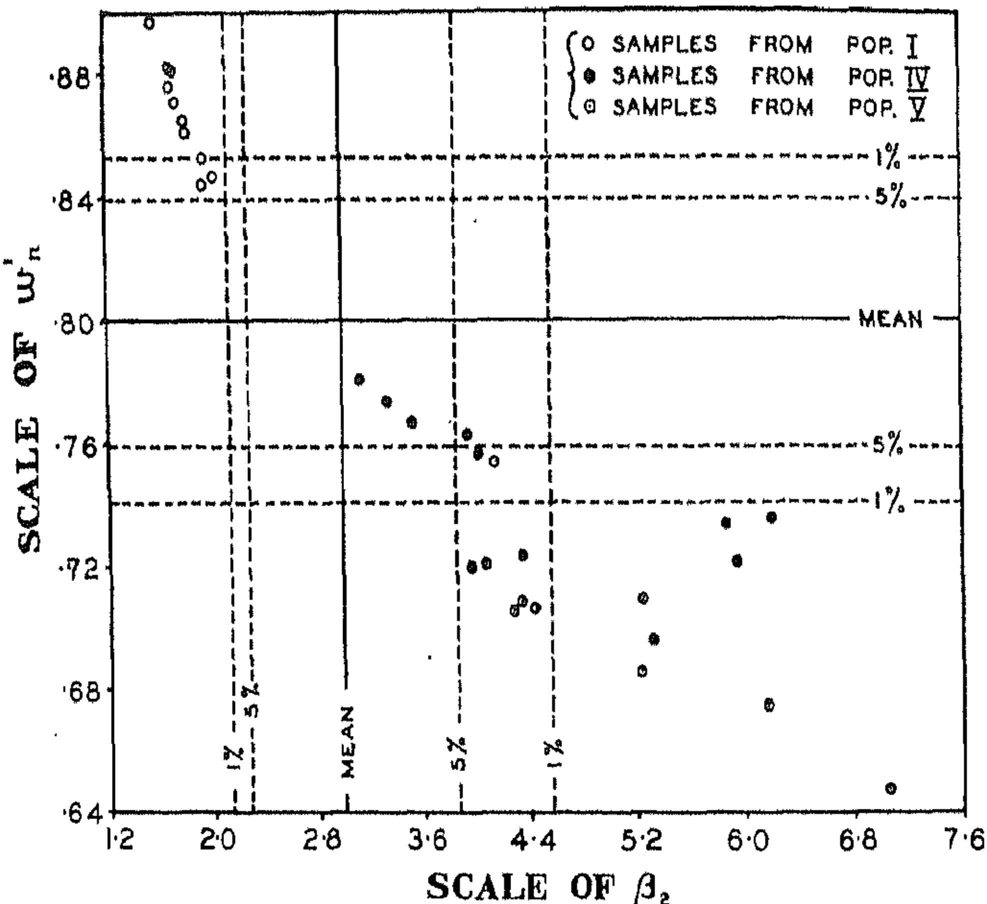
Upper limits: 5°/。 at 3·86, 1°/。 at 4·57.

* See Mr Geary's Section III below.

† First approximation values of the standard errors of β_1 and β_2 for samples from different Pearson type curves may be obtained from *Tables for Statisticians and Biometricians*, Part I. Tables XXXVII and XXXVIII.

‡ The formulae used are (5) to (8) of my paper in *Biometrika*, Vol. XXIII. (1931), p. 428, which give mean $\beta_2 = 2\cdot922$, $\sigma_{\beta_2} = 0\cdot510$, $B_1(\beta_2) = 1\cdot971$, $B_2(\beta_2) = 7\cdot460$. The calculations were kindly undertaken by Mr K. Williams.

FIG. 4. COMPARISON OF β_2 & w_n' IN SAMPLES OF 76.



Turning to the diagram it is seen that there is again a high correlation between the two criteria β_2 and w_n', which is less intense for the leptokurtic populations than for the rectangular. Counts of the points falling beyond different limits leads to Table VII on the opposite page, analogous to Table V. In view of the uncertainty of the position of the β_2 limits, it would not be justifiable to make any critical comparison of relative efficiency of the two tests in this case, but there is certainly no evidence that the w_n' test is less efficient than the β_2 test.

A noticeable feature of Table VI is the bias both in β_2 and w_n' for samples from leptokurtic populations. For the case of normal populations this bias can be corrected by the use of multiplying factors $\frac{n'+1}{n'-1}$ (exactly for β_2) and $e^{-\frac{1}{4(n'-1)}}$ (approximately for w_n'), but these factors will clearly differ for other populations. It will be seen that for Population IV the mean of the 10 sample values of β_2

TABLE VI.

Samples of 76; Values of Criteria.

Sample No.	Population I (0·0, 1·8)			Population IV (0·0, 7·1)			Population V (0·0, 5·9)		
	$\sqrt{\beta_1}$	β_2	w_n'	$\sqrt{\beta_1}$	β_2	w_n'	$\sqrt{\beta_1}$	β_2	w_n'
1	+·090	1·821	·865	+·190	4·038	·757	−·452	5·250	·710
2	+·315	1·839	·861	−1·425	6·200	·736	+·443	6·189	·675
3	+·282	1·709	·876	−·797	5·957	·722	+·822	3·509	·767
4	+·030	1·582	·897	−·755	3·937	·763	+·454	4·441	·707
5	−·232	1·946	·844	+·575	5·874	·734	+·292	5·240	·686
6	+·198	1·709	·882	+·663	3·143	·781	−·169	4·285	·705
7	−·232	1·933	·853	+·570	3·334	·774	−·680	4·093	·721
8	+·315	1·735	·881	+·033	5·323	·697	−·219	4·336	·709
9	+·226	2·044	·847	−·688	4·350	·724	+·106	4·148	·754
10	−·032	1·772	·871	−·442	3·980	·720	−·390	7·149	·648
Mean	+·097	1·809	·868	−·208	4·614	·741	+·021	4·864	·708

TABLE VII.

Samples of 76; Comparison of Criteria.

	Population I		Population IV		Population V	
	Using β_2	Using w_n'	Using β_2	Using w_n'	Using β_2	Using w_n'
Number of sample points:						
Within 5 °/. limits	0	0	2	3	1	1
Between 5 °/. and 1 °/. limits	0	2	4	1	5	1
Beyond 1 °/. limits	10	8	4	6	4	8

TABLE VIII.

Samples of 76; Comparison of Criteria.

Estimate of standard error of:	Population I	Population IV	Population V	Exact values for Normal population
w_n'	·017	·027	·035	·024
β_2	·137	1·129	1·106	·510
$\sqrt{\beta_1}$	·221	·710	·455	·270

(4·61) falls short of the population value (7·1) by over 33 °/ₒ*. The bias in w_n', here in the opposite direction, is relatively of much less importance. In the same way, while it will be seen that the standard error of w_n' increases as the population becomes more leptokurtic, this increase is very definitely less than that in the standard error of β_2. Let the reader examine, for example, the sets of 10 samples in the Tables on p. 343. The results given there suggest that in samples of moderate size w_n' provides a better estimate than β_2 of its corresponding population parameter. The problems of estimation and of testing the hypothesis of normality are not however identical, and much more work is necessary before a final comparison of the merits of these two criteria can be made.

3. *Conclusions.*

(1) In testing for departure from normality it would no doubt be desirable to have a single test sensitive both as regards skewness and kurtosis. As this is not available it is necessary to apply two separate tests.

(2) To detect skewness the best criterion would appear to be $\sqrt{\beta_1}$, for which tables of 5 °/ₒ and 1 °/ₒ probability levels, believed to be sufficiently accurate for all ordinary purposes are available†.

(3) As a test of whether the population sampled is platykurtic or leptokurtic, either β_2, w_n or w_n' may be used. Out of these there are strong practical grounds for choosing w_n', the ratio of mean deviation to standard deviation, unless the sample is very large when β_2 may be used. The reasons for this are as follows:

(*a*) Until further investigation has been carried out, the accuracy of the tables of 5 °/ₒ and 1 °/ₒ probability levels for β_2 cannot be established.

(*b*) While the sampling distribution of w_n is known and accurate tables have been given by Mr Geary, the calculation of this criterion is very troublesome except in the case where the population mean is given.

(*c*) There are strong grounds for believing that if $n' > 40$, w_n' may be referred to the w_n tables of 5 °/ₒ and 1 °/ₒ probability levels.

(*d*) The calculation of w_n' is straightforward and shorter than that of β_2; if the data are grouped an approximate adjustment to obtain the mean deviation may be used‡.

* This bias may be explained as follows. The high value of β_2 for the distribution law of equation (1), is due to the very long drawn-out tails. The chance however of obtaining an observation falling in these tails in samples even as large as 70 is small, so that the sample β_2's have a very skew sampling distribution with mode well below the population value. The standard error is however infinite.

† *Tables for Statisticians and Biometricians*, Part II. Table XXXVII *bis* for $n' = 50, 75, 100 \ldots$ etc. and *Biometrika*, Vol. XXVII. (1934), pp. 269—271, for $n' = 20, 30, \ldots 90, 100$.

‡ If the centre of the group containing the mean, $\bar{x}$, of the sample be taken as the origin of x, if n_t observations fall into the tth group having central value x_t, if n_0 observations fall into the group containing the mean and if the mean is at a distance θ from the origin $(-\frac{1}{2} \leqslant \theta \leqslant \frac{1}{2})$ where the unit is the group breadth, then we may take as an approximation

$$\Sigma n_t |x_t - \bar{x}| = \Sigma n_t |x_t| + \{\theta \times \text{number of observations in groups below the origin group}\}$$
$$- \{\theta \times \text{number of observations in groups above the origin group}\} + n_0 \left(\tfrac{1}{4} + \theta^2\right).$$

(*e*) *There is clear experimental evidence that in the case of leptokurtic populations the sample* β_2 *provides a biased estimate of the population value. This bias is far smaller in the case of* w_n'.

(4) *A high negative correlation exists between the sample values of* β_2 *and* w_n'*. *This means that in testing the hypothesis of normality, in any given case a conclusion drawn from the value of one criterion is unlikely to differ much from that which would be drawn from a knowledge of the other. If the uncertainty regarding the probability levels of* β_2 *could be removed, it is likely that for large samples this criterion would be found somewhat more efficient than* w_n' *in detecting departure from normality. This point cannot however be regarded at present as established.*

* See Section III added by Mr Geary giving the value for this correlation when the population sampled is Normal.

[NOTE. *The object of publishing the following Appendix containing* 72 *random samples of various sizes from diverse forms of curves is to put into the hands of those desiring to test other, old or new, criteria ready-made samples and save them from the heavy labour of obtaining afresh their own samples.* ED.]

APPENDIX.

FULL DATA CONCERNING POPULATIONS AND SAMPLES USED IN A PAPER BY E. S. PEARSON ENTITLED: "A COMPARISON OF β_2 AND MR GEARY'S w_n CRITERIA."

For the purpose of future reference and comparison, it has appeared useful to publish in this appendix full details of the populations sampled and the method of sampling, and also to give the complete data of the samples themselves.

In sampling with Tippett's Random Numbers (*Tracts for Computers*, No. XV) the method of procedure described on pp. iv—v of the introduction to the Tract has been employed. The frequency distributions of the 10,000 numbers representing each population are shown in Table X, except in the case of Population I (Rectangular); here the sampling was arranged so that there was an equal chance (i.e. $\frac{1}{33}$) of the variable, x, assuming values $-16, \ldots -1, 0, +1, \ldots +16$. This was effected by assigning the numbers 01—03 to $x=-16$, 04—06 to $x=-15$, ... 97—99 to $x=+16$, omitting the number 00 from consideration.

In apportioning the 10,000 numbers it is inevitable that difficulty should arise in the case of very leptokurtic distributions. This difficulty was discussed in a note contributed to Dr le Roux's paper, to which the reader is referred*. It means that the actual populations sampled cannot correspond exactly to those defined by the theoretical curves, but from the point of view of the present investigation this is of no consequence, since the objective has been to investigate the efficiency of various criteria in detecting certain wide departures from normality.

The frequency constants shown in Table IX have been calculated from the actual distributions sampled, i.e. from those given in Table X, without applying

TABLE IX.

Frequency Constants of Sampled Populations.

	I	II	Normal	III	IV	V	VI	VII
Mean	0·000	0·000	0·000	0·000	0·000	0·000	0·142	0·147
Standard Deviation	9·522	10·544	10·004	5·675	6·455	3·546	5·008	12·370
Mean Deviation	8·250	8·629	7·986	4·378	4·757	2·483	3·961	9·860
$\sqrt{\beta_1}$	0·000	0·000	0·000	0·000	0·000	0·000	0·701	0·995
β_2	1·798	2·500	3·000	4·116	7·052	5·884	3·725	3·834

* *Biometrika*, Vol. XXIII. pp. 158—159.

any corrections for grouping or abruptness except in the case of the mean deviation where the expression given in the footnote ‡ on p. 344 above was used. The unit employed in the case of the mean, standard deviation and mean deviation is the unit of x shown on the x-scales of Table X. It will be seen that this varies from about $\frac{2}{7}$ to $\frac{1}{12}$ of the standard deviation; it would, no doubt, have been desirable to arrange a finer grouping unit in the case of Populations III—VI, but the data were collected from sampling experiments initiated at different times and with different objectives.

The sampling results are shown in Tables XI—XIII; for Experiment A, the 50 individual values of x are recorded for each sample. These values are tabled in a completely random order; in fact the order is that used in calculating the first of the 4 sets of values of w_n given in Table II. The rearrangements needed to obtain the other 3 values of w_n were carried out by writing the 50 x-values on small counters, thoroughly mixing and drawing out randomly in order.

For Experiments B and C the tables show the frequency of occurrence in each sample of the different values of x. Thus in Table XII the columns each total 300 and in Table XIII they total 76.

The sample frequency constants tabled in the preceding paper have been calculated directly from these figures without further grouping. No Sheppard's Corrections have been applied, either to the second or fourth moments; the mean deviation was calculated from the expression given on p. 344 above*.

* Further reflection suggests that if Sheppard's Corrections are not used, the correction to the mean deviation to adjust the contribution from the central group is not really justifiable. The consistent course would, no doubt, have been to suppose that x could assume certain discrete values only; the resulting alterations to the values of the mean deviation would, however, be too small to be of practical significance.

TABLE X.

Grouped Frequencies in Populations Sampled.

x	Population frequencies for symmetrical distributions					Population frequencies for skew distributions					
						VI, $\sqrt{\beta_1}=0·7$, $\beta_2=3·75$		VII, $\sqrt{\beta_1}=1·0$, $\beta_2=3·8$			
	II $\beta_2=2·5$	Normal	III $\beta_2=4·1$	IV $\beta_2=7·1$	V $\beta_2=5·0$	x	Frequency	x	Frequency	x	Frequency
0	348	398	774	758	1812	−12	1	−17	141	+23	62
± 1	348	397	758	749	1350	−11	9	−16	245	+24	57
± 2	345	391	710	690	905	−10	37	−15	298	+25	52
± 3	340	381	637	615	606	− 9	98	−14	332	+26	47
± 4	332	368	550	529	406	− 8	187	−13	353	+27	43
± 5	322	352	459	449	273	− 7	327	−12	367	+28	39
± 6	311	334	369	355	183	− 6	472	−11	375	+29	35
± 7	298	312	291	282	122	− 5	611	−10	377	+30	32
± 8	284	289	222	221	82	− 4	726	− 9	376	+31	28
± 9	268	266	168	170	55	− 3	802	− 8	373	+32	26
±10	251	242	123	130	37	− 2	839	− 7	366	+33	23
±11	234	218	91	100	25	− 1	834	− 6	358	+34	21
±12	214	195	69	77	16	0	796	− 5	349	+35	18
±13	196	171	47	58	11	+ 1	734	− 4	339	+36	16
±14	177	150	34	45	8	+ 2	651	− 3	327	+37	15
±15	159	129	24	35	5	+ 3	569	− 2	316	+38	13
±16	139	111	18	27	3	+ 4	482	− 1	303	+39	11
±17	122	94	12	21	2	+ 5	400	0	290	+40	9
±18	104	79	9	16	2	+ 6	325	+ 1	277	+41	9
±19	88	66	7	13	1	+ 7	260	+ 2	264	+42	8
±20	74	54	5	10	1	+ 8	205	+ 3	252	+43	6
±21	60	44	3	8	—	+ 9	159	+ 4	239	+44	6
±22	46	36	3	7	—	+10	121	+ 5	226	+45	5
±23	37	28	1	5	1	+11	92	+ 6	213	+46	4
±24	27	23	2	5	—	+12	69	+ 7	202	+47	3
±25	19	17	1	3	—	+13	51	+ 8	190	+48	3
±26	14	14	1	3	—	+14	37	+ 9	178	+49	3
±27	8	10	—	2	—	+15	27	+10	168	+50	2
±28	5	8	1	2	—	+16	20	+11	157	+51	2
±29	3	6	—	2	—	+17	14	+12	146	+52	1
±30	1	5	—	2	—	+18	9	+13	137	+53	1
±31	—	3	—	1	—	+19	7	+14	128	+54	1
±32	—	2	—	1	—	+20	5	+15	119	+55	1
±33	—	2	—	—	—	+21	4	+16	110	+56	1
±34	—	1	1	1	—	+22	2	+17	102	+57	—
±35	—	1	—	1	—	+23	2	+18	94	+58	1
±36	—	1	—	—	—	+24	1	+19	87	+59	—
±37	—	—	—	1	—	+25	—	+20	80	+60	—
±38	—	—	—	1 at ±40	—	+26	1	+21	74	+61	—
±39	—	1	—	1 at ±47	—	+27	—	+22	68	+62	1
±40	—	—	—	1 at ±58	—	+28	1	continued above			
Totals	10,000	10,000	10,000	10,000	10,000	Total	10,000			Total	10,000

N.B. For Population I (Rectangular) see special reference in text.

TABLE XI (*continued*).

Population III			Population VI						Population VII						
+ 5	− 1	+ 1	− 6	− 1	− 4	+ 4	+ 1	− 2	0	−12	0	− 5	+27	− 8
0	+ 4	+23	− 4	+ 3	+10	+ 9	0	− 2	0	− 5	−12	11	+ 1	+35
+ 1	− 3	− 2	− 6	0	− 5	− 6	− 5	+ 7	−17	+21	− 6	− 3	− 8	+ 5
−12	+ 1	− 7	+ 5	− 3	+ 7	− 2	− 2	+ 1	− 9	+11	+20	+ 4	+10	− 3
+ 4	+ 6	+ 3	− 6	+ 3	+ 9	+ 2	0	− 2	+11	15	−17	− 9	− 7	−11
+ 1	− 6	− 1	− 2	− 6	− 3	+11	− 6	− 1	−14	− 5	+ 9	− 8	+ 9	− 3
− 5	+ 1	− 2	+ 3	− 7	− 5	− 2	+ 2	− 2	−11	− 4	12	+ 1	+ 2	−15
− 5	0	0	−11	+ 1	+11	0	+ 5	+ 1	+11	+36	+ 4	−10	+17	+ 9
+ 5	− 3	+ 3	+ 3	− 6	+ 3	− 6	− 7	+ 2	− 2	−15	4	+20	+ 7	+33
− 3	+ 1	+ 4	+ 1	+10	+15	− 3	− 5	0	− 8	+31	+ 3	−12	− 8	+ 4
− 4	+ 7	+11	− 2	− 5	+ 3	− 1	− 1	0	+19	+19	+19	+ 6	−11	− 9
+ 4	− 1	− 2	+ 7	− 1	+ 8	− 1	− 8	− 3	+ 2	− 4	− 4	− 9	− 1	+ 4
+ 3	+14	+ 4	− 1	− 3	− 2	− 4	0	− 5	+15	15	−10	0	− 6	+ 9
+ 1	− 7	+ 7	−10	− 8	− 3	0	0	+10	−13	−12	−16	− 1	+ 9	−10
− 3	+ 4	+ 8	− 2	− 3	− 7	− 2	+14	− 1	−13	+ 7	−10	+18	+12	− 6
+ 7	− 1	− 4	− 3	+12	− 5	− 4	+ 3	− 4	− 5	+ 5	−13	+ 5	− 4	+ 1
− 1	− 4	− 7	− 8	− 1	− 2	+ 3	0	+11	−12	+22	− 9	− 1	5	+ 2
+ 1	− 1	− 5	− 1	− 4	0	− 1	+ 7	0	0	−11	− 9	− 4	− 7	−12
0	− 4	+ 2	+ 1	− 4	− 4	+ 3	− 2	− 4	−13	− 6	−10	+ 2	+ 6	− 9
− 5	− 4	+ 4	+ 1	− 5	− 2	+ 4	− 6	− 4	−11	+12	−16	− 7	+ 6	+ 9
0	− 9	+ 2	+ 3	0	− 5	+ 6	− 6	+ 7	−15	− 9	−15	+ 3	+19	+16
+ 2	+ 5	+ 2	− 5	− 1	− 2	+ 3	− 5	+ 7	− 1	−10	− 1	− 9	0	−12
− 2	+ 5	+ 1	+ 2	+10	− 3	+ 6	− 1	+ 7	−10	− 9	0	0	− 6	+11
+ 7	+ 2	+ 4	+ 3	+ 3	− 6	0	− 4	− 9	−14	− 4	−11	− 8	− 3	+ 6
+ 1	− 2	+ 1	+ 1	− 4	− 6	0	+ 4	0	0	− 2	+19	− 3	−11	− 2
−11	+ 1	− 8	+ 4	+ 3	0	+ 4	+ 3	+13	+ 4	− 2	− 2	+10	+ 1	+30
+ 3	− 7	+ 6	+ 3	+ 3	+ 2	+ 3	− 3	+ 5	−10	−12	+ 5	+ 1	+25	− 8
− 3	0	− 6	+ 8	− 5	− 2	− 6	+ 2	− 2	− 3	+10	+ 3	+ 5	−11	+20
− 5	− 1	+ 1	+ 6	− 4	− 4	+10	+11	+ 8	−11	− 9	+ 5	+18	− 9	−16
− 1	− 9	+ 6	− 1	− 4	− 4	− 8	+ 1	− 3	−15	− 7	− 1	+10	+ 7	− 8
+ 5	− 4	− 1	+ 9	0	− 5	+10	+ 6	− 5	− 9	− 6	− 3	0	− 8	−14
+ 4	+12	+13	0	+ 2	0	− 2	0	− 1	− 2	− 6	+ 3	+ 4	−13	−16
+ 9	+ 2	+ 4	− 5	− 6	+ 6	+ 2	+ 2	− 3	+14	+ 9	+ 3	− 8	+ 9	−15
+ 2	+ 8	− 3	+ 1	− 3	− 2	− 2	− 1	+ 2	− 7	+10	+27	+27	−15	− 1
− 3	+ 4	0	+ 2	− 5	+ 2	+ 6	− 1	+ 4	− 4	−13	− 6	− 7	+19	−17
− 1	+ 1	+ 1	− 7	0	+ 2	+ 4	+13	− 7	− 7	−11	−17	− 4	+25	+ 7
+ 3	+ 7	− 1	+ 5	− 7	+ 4	− 1	+ 4	+11	+ 9	0	+10	− 6	− 7	+ 3
− 2	+ 3	+ 1	− 6	+ 2	0	+ 3	− 6	+ 6	+ 1	− 6	+ 9	− 4	+ 3	− 6
0	0	− 2	0	+ 5	+ 1	− 7	− 6	− 3	− 4	− 8	+ 6	+18	+30	+ 6
− 3	+ 7	− 2	− 3	+13	− 8	+ 3	+ 5	+15	− 2	− 3	− 3	− 3	− 5	−13
− 3	+15	−12	+ 8	− 7	− 7	+ 3	+ 4	− 5	+ 5	−11	−17	− 6	− 3	− 6
+ 6	+ 8	− 2	− 4	− 3	+ 1	− 1	+ 2	− 5	− 8	− 7	− 6	− 7	+ 3	−13
+ 3	+ 1	+10	+ 7	− 8	+11	− 8	− 5	+ 1	0	− 1	+ 6	−12	−10	−12
−11	+ 2	+ 5	+12	0	− 2	0	+ 3	+ 5	+20	−13	+ 8	−12	−12	+ 4
0	− 3	+ 4	− 1	+ 3	+ 3	− 9	0	− 9	−11	+ 3	−13	−14	− 7	−14
− 6	− 5	+ 4	− 3	− 3	− 6	+ 1	− 3	+ 8	+29	+14	−16	+16	− 9	− 5
−10	0	− 7	+ 5	0	− 6	− 3	− 3	+10	+ 6	−16	−16	−16	−11	+ 5
− 2	+ 6	+ 2	− 3	− 6	+ 9	− 6	+ 9	+ 7	+29	− 8	− 4	− 9	+11	+14
− 6	+ 6	− 2	− 3	+ 4	0	− 4	− 6	+ 5	− 8	+17	−16	−11	+ 7	− 7
+10	− 3	− 3	− 7	+10	0	+ 5	− 4	− 4	− 6	+ 6	−10	+20	− 8	+30

TABLE XII.

Experiment B. Frequency of Different Values of x in Samples of 300.

x	6 Samples from Population II						6 Samples from Population III						x
−28	—	—	—	—	—	—	—	—	—	—	—	—	−28
−27	—	—	1	1	1	—	—	—	—	—	—	—	−27
−26	2	—	1	1	—	—	—	—	—	—	—	1	−26
−25	2	1	—	—	—	—	—	—	—	—	—	—	−25
−24	2	—	—	—	2	—	—	—	—	—	—	—	−24
−23	—	3	—	1	1	—	—	—	—	—	—	—	−23
−22	1	1	1	2	1	2	—	1	—	—	—	—	−22
−21	3	2	2	2	4	1	—	—	—	—	—	—	−21
−20	1	3	2	3	4	2	—	—	—	—	—	1	−20
−19	2	6	2	6	—	1	—	—	1	1	—	1	−19
−18	1	3	1	4	3	6	—	—	—	1	—	1	−18
−17	3	2	4	4	4	4	1	—	—	1	—	—	−17
−16	10	2	3	3	9	3	1	1	—	—	—	—	−16
−15	7	5	6	4	6	3	—	1	—	—	1	2	−15
−14	8	5	4	5	3	5	—	1	3	2	1	1	−14
−13	7	12	5	6	8	5	—	—	—	1	—	—	−13
−12	3	3	5	5	8	5	2	3	2	3	—	5	−12
−11	7	7	5	12	9	7	—	1	4	—	—	4	−11
−10	11	6	9	4	6	4	8	3	3	6	2	2	−10
− 9	10	5	11	7	9	4	12	6	1	7	7	7	− 9
− 8	6	10	7	7	12	12	4	7	—	7	4	8	− 8
− 7	9	7	7	13	11	5	8	12	11	8	11	4	− 7
− 6	7	11	14	9	13	10	14	9	8	14	11	12	− 6
− 5	11	5	9	7	11	12	10	10	13	12	9	14	− 5
− 4	9	8	13	8	7	12	19	15	14	10	19	16	− 4
− 3	7	17	10	12	8	15	17	18	17	14	19	15	− 3
− 2	6	7	13	7	10	13	15	19	27	14	24	15	− 2
− 1	9	10	11	6	8	7	29	23	23	24	24	17	− 1
0	9	11	6	11	10	10	18	28	21	28	21	29	0
+ 1	8	15	14	7	15	8	16	23	30	28	21	30	+ 1
+ 2	6	4	8	9	6	8	25	16	31	15	17	18	+ 2
+ 3	12	7	8	12	8	18	16	16	24	15	32	19	+ 3
+ 4	9	8	12	11	11	14	18	22	10	22	15	16	+ 4
+ 5	8	8	11	13	15	11	17	18	20	23	14	19	+ 5
+ 6	8	14	10	9	8	10	14	16	7	13	13	14	+ 6
+ 7	11	2	8	6	10	8	10	8	5	7	11	8	+ 7
+ 8	6	6	10	13	5	8	3	11	8	8	8	4	+ 8
+ 9	10	11	7	7	5	8	2	3	2	6	6	4	+ 9
+10	12	9	7	5	8	10	3	1	5	3	2	7	+10
+11	10	8	2	6	12	6	6	5	4	1	2	1	+11
+12	6	8	6	9	5	4	5	2	1	1	2	1	+12
+13	3	6	4	6	1	8	1	—	—	2	—	—	+13
+14	6	6	6	5	3	4	—	—	3	—	1	1	+14
+15	3	7	6	4	4	6	—	—	—	1	—	—	+15
+16	3	7	4	8	5	7	—	—	1	—	1	1	+16
+17	7	3	6	7	3	3	2	—	—	—	1	—	+17
+18	4	2	6	2	2	4	—	—	—	1	—	1	+18
+19	6	1	4	2	1	1	2	1	1	—	—	1	+19
+20	1	2	2	3	2	1	—	—	—	1	1	—	+20
+21	2	1	1	3	1	3	—	—	—	—	—	—	+21
+22	3	2	1	—	1	—	—	—	—	—	—	—	+22
+23	1	—	2	1	1	—	1	—	—	—	—	—	+23
+24	—	—	1	—	—	—	1	—	—	—	—	—	+24
+25	—	1	1	—	—	—	—	—	—	—	—	—	+25
+26	1	1	1	1	—	1	—	—	—	—	—	—	+26
+27	—	—	—	1	—	1	—	—	—	—	—	—	+27
+28	1	—	—	—	—	—	—	—	—	—	—	—	+28

TABLE XIII.

Experiment C. Frequency of Different Values of x in Samples of 76.

x	10 Samples from Population I										10 Samples from Population IV										10 Samples from Population V										x
−30																															−30
−28													1																		−28
* *																															* *
−25																															−25
−24												2																			−24
−23														1				2													−23
−22																															−22
−21																															−21
−20													1																		−20
−19																															−19
−18												1		1	1					1											−18
−17														1																	−17
−16	2	1			1	3	2	2	3	2	1								1												−16
−15	2		2	1	2	3		1		2					1			1	1	1											−15
−14	2	4	3	2	4	3	2	4	1	1	1		1	1						1										1	−14
−13	4	1	5	4	–	2	5	5	2		1				1			1	1	1	1				1						−13
−12	2	3	5	3	1	2	3	6	4	3		1		1			2		1					1	1		1				−12
−11	3	4	2	3	3	4	4	2	5	2		1	1	1					1			1					—	1			−11
−10	3	1	3	2	2	1	2	4	2	3	1			2	3		1	2									2				−10
− 9	1	7	3	3	2	3	1	—	3	3	1	2	2	2		1	1	1		2		1					1	1	1		− 9
− 8	4	2	4	4	2	5	—	1	1	5	1	1	2	—	1	2	2	2	2	1						1	2	1	—	1	− 8
− 7	3	4	1	2	2	5	—	4	2	1	1	4	4	1		1	3	2	1	2	2	3		1	1	1	1			1	− 7
− 6	1	—	4	2	–	1	2	4	4	3	3	3	1	2	2	3	1	–			1	1	–	2		2	1			1	− 6
− 5	2	3	3	1	2	2	2	1	4	2	5	3	3	5	1	4	—	7		2	2		2	3	3	1	5	2	5	1	− 5
− 4	1	3	1	1	4	2	2	2	1	2	3	8	3	1	6	2	6	1	6	2		1	2	3	6	1	2	3	2	5	− 4
− 3	5	6	2	4	1	3	4	3	3	2	4	3	3	7	6	8	2	6	3	2	6	2	6		3	5	1	4	4	7	− 3
− 2	2	2	2	4	—	—	3	2	3	2	3	2	7	2	2	4	6	5	6	6	5	7	6	7	5	6	4	3	6	9	− 2
− 1	1	2	2	1	1	3	1	—	3	2	8	10	2	4	7	4	4	4	3	7	7	8	12	11	8	12	6	10	12	9	− 1
0	3	4	2	1	3	1	3	4	3	4	3	6	3	7	5	7	9	6	11	9	10	20	13	10	13	20	21	15	12	14	0
+ 1	3	2	1	2	5	2	2	—	2	1	5	4	9	3	4	7	1	10	5	8	13	9	12	12	10	9	12	15	11	13	+ 1
+ 2	2	—	2	3	1	2	5	1	1	4	6	6	9	6	2	8	4	3	14	8	6	4	3	8	10	9	5	4	9	6	+ 2
+ 3	2	—	5	2	3	1	2	3	4	2	6	5	6	8	7	3	8	3	4	3	5	6	5	4	4	1	4	3	4	3	+ 3
+ 4	3	1	—	1	5	3	5	1	3	1	7	5	1	2	4	1	3	4	4	1	3	6	6	3	2	5	3	4	5	2	+ 4
+ 5	1	2	1	—	4	2	5	1	1	3	2	3	2	6	4	3	1	4	3	1	3	1	2	2	2	1	3	4	2	1	+ 5
+ 6	5	3	1	—	2	—	1	2	4	1	3	2	4	2	6	2	5	2	2	5	3	2	2	2	2	1	1	2	1		+ 6
+ 7	2	3	1	5	2	2	7	4	2	2	3	3	2	2	4	5	3	2	1	3	1	1		1					1	1	+ 7
+ 8	1	1	4	2	3	2	2	2	2	—	3	1	5	1	3	2	1	1	2	2	1	1	2	1		1		2			+ 8
+ 9	1	2	1	2	4	2	—	2	2	5	—	—	—	1	1	1	2	1	1	3			2	—	1			2		—	+ 9
+10	3	1	1	1	2	4	2	1	1	4	1	—	1	4	1	3	2	1	1	3	1	—	1	2	1		1	—	1		+10
+11	2	—	2	2	1	2	3	1	2	3	—	—	1	1	—	—	1	1	1	—	—	1		1						1	+11
+12	2	2	3	2	—	1	3	2	—	4	—	—	—	—	1	—	2	1	—	—				1	1						+12
+13	1	3	—	6	1	2	1	4	1	1	2	—	—	—	—	1	1	—	1	—				1	2						+13
+14	3	3	4	3	6	1	1	1	2	3	—	—	—	1	—	1	—	—	—	1											+14
+15	3	4	5	4	4	6	1	2	1	1	—	—	—	—	—	2	—	—	—	—											+15
+16	1	2	1	3	3	1		1	4	2	—	—	—	—	—	—	—	1	—	1		1									+16
+17	—	—	—	—	—	—	—	—	—	—	—	—	1	—	1	—	2	—	—	—	—	—	—	—	—	—	—	—	—	—	+17
+18	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	+18
+19	—	—	—	—	—	—	—	—	—	—	2	—	—	—	1	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	+19
+20	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	+20
+21	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	+21
+22	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	+22
+23	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	2	—	—	—	—	—	—	—	—	—	—	—	—	+23
+24	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	+24
+25	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	+25
* *	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	* *
+30	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	+30

III.

NOTE ON THE CORRELATION BETWEEN β_2 AND w'.

By R. C. GEARY, M.Sc.

IN the present state of our knowledge, the problem of determining which of two tests of normality is the more efficient, can be dealt with only by numerical methods. Having obtained the probability points for the rival tests for normal samples of different sizes, we should calculate the values of each of the test functions for a large number of distributions actually found in practical work and ascertain which of the tests gives the greater number of values lying in the respective regions of improbability for normal samples. As between the β_2 and w' tests*, investigation must be confined to samples so large that one may have confidence in the probability points for β_2 for normal samples, determined by E. S. Pearson.

Dr Pearson's discovery, illustrated on Figs. 3 and 4 of his paper, of marked statistical relationships between β_2 and w' in samples drawn from different parent universes, has obviously an important bearing on this problem. If there were a rigid algebraical relation between the tests, they would be equally efficient. The analysis which follows is subject to the qualification that the statistical relationship may be closer than the coefficient of correlation indicates. The coefficient of correlation -1 will be found only when the relation is linear. Fig. 4 suggests that the relationship between β_2 and w' for samples of given size from a given parent population may approximate rather to a curvilinear type.

The only universes for which an algebraical relation subsists between β_2 and w' are those in which there are two categories only—measuring say 1 and 0. Suppose that the sample proportions are p (measuring 1) and q (measuring 0). Then

$$m_1 = p,$$
$$m_2 = pq,$$
$$m_4 = pq(1-3pq).$$

The mean deviation $\quad d = 2pq.$

Then $\quad w' = d/\sqrt{m_2} = 2\sqrt{pq},$

and $\quad \beta_2 = m_4/m_2^2 = 1/pq - 3,$

so that $\quad \beta_2 = 4/w'^2 - 3.$

Accordingly as w' increases β_2 diminishes.

In large samples the sets of values of the elements which minimise β_2 maximise w', and vice versa. If the y_i represent the deviations from the sample mean, then

$$\beta_2 = \frac{n'\Sigma y_i^4}{(\Sigma y_i^2)^2} \text{ and } w' = \frac{\Sigma|y_i|}{\sqrt{n'\Sigma y_i^2}}.$$

* [At the desire of Mr Geary I have left w' standing throughout this section, although in the first section of his paper he has invariably used it with the appropriate subscript, here dropped. ED.]

If the sample is so large that the y's may be considered independent*, the *minimum* value of β_2, namely 1, is found when

$$|y_1| = |y_2| = \ldots = |y_{n'}|,$$

which gives the *maximum* value 1 of w'. The maximum value of β_2, namely n', and the minimum value $\frac{1}{\sqrt{n'}}$ of w' are found for the following series of values of the y's:

$$y_i \neq 0,$$
$$y_k = 0\ (k \neq i).$$

E. S. Pearson's Table I clearly indicates a tendency in the seven universes (to which the normal with $\beta_2 = 3$ and $w' = \cdot 798$ might be added to give an eighth) for a high β_2 to accompany a low w' and vice versa.

The correlation between β_2 and w' for normal samples may be determined as follows. If the sample of $n' = n + 1$ is

$$x_1,\ x_2,\ \ldots,\ x_{n'},$$

then
$$w' = \frac{\Sigma\,|x_i - \bar{x}|}{\sqrt{n\,(n+1)}\,s} = \sqrt{\frac{n+1}{n}}\,\frac{d}{s},$$

and
$$\beta_2 = \frac{(n+1)\,\Sigma\,|x_i - \bar{x}|^4}{n^2 s^4} = \left(\frac{n+1}{n}\right)^2 \frac{m_4}{s^4},$$

with
$$s^2 = \frac{1}{n}\,\Sigma\,|x_i - \bar{x}|^2,$$

Indicating by square brackets, [ ], universal means about fixed origin, the coefficient of correlation required is

$$r = \frac{[w'\beta_2] - [\beta_2]\,[w']}{\sigma_{w'}\sigma_{\beta_2}}.$$

The values of $[\beta_2]$ and σ_{β_2} are known†:

$$[\beta_2] = \frac{3n}{n+2};\quad \sigma^2_{\beta_2} = \frac{24\,(n+1)\,(n-1)\,(n-2)}{(n+2)^2\,(n+4)\,(n+6)}.$$

Since w' and s are independent,

$$[w'] = \sqrt{\frac{n+1}{n}}\,\frac{[d]}{[s]},\quad [w'^2] = \frac{n+1}{n}\,\frac{[d^2]}{[s^2]},$$

with
$$[s] = e^{-\frac{1}{4n} + \frac{1}{24n^3} - \frac{1}{20n^5} + \frac{17}{112n^7} - \cdots},\quad [s^2] = 1,$$

and
$$[d] = \sqrt{\frac{2n}{\pi\,(n+1)}};\quad [d^2] = \frac{n}{(n+1)^2}\left(1 + \frac{2}{\pi}\sqrt{n^2-1} + \frac{2}{\pi}\sin^{-1}\frac{1}{n}\right).$$

The last of these formulae is due to Helmert‡. The value of $\sigma_{w'}$ will be found from

$$\sigma^2_{w'} = [w'^2] - [w']^2.$$

* A solution of the problem of determining the sets of values of the n' variates which give limiting values of β_2 and w' for all sizes of samples, and the respective limiting values, would have some mathematical interest.

† Readily derivable from formulae given by R. A. Fisher, *Proc. Royal Society, London*, Series A, 130 (1931), p. 25.

‡ *Astronomische Nachrichten*, Bd. 88, No. 2096 (1876).

There remains the calculation of

$$[w'\beta_2] = \left(\frac{n+1}{n}\right)^{\frac{5}{2}} \frac{[dm_4]}{[s^5]},$$

in which

$$[s^5] = \frac{(n+1)(n+3)}{n^2}[s].$$

Also

$$[dm_4] = \frac{1}{(n+1)^2}\{(n+1)[y_1^5] + n(n+1)[y_1^4 y_2]\},$$

with

$$y_1 = |x_1 - \bar{x}| \text{ and } y_2 = |x_2 - \bar{x}|.$$

The distribution of y_1 is

$$\frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\, e^{-y_1^2/2\sigma^2}\, \delta y_1 \, (\infty \geqslant y_1 \geqslant 0),$$

with

$$\sigma^2 = \frac{n}{n+1},$$

so that

$$[y_1^5] = 8\sigma^5 \sqrt{\frac{2}{\pi}}.$$

The joint distribution of y_1 and y_2 is

$$\frac{1}{\pi\sigma^2\sqrt{1-\rho^2}}\left\{ e^{-\frac{1}{2\sigma^2(1-\rho^2)}(y_1^2+y_2^2-2\rho y_1 y_2)} + e^{-\frac{1}{2\sigma^2(1-\rho^2)}(y_1^2+y_2^2+2\rho y_1 y_2)} \right\} \delta y_1 \delta y_2.$$

The value of ρ, the coefficient of correlation between $x_1 - \bar{x}$ and $x_2 - \bar{x}$, is $-\frac{1}{n}$. From this distribution it has been found that the value of $[y_1^4 y_2]$ is

$$\sigma^5 \sqrt{\frac{2}{\pi}}\,(3 + 6\rho^2 - \rho^4).$$

All the data are now available. Upon reduction the coefficient of correlation r between w' and β_2 for normal samples takes the simple form

$$r = -\sqrt{\frac{2}{\pi}}\, \frac{(n-1)(n-2)}{[s]\, n\, (n+2)(n+3)\, \sigma_{w'} \sigma_{\beta_2}}.$$

The values of $[s]$, $\sigma_{w'}$ and σ_{β_2} in the denominator have been given above. The presence of the factors $n-1$ and $n-2$ in the numerator might have been anticipated. If either were missing, r would assume the impossible value ∞ in samples of 2 or 3, since the factors $\sqrt{n-1}$ and $\sqrt{n-2}$ occur in σ_{β_2} in the denominator. In samples of 2, of course, both β_2 and w' assume the same value 1 for all samples, while β_2 assumes the same value $\frac{3}{2}$ for all samples of 3.

When n tends towards infinity the coefficient of correlation tends towards

$$r_\infty = -\frac{1}{\sqrt{12(\pi - 3)}} = -\cdot 7673\ldots.$$

The values of r for certain sizes of sample are as follows:

Size of Sample, n'... ...	6	11	26	51	101
Coefficient of Correlation, r	$-\cdot 921$	$-\cdot 851$	$-\cdot 809$	$-\cdot 790$	$-\cdot 778$

ON THE VALIDITY OF A CERTAIN PEARSON'S FORMULA.

By JAN WIŚNIEWSKI.

§ 1. In the memoir "On the Measurement of the Influence of broad Categories on Correlation" printed in Vol. IX. of *Biometrika* (pp. 116 to 139) Karl Pearson makes some statements which seem to need qualification.

The notations are as follows: variables x, y; the class-marks of the corresponding class intervals C_x, C_y. The first statement we shall consider is the assumption that in each class-interval the actual arithmetic average of the variable x or y is equal to the corresponding class-mark C_x or C_y. Of course, this is an approximation only and within its degree of accuracy only holds the ensuing formula (i) of the paper referred to:

$$r_{xC_x} = \frac{\sigma_{C_x}}{\sigma_x} \quad \text{.......................................(i).}$$

It is more than sure that Karl Pearson was aware of the approximative character of form (i) when proposing it; we merely point to this circumstance in connection with the numerical example that will follow.

The main point of the present note is to show the limited validity of another formula of the memoir discussed, viz. form (ix):

$$r_{xy} = \frac{r_{C_yC_x}}{r_{xC_x} r_{yC_y}} \quad \text{.................................(ix).}$$

(ix) is based upon (iv) and (viii),

$$r_{xy} = \frac{r_{yC_x}}{r_{xC_x}} \quad \text{.....................................(iv),}$$

$$r_{yC_x} = \frac{r_{C_yC_x}}{r_{yC_y}} \quad \text{..............................(viii).}$$

The derivation of (iv) is the following (to quote Pearson's words): "...since a given x will have a constant class-mark, the correlation of y and C_x for a constant x is zero; that is to say that the partial correlation coefficient

$$_x\rho_{yC_x} = \frac{r_{yC_x} - r_{xy} r_{xC_x}}{\sqrt{1 - r^2_{xy}}\sqrt{1 - r^2_{xC_x}}} = 0\text{"} \quad \text{......................}(a).$$

§ 2. Formula (a) and its derivation need a thoroughgoing discussion. First, we shall make some observations concerning partial correlation in general. According to the writer's view, partial correlation may be given two interpretations: the "layer" interpretation and the "projection" interpretation.

First, we shall present the more usual "layer" interpretation. Suppose we are dealing with the variables u, v, t. The points having coordinates u_i, v_i, t_i are

distributed in a three-dimensional space. Now if we want to find the partial correlation between u and v we divide the space into layers by means of planes parallel to the u, v plane. In each layer the variability of t is greatly reduced so that we consider it negligible. Within each layer we make a projection of the u_i, v_i, t_i points on a plane parallel to the u, v plane and calculate the correlation found for the distribution of the projected points (of course the projection is made perpendicularly to the said plane). At last, we compute a kind of average* of the correlation coefficients in each layer and this is the partial correlation coefficient, provided the regressions of u on t and of v on t are linear. If they are not strictly so, the partial correlation coefficient calculated according to the familiar formulae is at any rate an approximation to the coefficient calculated in the manner described above.

The "projection" interpretation is arrived at from the former when we assume that layers are made thinner and thinner. Instead of making a projection within each layer we make one of all u_i, v_i, t_i points on the u, v plane. But the projection is no more perpendicular to this plane. It is made in such a manner that the coordinates of each projected point are $u_i - \bar{u}_i$, $v_i - \bar{v}_i$ where $\bar{u}_i$ and $\bar{v}_i$ are values obtained from regression equations† on t. If we assume those equations to be linear, we again get the classical formula $(r_{uv} - r_{ut}r_{vt})/\sqrt{(1-r^2_{ut})(1-r^2_{vt})}$ equivalent to the simple correlation coefficient calculated for the projected points. If t is a discrete variable and the layers are made so thin as to comprise only one value of t, the "layer" and the "projection" interpretations are identical.

§3. Now we can analyse the special case of ${}_x\rho_{yC_x}$. The distribution of the y_i, C_{xi}, x_i points is very peculiar indeed. Assume that x and y are continuous variables, i.e. they can take any given value within the field of variation. C_x, on the other hand, can take only special values. Moreover, C_x is a definite function of x, viz.

$$C_x = h\left[\frac{x}{h}\right] + \frac{h}{2} \quad \dots\dots\dots\dots\dots\dots\dots\dots(b),$$

where $[z]$ means the largest integer smaller than z, and h is the class-interval.

For this reason all y_i, C_{xi}, x_i points must be situated on plane "stripes" parallel to the x, y plane. Each "stripe" is bounded by two parallel straight lines which are determined as follows:

$$C_x = C_{xi};\ x = C_{xi} + \tfrac{1}{2}h \quad\text{and}\quad C_x = C_{xi};\ x = C_{xi} - \tfrac{1}{2}h.$$

The "layer" interpretation fails in this case completely. As a matter of fact it is only natural to take x-layers identical with its class-intervals. Then within each

* This kind of average resembles what Irving Fisher terms "aggregatic average." In our case, denoting with $\bar{u}_k$ and $\bar{v}_k$ the arithmetic averages of u and v for the layer k, we have

$$\rho = \frac{\sum\limits_k\sum\limits_i (u_i - \bar{u}_k)(v_i - \bar{v}_k)}{\sqrt{\sum\limits_k\sum\limits_i (u_i - \bar{u}_k)^2 \sum\limits_k\sum\limits_i (v - \bar{v}_k)^2}}.$$

† In general, these equations need not be *perfectly* fulfilled.

layer all points will lie on a straight line parallel to the y-axis. This means that the correlation coefficient for each layer becomes indefinite and so is the partial correlation coefficient calculated according to the "layer" interpretation.

We would obtain the same result from the "projection" interpretation if we assumed that the values of $\bar{C}_{xi}$ are determined from equation (b), which is really the case. By $\bar{C}_{xi}$ is meant the value of C_{xi} obtained from a regression equation.

It seems that the only way to get for ${}_x\rho_{yC_x}$ a definite value is to start from the "projection" interpretation and to assume

$$\bar{C}_{xi} = r_{xC_x} \frac{\sigma_{C_x}}{\sigma_x}(x_i - M_x) + M_{C_x} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(c),$$

where M_x and M_{C_x} denote the arithmetic averages of x and C_x respectively. As we stated, each y_i, C_{xi}, x_i point must lie on a "stripe" conforming to the description given above. More particularly, for a given x_i all points lie on a straight line which is defined by $x = x_i$, $C = C_{xi}$ as determined from (b). Such a straight line is, of course, situated on a given "stripe" and parallel to its boundaries. What will be the projection of the points lying on such a line, on the y, C_x plane? The projections will have a common C_x coordinate equal to $C_{xi} - \bar{C}_{xi}$, where $\bar{C}_{xi}$ is determined from (c), and y's equal to $y_i - \bar{y}_i$, where $\bar{y}_i$ is found from the regression equation of y on x. As $C_{xi} - \bar{C}_{xi}$ takes different values* the projected points will not lie on one straight line parallel to the y-axis (as in the "layer" interpretation case). If y is not a one-valued function of x, the projected points also will not lie on one straight line parallel to the C_x-axis and, consequently, ${}_x\rho_{yC_x}$ will take a definite value. If the regression of y and x is strictly linear, the mathematical expectation of $y_i - \bar{y}_i$ for each $C_{xi} - \bar{C}_{xi}$ is zero and therefore ${}_x\rho_{yC_x}$ will equal zero. If the regression is only "practically" linear, as is assumed by Pearson, equation (a) still will give a good approximation.

The conclusion is that (iv) is true provided the regression of y on x is linear. This is a sufficient condition but not absolutely necessary. We can fancy cases where the regression in question is not strictly linear and yet ${}_x\rho_{yC_x} = 0$.

§4. The derivation of (viii) is similar; namely, Pearson assumes that the correlation of C_x and C_y for a constant y is zero and, accordingly,

$$_y\rho_{C_xC_y} = \frac{r_{C_xC_y} - r_{yC_x} r_{yC_y}}{\sqrt{1 - r^2_{yC_x}}\sqrt{1 - r^2_{yC_y}}} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(d).$$

In order to examine the validity of (d) we shall start from the "projection" interpretation and assume that $\bar{C}_{xi}$ and $\bar{C}_{yi}$ are determined from equations analogous to (c).

The distribution of the C_{xi}, C_{yi}, y_i points in space is still more peculiar than in the foregoing case. Each point lies on a rectilinear segment parallel to the y-axis and limited by two points, having the coordinates $C_x = C_{xi}$; $C_y = C_{yi}$;

* If x is a continuous variable $C_{xi} - \bar{C}_{xi}$ is also one. For practical purposes we may assume, however, that $|C_{xi} - \bar{C}_{xi}| < h$.

$y = C_{yi} \pm \frac{1}{2}h$ (supposing the class-intervals of x and of y to be the same). For a given C_{yi} all such segments lie on a plane "stripe" analogous to those described in the preceding paragraph. The difference is that *in the former case the points lay* distributed more or less evenly throughout the stripe while in the present one they can occupy only determined positions on rectilinear segments parallel to the y-axis.

Correspondingly the projection of the C_{xi}, C_{yi}, y_i points on the C_x, C_y plane will differ from that of the former case. Let us denote, for the sake of simplicity,

$$u_i = C_{yi} - \bar{C}_{yi} = C_{yi} - M_{C_y} - r_{yC_y} \frac{\sigma_{C_y}}{\sigma_y}(y_i - M_y) \quad \ldots\ldots\ldots\ldots\ldots(e),$$

$$v_i = C_{xi} - \bar{C}_{xi} = C_{xi} - M_{C_x} - r_{yC_x} \frac{\sigma_{C_x}}{\sigma_y}(y_i - M_y) \quad \ldots\ldots\ldots\ldots\ldots(f),$$

from which follows

$$v_i = C_{xi} - M_{C_x} + \frac{r_{yC_x}\sigma_{C_x}}{r_{yC_y}\sigma_{C_y}}\{u_i - (C_{yi} - M_{C_y})\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(g).$$

(g) is evidently linear with respect to u. For the whole stripe we have a constant C_y and for each segment within a stripe a constant C_x. Therefore the projection of the segments of a stripe will be a set of segments parallel to each other and having the same field of variation of u (because the field of variation of y is the same for all segments, viz. from $C_y - \frac{1}{2}h$ to $C_y + \frac{1}{2}h$). As r_{yC_y} is positive and near to unity, the angular coefficients of the projected segments will depend mainly on r_{yC_x}.

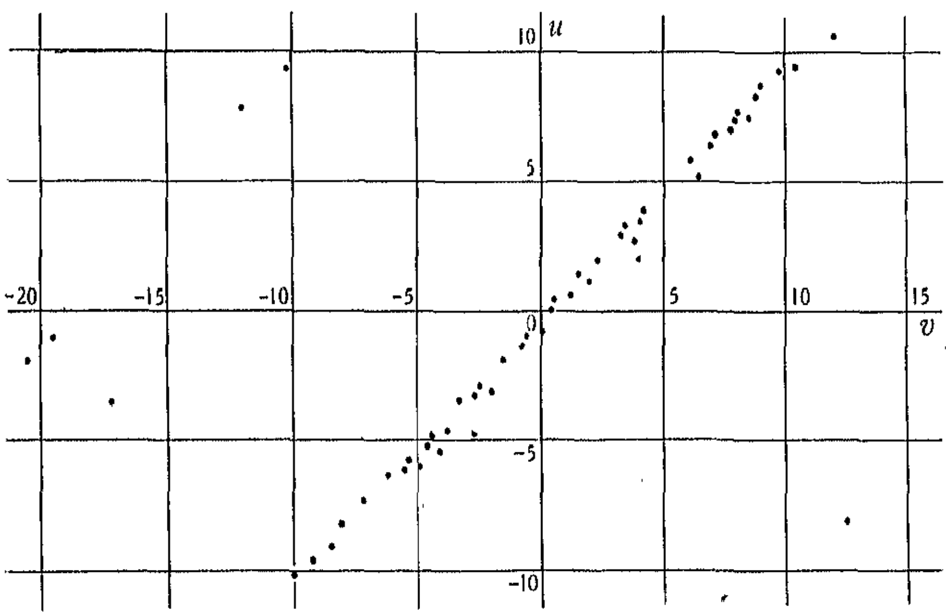


Chart 1. Distribution of observations from a numerical example.

Data projected according to equations (e) and (f). The dots on this chart do not number 75, as they should, because some of them represent several observations.

All segments of our distribution will also give a set of parallel segments*, but the segments taken from different stripes will have fields of variation of u of the same length, to be sure, yet somewhat shifted—suppose both C_{yi} and y_i in (e) increase by h, then u_i will increase by $h\left(1 - r_{yC_y}\dfrac{\sigma_{C_y}}{\sigma_y}\right)$.

What will be the correlation calculated for the projected distribution? Off-hand we cannot say anything. If the points were distributed evenly in each segment, the correlation would surely follow the sign of r_{yC_x}, the absolute value of ρ approaching zero as the number of segments for each C_y would increase. But the premise given above is not general; indeed, if we really have to do with "broad categories," we might expect that the correlation between y and C_x will be observable already within given y-intervals, i.e. for a given C_y. The coefficient of correlation for the points thus projected is evidently

$$_y\rho_{C_yC_x} = \frac{r_{C_yC_x} - r_{yC_y}r_{yC_x}}{\sqrt{1 - r^2_{yC_y}}\sqrt{1 - r^2_{yC_x}}} \quad \text{.......................}(h).$$

This cannot be zero except when

$$r_{yC_x} = r_{C_yC_x} = 0 \quad \text{...............................}(j),$$

or

$$\frac{r_{C_yC_x}}{r_{yC_x}} - r_{yC_y} = 0 \quad \text{...............................}(k).$$

Thus we are brought back to our starting point. (k) is clearly equivalent to (viii). Therefore we see that Pearson's contention (viii) cannot be generally proved in an analytical way. In the best case we can regard (viii) as an approximation of unknown precision. The application of the reasoning used at the end of § 3 is here impossible as the regression of C_x on y will *not*, in general, be strictly linear; consequently, the mathematical expectation of $C_{xi} - \bar{C}_{xi}$ for a given $C_{yi} - \bar{C}_{yi}$ will not necessarily equal zero. Especially, if some observations are distributed in such a manner that in a certain number of y-rows one cell only is filled in each, the aforesaid mathematical expectation cannot be generally zero. In such cases the sign of (h) will surely follow that of $r_{C_yC_x}$.

The above conclusions can also be derived from a discussion of (k). If the distribution is such as just described $r_{C_yC_x}$ should nearly equal r_{yC_x}. On the other hand, we may expect that in such cases the classification is really done in "broad categories," thus the value of r_{yC_x} will be comparatively small and the left-hand side of (k) will be positive instead of zero. We may also resort to the graphic presentation given in Chart 2. Suppose in some y-rows one cell only is filled in each, then this will be surely that cell through which passes the line of regression of C_x on y, this regression being assumed linear. The points of this cell, when projected, will lie on a segment passing near to the (0, 0) point. If the bulk of the

* Such a projection is shown in Chart 1. It can be clearly seen that the projected points form a set of parallel segments. [Surely any reasonable test for outliers would throw out the five points to the left and the single point to the right of Chart 1? ED.]

observations belongs to such cells we shall have a picture like that in Chart 2—viz. a great mass of points situated on a narrow stripe passing near to the origin of the system. Those points will clearly contribute to produce a correlation coefficient of large numerical value and of the sign of $r_{G_yG_x}$. If, on the other hand, the majority of points belong to other cells, their projections will give a picture of the scattered points in Chart 2, those points contributing to lower the numerical value of ${}_y\rho_{G_yG_x}$.

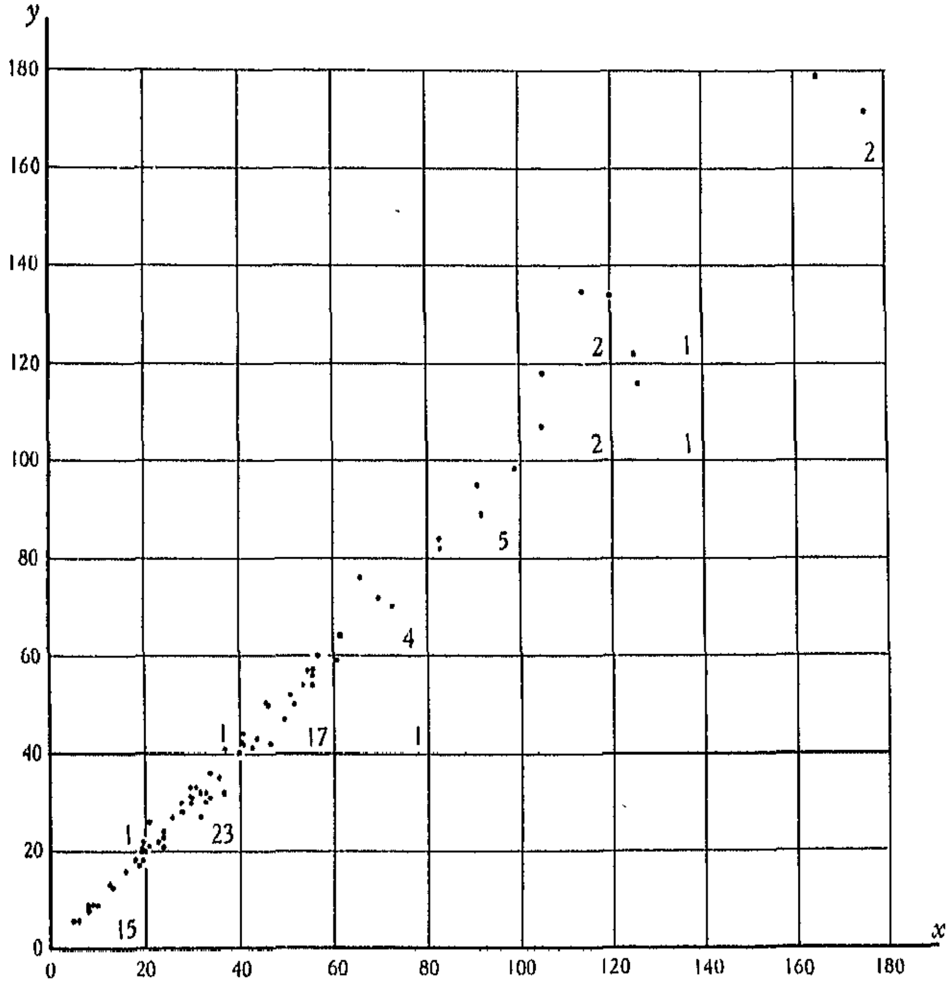


Chart 2. Distribution of observations from the numerical example. Original data.

It should also be noticed that our conclusions would be unchanged if instead of (e) we simply took $u_i = G_{yi} - y_i$.

§5. To sum up: Pearson's formulae (viii) and (ix) are not generally true, and their degree of approximation is unknown. How serious blunders can arise, in extreme cases, from their application, the following example will show:

Numerical Example. We will study the distribution of rural districts in a certain Polish county according to the number of dwelling houses (*Statistique de la Pologne* publiée par l'Office Central de Statistique de la Rép. Polonaise, Série B, fasc. 8-b, pp. 11—12. Arrondissement de Chodzież, communes rurales. Nombre de bâtiments d'habitation). x-data from 1921 census, y-data from 1931 census. The original data are given in the table below:

x	y	x	y	x	y	x	y	x	y
23	22	5	6	33	30	10	17	36	35
41	42	56	57	62	64	40	40	30	31
52	50	41	44	41	42	28	28	21	26
30	30	66	76	8	9	83	82	31	33
24	24	13	13	37	32	73	70	114	135
83	84	18	18	120	134	54	54	30	31
34	36	20	20	34	31	65	57	105	118
20	22	9	9	26	27	46	50	13	13
37	41	20	20	32	27	30	33	70	72
29	18	28	30	176	172	46	50	33	32
10	9	8	8	44	43	105	107	56	56
91	95	56	54	57	60	99	98	24	23
51	52	61	59	6	6	50	47	126	122
32	32	165	179	43	41	24	21	92	89
126	110	16	16	47	42	20	20	21	21

The data are also charted and given in graphic form (Chart 2). The cross-classification has been executed in intervals of twenty houses, including the upper class-limit. The figures given in the graph are the frequencies in the several cells.

The values of the parameters of the distribution are:

	x	C_x	y	C_y
Arithmetic average M	47·93	48·10	48·71	48·63
Standard deviation σ	35·92	36·66	37·45	36·84

$N = 75$, $r_{xy} = {}·9934$, $r_{C_xC_y} = {}·9879$, $r_{xC_x} = {}·9871$, $r_{yC_y} = {}·9877$, $r_{yC_x} = {}·9810$.

All r's are computed directly, i.e. making no use of the quoted Pearson's formulae. Now in the light of this example we shall test the validity both of the formulae mentioned and of the limitations we suggested. The particular example we chose seems to be well adapted for this purpose.

M_x and M_{C_x} differ widely, so formula (i) cannot be expected to give good results. Indeed, r'_{xC_x}* actually comes out greater than 1! On the other hand, the difference between M_y and M_{C_y} is rather slight and correspondingly $r'_{yC_y} = {}·9837$, which is not far from the value calculated directly.

As the regression of y on x is very nearly linear, form (iv) should give correct results. As a matter of fact, $r_{yC_x}/r_{xC_x} = {}·9938$. This discrepancy between this value

* We denote by r' those coefficients which are calculated indirectly, i.e. by means of Pearson's formulae.

and the one given above is within the margin of error due to rounding off decimals.

Because of the existence of many y-rows containing only one cell that is not empty, formulae (viii) and (ix) must register failures. Thus ${}_{x}\rho_{o_y o_x}$ instead of zero equals ·626, $r_{o_y o_x}/(r_{x o_x} r_{y o_y}) > 1$; $r_{o_y o_x}/(r'_{x o_x} r'_{y o_y}) = {}$·9919 which happens to be in the right direction, but we wonder whether one would not hesitate to use r''s greater than unity.

Lastly, we shall test Pearson's formula (xv) devised for the case when, as usually, we know very little about the distribution within class intervals.

$$r''^2_{x o_x} = 1 - \frac{h_x^2}{12\sigma_x^2} \quad \text{..............................(xv)},$$

where h is the class-interval. Here again $r_{o_y o_x}/(r''_{x o_x} r''_{y o_y}) > 1$.

One might object that the differences between theory and application are so minute as not to deserve attention, but the corrections to crude coefficients as proposed by Pearson are themselves, in general, of a rather small order.

ON THE CORRECTIONS FOR BROAD CATEGORIES, BEING A NOTE ON MR WIŚNIEWSKI'S MEMOIR.

By K. PEARSON.

I PROPOSE to discuss Mr Wiśniewski's criticism of my method of correcting for broad categories, by dealing first with the numerical example which he provides. I may state that I consider that example a very extreme one to test the accuracy of my corrections upon, and the reasons for this will appear in the course of my analysis.

Mr Wiśniewski's data consist of the numbers of inhabited houses in certain rural districts or arrondissements in Poland at the date of two censuses. There is no district without a house, the lowest number being five, and the distributions at both censuses are crowded into the ranges 5 to 60. Thus the frequency distributions of the two variates are both extremely skew, we are ignorant of where they start, and the sample consists of 75 rural districts only, so that the standard errors of the constants involved are very large, and no allowance is made for any of these peculiarities. The case is, however, said to be singularly suitable for testing my formulae.

In Table I the actual data are given, where I must remark that the groups of 20 houses are included in the group 0 to 20, and the group 20 to 40 does not include the 20 and does include the 40 and so on.

Hence, if this discrete series is to be replaced by a continuous variate we must take our groups to be 20·5 to 40·5 and so on. Mr Wiśniewski does not appear to notice that my formulae are deduced on the assumption of *continuous* and not discrete variables. Even when we replace discrete by continuous variables the start of the continuous variables is left in the air. Now I show in my original paper that my correction for the correlation coefficient reduces to Sheppard's correction, when the sub-ranges are all equal, as they are in this case (except the doubtful first category) and when we accept the hypothesis of Sheppard, i.e. *that high contact at the terminals of the distribution holds*. Now in Mr Wiśniewski's data there is no high contact at the zero ends of the frequency distributions, but on the contrary great abruptness. Hence the standard deviations involved in the value of the correlation must be corrected in a manner different from that involved in taking

$$r_{x}\sigma_{x} = \sqrt{1 - \frac{h^2}{12\sigma_x^2}},$$

a relation which depends on the high contact hypothesis. It is overlooking this point which leads him to obtain a corrected correlation coefficient slightly over unity.

Taking the data as given in Table I and working with the 75 individual cases, I find, x being the horizontal and y the vertical variate,

Means $\bar{x} = 47{\cdot}933{,}333$, $\bar{y} = 48{\cdot}706{,}667$,

Standard Deviations

$$\sigma_x = 35{\cdot}914{,}466, \quad \sigma_y = 37{\cdot}454{,}959,$$

Product Moment Coefficient $= 1336{\cdot}353{,}777$,

providing the correlation coefficient

$$r_{xy} = {\cdot}993{,}442.$$

Mr Wiśniewski gives the values

$$\bar{x} = 47{\cdot}93, \quad \bar{y} = 48{\cdot}71, \quad \sigma_x = 35{\cdot}92, \quad \sigma_y = 37{\cdot}45, \quad r_{xy} = {\cdot}9934,$$

which nearly agree with the above values, although they are taken to an inadequate number of figures having regard to later work.

Now the problem before us is to obtain these results, or results close to them from a contracted table, i.e. one of broad categories, and so save the tedious process of dealing with the individual 75 entries in computing the coefficients. But the corrections for grouping observations together have hitherto been based on *continuity* of the variates, except in those proposed by H. C. Carver for grouping discrete variates*, and these fail in the present case, as we shall see, precisely as Sheppard's fail, and for the same reason—they disregard the abruptness. In order accordingly to correct the contracted table we are bound to suppose the frequencies represented on it result from the sampling of a *continuous* distribution. In the next place we are at liberty to choose our grouping, and the safest grouping experience shows is that which starts close to the lowest observation in the data; in this case 5, 5. Let us take our grouping accordingly in sub-ranges of 20 from 4·5. We then reach Table II.

TABLE II.

x-variate

y-variate	*4·5—24·5*	*24·5—44·5*	*44·5—64·5*	*64·5—84·5*	*84·5—104·5*	*104·5—124·5*	*124·5—144·5*	*144·5—164·5*	*164·5—184·5*	Totals
4·5—24·5	21	—	—	—	—	—	—	—	—	21
24·5—44·5	1	23	1	—	—	—	—	—	—	25
44·5—64·5	—	—	13	—	—	—	—	—	—	13
64·5—84·5	—	—	—	5	—	—	—	—	—	5
84·5—104·5	—	—	—	—	3	—	—	—	—	3
104·5—124·5	—	—	—	—	—	2	2	—	—	4
124·5—144·5	—	—	—	—	—	2	—	—	—	2
144·5—164·5	—	—	—	—	—	—	—	—	—	—
164·5—184·5	—	—	—	—	—	—	—	—	2	2
Totals	22	23	14	5	3	4	2	—	2	75

* *Annals of Mathematical Statistics*, Vol. I, p. 111.

The peculiar nature of the distribution upon which Mr Wiśniewski proposes to test my corrections will now be obvious to the reader. There is extreme abruptness at the low end of the range for both variates, and the usual, or Sheppard's corrections, are bound to fail. However, let us see how the results work out. We will take our working axes through 54·5, 54·5, and our working unit 20 houses. Summations being from working origin. We find

$\frac{1}{75} S(x) = -\cdot 32, \quad \frac{1}{75} S(y) = -\cdot 32, \quad \frac{1}{75} S(x^2) = 3\cdot 573{,}333, \quad \frac{1}{75} S(y^2) = 3\cdot 546{,}666,$

and $\qquad \frac{1}{75} S(xy) = 3\cdot 52, \quad$ and $q_{xy} = 3\cdot 52 - \cdot 1024 = 3\cdot 4176.$

Hence, making no corrections, we have

$$\sigma_x^2 = 3\cdot 470{,}933, \qquad \sigma_y^2 = 3\cdot 444{,}267, \qquad q_{xy} = \frac{S(xy)}{75} - \bar{x}\bar{y} = 3\cdot 4176,$$

$$\sigma_x = 1\cdot 863{,}044, \qquad \sigma_y = 1\cdot 855{,}874, \qquad \sigma_x \sigma_y = 3\cdot 457{,}575.$$

Accordingly the crude uncorrected r_{xy} is given by

$$r_{xy} = q_{xy}/\sigma_x \sigma_y = \cdot 988{,}438.$$

Now Sheppard has shown that if the sub-ranges are equal and sufficiently small as compared with σ_x and σ_y* and there is high contact at both ends for continuous variates, then q_{xy} requires no correction and $\frac{1}{12}$ (sub-range)^2 must be subtracted from σ_x^2 and σ_y^2. I have shown that under like conditions my class index corrections $r_{x\theta_x}$ and $r_{y\theta_y}$ reduce to $\sqrt{1 - \frac{h^2}{12\sigma_x^2}}$ and $\sqrt{1 - \frac{k^2}{12\sigma_y^2}}$, where h and k are the x and y sub-ranges respectively. These lead if the conditions stated above be supposed to hold to precisely the same result as Sheppard's corrections.

Using Sheppard, we find

$$\sigma_x = 1\cdot 840{,}5433, \qquad \sigma_y = 1\cdot 833{,}2850, \qquad \sigma_x \sigma_y = 3\cdot 374{,}240,$$

and accordingly

$$r_{xy} = 3\cdot 4176/3\cdot 374{,}240 = 1\cdot 012{,}85.$$

Thus these corrections fail, and the reason for failure lies not so much in the fact that h is more than half σ_x, as in the absence of high contact at the start of the ranges of both x and y. Before we turn to the correction for abruptness, let us see what Carver's formulae will provide.

Now if there be u discrete values in the sub-range, Carver subtracts

$$\frac{1}{12}\left(1 - \frac{1}{u^2}\right) = \frac{1}{12}\left(1 - \frac{1}{400}\right)$$

from the standard deviation squared, i.e. ·0831,2500 instead of ·0833,3333, like Sheppard. This gives us

$$\sigma_x = 1\cdot 840{,}5999, \qquad \sigma_y = 1\cdot 833{,}3418, \qquad \text{and } \sigma_x \sigma_y = 3\cdot 374{,}4487.$$

Thus the corrected $\qquad r_{xy} = 3\cdot 4176/3\cdot 374{,}4487 = 1\cdot 012{,}78.$

* This is certainly not fulfilled in the present data.

We see again that the correlation is greater than unity, and with twenty discrete values in the sub-range Carver's formula scarcely improves on Sheppard's, which treats them as continuous. Thus the failure does not arise from treating discrete grouping as continuous.

The source of error clearly lies in neglecting the abruptness. Allowance for abruptness has been made by Pairman and Pearson* and put into a convenient form by Sandon†, but possibly the best summary is that provided in Part II, p. cxciii of the *Tables for Statisticians and Biometricians*. Assuming high contact at the end of the range and moments to be taken about the start, we have to correct the means and the standard deviations of the two variates by the formulae

$$\mu_1' = \nu_1' - hH_{1,a}, \qquad \mu_2' = \nu_2' - \tfrac{1}{12}h^2 - h^2 H_{2,a},$$

where
$$H_{1,a} = \frac{11{,}153n_1 - 12{,}566n_2 + 10{,}176n_3 - 4{,}586n_4 + 863n_5}{60{,}480N},$$

$$H_{2,a} = \frac{1{,}830n_1 - 4{,}358n_2 + 4{,}110n_3 - 1{,}962n_4 + 380n_5}{60{,}480N}.$$

Here n_1, n_2, n_3, n_4 and n_5 are the first five sub-frequencies from the abrupt terminal, and N is the total frequency $= 75$ in our case. Noting, that for x,

$$n_1 = 22, \quad n_2 = 23, \quad n_3 = 14, \quad n_4 = 5, \quad n_5 = 3,$$

and that of y, $n_1 = 21, \quad n_2 = 25, \quad n_3 = 13, \quad n_4 = 5, \quad n_5 = 3,$

we find
$${}_xH_{1,a} = +\cdot 0172{,}9960, \qquad {}_xH_{2,a} = -\cdot 0024{,}4797,$$
$${}_yH_{1,a} = +\cdot 0070{,}5688, \qquad {}_yH_{2,a} = -\cdot 0056{,}7901.$$

Accordingly,
$${}_x\mu_1' = 2\cdot 5 - \cdot 32 - \cdot 0172{,}9960 = 2\cdot 1627{,}0040,$$
$${}_y\mu_1' = 2\cdot 5 - \cdot 32 - \cdot 0070{,}5688 = 2\cdot 1729{,}4312.$$

The corresponding values from the origin (2·5, 2·5) are

$${}_x\mu_1'' = -\cdot 3372{,}9960, \qquad {}_y\mu_1'' = -\cdot 3270{,}5688,$$

and accordingly the corrected q_{xy} is given by

$$q_{xy} = \tfrac{1}{75} S(xy) - {}_x\mu_1'' \times {}_y\mu_1'' = 3\cdot 52 - \cdot 1103{,}1615 = 3\cdot 4096{,}8385.$$

Again

$${}_x\nu_2' = \tfrac{1}{75} S(x^2) + 5\,\frac{S(x)}{75} + 6\cdot 25 = 3\cdot 5733{,}3333 - 1\cdot 60 + 6\cdot 25 = 8\cdot 2233{,}3333,$$
$${}_x\mu_2' = 8\cdot 2233{,}3333 - \cdot 0833{,}3333 - (-\cdot 0024{,}4797) = 8\cdot 1424{,}4797,$$
$$\sigma_x^2 = {}_x\mu_2' - {}_x\mu'^2 = 8\cdot 1424{,}4797 - 4\cdot 6772{,}7302 = 3\cdot 4651{,}7495,$$

and
$$\sigma_x = 1\cdot 8614{,}9804.$$

Again

$${}_y\nu_2' = \tfrac{1}{75} S(y^2) + 5\,\frac{S(y)}{75} + 6\cdot 25 = 3\cdot 5466{,}6667 - 1\cdot 60 + 6\cdot 25 = 8\cdot 1966{,}6667,$$
$${}_y\mu_2' = 8\cdot 1966{,}6667 - \cdot 0833{,}3333 + \cdot 0056{,}7901 = 8\cdot 1190{,}1234,$$
$$\sigma_y^2 = {}_y\mu_2' - {}_y\mu_1'^2 = 8\cdot 1190{,}1234 - 4\cdot 7216{,}8180 = 3\cdot 3973{,}3054,$$

and
$$\sigma_y = 1\cdot 8431{,}8489.$$

* *Biometrika*, Vol. XII. pp. 233, 239.

† *Biometrika*, Vol. XVI. pp. 193—195.

Thus $$\sigma_x \sigma_y = 3\cdot4310{,}8506,$$

and $$r'_{xy} = \frac{q_{xy}}{\sigma_x \sigma_y} = \frac{3\cdot4096{,}8385}{3\cdot4310{,}8506} = \cdot993{,}763,$$

and this differs from the true value ·993,442 by ·000,321. Even this might have been got rid of had a more elaborate attempt been made to determine the probable start of the marginal frequency distributions.

Now let us examine what Mr Wiśniewski has done. He begins by assuming that my class-marks C_x and C_y are the midpoints of the sub-ranges. On the very first page of my memoir, I start by saying that I am going to take the mean $\bar{x}_s$ of the group of individuals in the sth class as my class-mark. I then proceed later in the memoir to discuss the conditions under which it is possible to replace the summations involving $\bar{x}_s$, $\bar{y}_t$ by corresponding summations of the midpoints x_s, y_t of the sub-ranges as class-marks. Mr Wiśniewski pays no attention to these conditions, never enquiring whether they are satisfied or not, and thus reaches corrected values of the correlation which exceed unity, which they may easily do as the uncorrected correlation of his suitable case is already ·988 and over. I can only think that he has not read, or if he has done, not understood the meaning of my memoir. On pp. 120 and 128 it is stated that the vanishing of certain summations depends upon high contact at the terminals or on the truth of Sheppard's hypothesis. When this hypothesis does not hold, then, as is well known, corrections have to be made for "abruptness."

It is idle to say how unusable these formulae are if we take as an illustration a case expressly excluded by the author of them. But in these excluded cases, if proper corrections are made for the means and standard deviations, we find that the short table gives a result close to the true value obtained from the full data.

Mr Wiśniewski never tells us what he means by a "class-mark." I understand by it a term which is common to every individual in a broad category, and any quantity may be used as the class-mark which is constant for every member in the group. Thus the mean $\bar{x}$ of the group may be used, or the mid-point of the sub-range, or the median of the values of x in the sub-range. I start my discussion by saying I will take the first of these as convenient, but pass on in the course of the paper to consider the conditions under which we may use other values to represent the "class-mark."

I now come to the criticism of my formulae and the degree of their approximation, using of course the value of C_x for which they are established.

In the first place, let us consider Formula (i) of the previous memoir

$$r_{xC_x} = \frac{\sigma_{C_x}}{\sigma_x} \qquad \text{(i)}.$$

Let x and C_x ($=\bar{x}_s$) be measured from the mean of the variate x, and let n_s be the frequency in the sth class, Σ a summation within the class, and S a summation for all classes. Then by definition

$$r_{xC_x} = \frac{S(C_x \Sigma x_s)}{N\sigma_x\sigma_{C_x}} = \frac{S(C_x n_s C_x)}{N\sigma_x\sigma_{C_x}} = \frac{S\left(\frac{n_s}{N} C_x^2\right)}{\sigma_x\sigma_{C_x}} = \frac{\sigma_{C_x}^2}{\sigma_x\sigma_{C_x}} = \frac{\sigma_{C_x}}{\sigma_x}.$$

The formula is thus *absolutely* true and not approximative at all, and Mr Wiśniewski's remark "It is sure that Karl Pearson was aware of the approximative character of this formula (i)" appears somewhat wide of the mark.

Let us turn now to a second formula for the partial correlation of

$$_{x}r_{y\,C_x} = 0.$$

Mr Wiśniewski gives a somewhat lengthy discussion of the meaning of a partial correlation coefficient*. I should interpret the relationship here with discrete x and y to be as follows: If we pick out for a given x all the corresponding y's and C_x's, then the product moment $S(y-\bar{y}_x)(C_x-(\bar{C}_x)_{x=\text{const.}})$ must vanish. This it clearly does. For C_x is a constant when x is constant, and $S(y)=n_x\bar{y}_x$, both factors of the product being independent for constant x. There is accordingly, in my mind, no doubt of the absolute truth of

$$_{x}r_{y\,C_x} = 0.$$

In the same manner

$$_{xy}r_{C_x C_y} = 0$$

is absolutely true, for the correlation of two constant quantities is zero. The question then arises as to how far we may replace these partial correlation coefficients by the customary expansion formulae for linear regression. This is discussed at considerable length in my memoir, and involves the relationship between the three following summations

$$S\left(\frac{n_{st}}{N}\bar{x}_{st}\bar{y}_{st}\right),\qquad S\left(\frac{n_{st}}{N}x_s y_t\right),\quad\text{and}\quad S\left(\frac{n_{st}}{N}\bar{x}_s\bar{y}_t\right),$$

where n_{st} is the frequency in the n_{st} cell, i.e. that common to the sth column and tth row, x_s, y_t are the midpoints of the sth column and tth row, and $\bar{x}_s$, $\bar{y}_t$ are the means of the x-constants of the sth column and of the y-constants of the tth row, namely, they are what I term on the first page of my memoir C_x and C_y, the class indices.

On p. 123, the fundamental equation (xvii) is reached, namely†,

$$r_{xy} = \frac{S\left(\dfrac{n_{st}}{N}\dfrac{\bar{x}_s}{\sigma_x}\dfrac{\bar{y}_t}{\sigma_y}\right)}{S\left(\dfrac{n_t\bar{x}_s^2}{N\sigma_x^2}\right)\times S\left(\dfrac{n_t\bar{y}_s^2}{N\sigma_y^2}\right)} \quad\text{........................(xvii)},$$

or, in our present notation,

$$r_{xy} = \frac{S\left(\dfrac{n_{st}}{N}C_x C_y\right)}{\sigma_{C_x}\sigma_{C_y}\times\dfrac{\sigma_{C_x}}{\sigma_x}\dfrac{\sigma_{C_y}}{\sigma_y}} = \frac{r_{C_x C_y}}{r_{xC_x}r_{yC_y}}. \quad\text{..............(xvii }bis\text{)},$$

which amounts to saying

$$_{xy}r_{C_x C_y} = 0$$

when that partial correlation coefficient is expressed in the ordinary linear form‡.

* As far as I can follow Mr Wiśniewski, he is only indicating the difference long recognised by biometricians between 'singular' and 'plural' partial correlations.

† All quantities are for the remainder of this note to be measured from the means of the x and y variates.

‡ See footnote p. 118 of my memoir.

But what are the expressly stated conditions under which (xvii *bis*) will be a close approximation to the true result?

(i) The most important is high contact at both end terminals of the total ranges, i.e. cases in which the Sheppard corrections are adequate. This condition has been entirely forgotten or disregarded by Mr Wiśniewski, and he chooses as a suitable example a case where there is abruptness of a high order at one terminal of both the x and y distributions!

(ii) In deducing (xvii *bis*) we have nowhere assumed that the regressions of C_x or C_y on x or y are linear. The regression of x on C_x and of y on C_y are a series of points on 45° lines. The regressions of C_x and C_y on x and y respectively are staircases, the midpoints of the treads being on 45° lines, approaching more and more nearly to straight lines as the sub-ranges h and k are reduced. But these do not for the present concern us*. The only assumption of linearity made is that of x on y (see top of p. 123 of my memoir). By the nature of things this must be closely fulfilled in Mr Wiśniewski's example.

(iii) Is there any other assumption made in deducing (xvii *bis*)? Yes, one that is again not referred to by Mr Wiśniewski. (xvii *bis*) is a very approximate formula, but it depends on fourth order terms being negligible as compared with second order terms. The term in question is

$$\frac{1}{576}\frac{hk}{\sigma_x\sigma_y} S\left(\frac{n_{st}}{N}\,\frac{n_{s+1}-n_{s-1}}{n_s}\,\frac{n_{t+1}-n_{t-1}}{n_t}\right).$$

Evaluating this term numerically for Mr Wiśniewski's data, we find that it amounts approximately to ·0005, or will only influence the fourth decimal place in the correlation coefficient, while the Sheppard corrections will modify the second decimal place. Accordingly we may consider our results to be sufficiently approximate, if we neglect this fourth order term in this example as suggested on p. 122 of the memoir.

It follows that the reason for Mr Wiśniewski's getting corrections which raise the corrected correlation above unity lies in his overlooking the condition (i) of applying the formula (xvii *bis*), namely, that there must be high contact at both terminals of the marginal totals. When the proper corrections for abruptness are made in the means and standard deviations, then the coefficient of correlation does not exceed unity, but on the contrary is close to the value obtained by the long process.

It may be as well here to show the relation of (xvii) to the closely approximate formula

$$r_{xy}=r_{yC_x}/r_{xC_x} \quad \text{..............................(xviii).}$$

* Except in so far as the first set provides another easy proof of (i). We have: Mean value of x for a given $C_x = C_x$ but it also equals since regression is linear $\frac{r_{xC_x}\sigma_x}{\sigma_{C_x}}\times C_x$. Accordingly r_{xC_x} must equal σ_{C_x}/σ_x.

Both formulae follow at once, if we assume linear regressions to hold for all the variables x, y, C_x, C_y. But these not being all linear, my memoir discusses the hypotheses under which they will hold with a very considerable degree of approximation, namely, (i) high contact, (ii) the regressions of x and y linear, and (iii) the product and squares of $\frac{1}{12}h^2/\sigma_x^2$, $\frac{1}{12}k^2/\sigma_y^2$ to be negligible as compared with those quantities themselves. Under these conditions, (xviii) will now be shown to be an approximate as (xvii *bis*).

Starting from (xvii) and using the approximate expressions $\bar{x}_s$ and $\bar{y}_t$ in terms of x_s and y_t on p. 119 of my memoir, we can at once write it in the form

$$r_{xy}\frac{\sigma_{C_x}}{\sigma_x} = r_{xy}r_{xC_x} = \frac{S\left(\frac{n_{st}}{N}C_x\bar{y}_t\right)}{\sigma_{C_x}\sigma_y}\frac{\sigma_y^2}{\sigma_{C_y}^2} = \frac{1+\frac{k^2}{12\sigma_y^2}}{\sigma_{C_x}\sigma_y}S\left\{\frac{n_{st}}{N}C_x\left(y_t+\frac{1}{24}k\frac{n_{t+1}-n_{t-1}}{n_t}\right)\right\}$$

$$= \left(1+\frac{k^2}{12\sigma_y^2}\right)\left[r_{C_xy}+\frac{1}{24}\frac{k}{\sigma_{C_x}\sigma_y}S\left(\frac{n_{st}}{N}C_x\frac{n_{t+1}-n_{t-1}}{n_t}\right)\right]$$

$$= \left(1+\frac{k^2}{12\sigma_y^2}\right)\left[r_{C_xy}+\frac{1}{24}\frac{k}{\sigma_{C_x}\sigma_y}S\left\{\frac{n_{st}}{N}\left(x_s+\frac{n_{s+1}-n_{s-1}}{n_s}\frac{1}{24}h\right)\right\}\right]\frac{n_{t+1}-n_{t-1}}{n_t}$$

$$= \left(1+\frac{k^2}{12\sigma_y^2}\right)\left[r_{C_xy}+\frac{1}{24}\frac{k}{\sigma_{C_x}\sigma_y}S\left(\frac{n_{st}x_s}{N}\frac{n_{t+1}-n_{t-1}}{n_t}\right)\right.$$

$$\left.+\frac{1}{576}\frac{hk}{\sigma_{C_x}\sigma_y}S\left(\frac{n_{st}}{N}\frac{n_{s+1}-n_{s-1}}{n_s}\frac{n_{t+1}-n_{t-1}}{n_t}\right)\right].$$

In the second summation we may replace σ_{C_x} by σ_x for $\sigma_{C_x}=\sigma_x\sqrt{1-\frac{1}{12}h^2/\sigma_x^2}$ and the $\frac{1}{12}h^2/\sigma_x^2$ may be neglected in a turn of the fourth order. Accordingly the third summation is negligible, and even in Mr Wiśniewski's case only contributes roughly ·0005 to the right-hand side of the equation.

Now consider the first summation, keeping t constant or sticking to a definite y, $\frac{S(n_{st}x_s)}{n_t}$ = mean x for a given $y_t=\bar{x}_{y_t}$, but if x and y have linear regression, then

$$\bar{x}_{y_t} = r_{xy}\frac{\sigma_x}{\sigma_y}y_t,$$

and the first summation becomes

$$-\frac{1}{24}\frac{k}{\sigma_{C_x}\sigma_y}r_{xy}\frac{\sigma_x}{\sigma_y}S\frac{1}{N}y_t(n_{t-1}-n_{t+1})$$

$$= -\frac{1}{24}k\sqrt{1+\frac{1}{12}\frac{h^2}{\sigma_x^2}}\frac{1}{\sigma_y^2}r_{xy}\left[S\frac{1}{N}(y_{t-1}n_{t-1}-y_{t+1}n_{t+1})+kS\frac{n_{t-1}}{N}+kS\frac{n_{t+1}}{N}\right].$$

If there be high contact at the terminals, *and in this case only*, the first summation vanishes, and the second and third equal unity. Accordingly, neglecting as before terms of the fourth order of small quantities, the first summation becomes

$$-\frac{1}{12}\frac{k^2}{\sigma_y^2}r_{xy},$$

which is equal to

$$-\frac{r_{xy}}{12}\frac{k^2}{\sigma_y^2}\frac{\sigma_{C_x}}{\sigma_x}\sqrt{1+\frac{1}{12}\frac{h^2}{\sigma_x^2}} = -\frac{r_{xy}}{12}\frac{k^2}{\sigma_y^2}\frac{\sigma_{C_x}}{\sigma_x}$$

to our degree of approximation. Thus

$$r_{xy}\frac{\sigma_{c_x}}{\sigma_x} = \left(1+\frac{k^2}{12\sigma_y^2}\right) r_{c_x y} - \frac{k^2}{12\sigma_y^2} r_{xy}\frac{\sigma_{c_x}}{\sigma_x},$$

or

$$r_{xy} r_{xc_x}\left(1+\frac{k^2}{12\sigma_y^2}\right) = \left(1+\frac{k^2}{12\sigma_y^2}\right) r_{yc_x},$$

or, finally,

$$r_{xy} r_{xc_x} = r_{yc_x},$$

which are to be proved. In other words, with our assumptions we get the two relations

$$r_{yc_x} = r_{xy} r_{xc_x} \text{ and } r_{xy} = r_{c_x c_y}/(r_{xc_x} r_{yc_y})$$

as close approximations; these relations are those of linear regression singular partial correlation coefficients.

The conditions of these results holding are stated in my original memoir, the *most important one being that of high terminal contacts, which has been entirely* overlooked by Mr Wiśniewski. It has long been known that with very high correlations r_{xc_x} and r_{yc_y} may occasionally over correct. But in almost every case where they do so, it will be seen, or it may be suspected, that there is abruptness at some one or other terminal which must be allowed for. We have shown how this may be done in the example selected as an illustration of failure by Mr Wiśniewski. I have, however, to thank him for his paper, because it has enabled me to emphasise again the conditions under which my formulae are approximations, and to indicate those formulae which are absolutely true. It seems to me on re-reading my memoir that I have clearly therein stated these points, but if Mr Wiśniewski has not grasped them, others may likewise fail to do so. I should welcome any alternative means of correction for broad categories when high contact fails, but more than twenty years' experience has shown me that the class index corrections give very reasonable results when high contact limits the frequency distribution.

THE ANGLO-SAXON SKULLS FROM BIDFORD-ON-AVON, WARWICKSHIRE AND BURWELL, CAMBRIDGESHIRE, WITH A COMPARISON OF THEIR PRINCIPAL CHARACTERS AND THOSE OF THE ANGLO-SAXON SKULLS IN LONDON MUSEUMS.

By J. C. BRASH, DORIS LAYARD AND MATTHEW YOUNG.

PART I. THE BIDFORD-ON-AVON SKULLS*.

By J. C. Brash and Matthew Young.

CONTENTS.

PART I.

APPENDIX TO PART I.

* By the courtesy of the Bidford Co-operative Society the skeletal remains are permanently preserved in the Museums of Human and Dental Anatomy in the University of Birmingham.

1. *Introduction.* The bones which are the subject of this communication were first brought to light in June 1921 during the cutting of a new road at Bidford-on-Avon, Warwickshire. In the course of this work a number of skeletons were disinterred along with various articles which indicated the site of an Anglo-Saxon Cemetery. Subsequently it was revealed that the cutting for the road traversed an extensive cemetery, but in 1921 the circumstances were such that no accurate note was preserved of the position of the various graves. The small number of bones available as the result of this initial discovery were merely collected together, many broken in the course of the digging, and, since the relation of the limb and other bones to the skulls was not determined, the exact number of graves opened during 1921 cannot be accurately known.

Arrangements were made, however, for a controlled excavation of the site, which was carried out during the following summers of 1922 and 1923. Details of the methods and course of the excavations, with a discussion of the site, its relation to other Anglo-Saxon cemeteries and of the evidence which suggests that the Bidford Cemetery is to be assigned to the early part of the sixth century, are to be found in two publications in *Archaeologia** by Mr John Humphreys and his collaborators in the work of excavation. The first of these papers includes a map of Bidford showing the exact site of the cemetery which is thus described: "The excavation took place on a site at the back of the High Street, on the north side, in a gravel plateau about 100 yards from the present bank of the river Avon, 150 yards from the Roman Rycknild Street, and 200 yards from the old ford, which lies to the east of the church. The site is on the extreme western border of Warwickshire, within a mile of the Worcestershire boundary." There are also plans of the cemetery, a table giving details of the individual graves and of the grave furniture found with each interment, and three excellent photographs of skeletons and grave furniture *in situ*. The papers in *Archaeologia* are mainly occupied with a detailed and splendidly illustrated description of the archaeological material provided in abundance by the excavations; and we cannot do better, for the purpose of linking the archaeological to the anthropological side of the discovery, than quote the paragraph in which the authors summed up the results of their detailed work.

"The evidence brought to light at Bidford, judging by the size of the graveyard already explored, the number of bodies, and the wealth and variety of the grave furniture, suggests a much earlier settlement in the heart of the Midlands than has generally been admitted, and the great number of cremation burials does not agree with the theory that the proportion of cremation burials decreases the further the settlements advanced up the valleys of the Thames and its tributaries. The materials show so strong a resemblance to those found in the Anglo-Saxon settlements in the Thames valley and its head-waters, that we are forced to conclude that the West Saxons, after penetrating the Cotswolds, descended into

* "An Anglo-Saxon Cemetery at Bidford-on-Avon, Warwickshire," by Humphreys, J., Ryland, J. W., Barnard, E. A. B., Wellstood, F. C., and Barnett, T. G. *Archaeologia*, 1924, LXXIII. 89–116.

"Second Report on the Excavations." *Idem, Ibid.* 1925, LXXIV. 271–288.

the Avon valley, and established themselves on the north bank of the river Avon at Bidford quite early in the sixth century."

"Notes on the Cranial and other Skeletal Characters" have already been published as Appendices to the general accounts in *Archaeologia*, and the purpose of the present paper is to provide data and measurements which may be of use to those who may in the future essay the task of surveying all Anglo-Saxon remains discovered in England and coordinating the scattered evidence of the physical features and interrelations of the various groups in different parts of the country*.

Acknowledgments of support by the Society of Antiquarians, the Birmingham Archaeological Society, and the University of Birmingham, and also to those who personally assisted the work, by giving help at Bidford, by contributions to the Excavation Fund, or by assistance in archaeological matters, have already been made in detail in the papers in *Archaeologia*

2. *Number of Interments.* A perusal of the two papers in *Archaeologia* makes it obvious that the Bidford Cemetery, from the archaeological point of view, must take an important place in the list of Anglo-Saxon discoveries. A preliminary statement of the numbers of skeletons which it has yielded will show that it is equally important in its anthropological aspect. In the table that follows, the probable age and sex distribution of the better-preserved remains are indicated; the number of interments represented by fragmentary bones only and the number of urns found are also stated.

BIDFORD-ON-AVON (1921–22–23).

		Age	♂	♀	Sex ?	Totals
Bones	Preserved ...	6	—	—	25	25
		12	—	—	21	21
		20	6	3	3	12
		"*M*"	32	20	20	72
		"*M*+"	16	15	9	40
		"Old"	3	5	2	10
	Fragmentary	"Young"	—	—	12	12
		"Adult"	7	13	*10*	30
Urns	Well-preserved		—	5	26	31
	Fragmentary		—	—	120+	120
		Totals	64	61	248+	373

"*M*" = Mature.

* Contour tracings of six (6) of the Bidford skulls have already been published by Professor F. G. Parsons, "Anglo-Saxon Skull Contours," *Royal Anthrop. Institute, Occasional Papers*, No. 9, 1928. Tracings of twelve (12) others are available in the Library of the Department of Anatomy, University of Birmingham. All these are indicated in the table of individual measurements. Tables of measurements of the limb bones are also available in the Department of Anatomy, University of Birmingham. [The contour tracings referred to above are of no service for comparison with the long series of contours published in *Biometrika*, as the former are *dioptographic*, and so are not true contours, i.e. plane sections of the skull. ED.]

The total number of interments uncovered during the excavations of 1922 and 1923, as determined by the discovery of well-preserved skeletons, of fragmentary bones, and of urns, was 229. The grand total of 373 is reached by adding 24 for the number of interments represented by the bones which were collected in 1921 *, during the cutting of the road which led to the discovery of the cemetery, and 120 for the minimum number of urns discovered in fragmentary condition throughout the period of the excavations. The number assigned to skeletons discovered in 1921 is also a minimum figure, since no record of the position of each skeleton was made and the bones were mixed; so that it has not been possible to assemble the skeletons with any certainty. The number is in fact derived from the minimum number of individuals determined from the skulls and mandibles alone, and agrees fairly well with the extent of the area excavated and the average spacing of interments determined in the controlled area in 1922--23. The total number of interments excavated in 1922—23, which yielded adult bones in a reasonably fit state for accurate observations and measurements, was about 45.

3. *Sexing of Bones.* The sexing of the bones has presented the usual difficulty, from the purely osteological point of view, of the residuum of doubtful specimens. Where whole skeletons have been preserved the pelvic bones have been taken as the most trustworthy indicators, and the information thus obtained of the range of variation of the other bones in skeletons of practically certain sex has been used to guide the decision where pelvic bones were not available. It has indeed been found that secondary sex characters are very well marked. A further and valuable check has been utilised in the character of the grave furniture where recorded, and this evidence alone has been taken for the purpose of the table in a number of instances. In no instance has there been serious conflict between the archaeological data and the osteological indications; the proportion of the sexes shown in the table may therefore be taken as approximately correct. A number which remain doubtful are indicated as such in the tables, but a sex diagnosis has been supplied for the doubtful skulls of which measurements could be taken.

4. *Age Distribution.* The system that has been adopted is to place the adults in groups according to indications supplied principally by the wearing of the teeth and the state of the cranial sutures, and to group the younger skeletons around the ages 6, 12 and 20, according to the state of the eruption of the teeth and the condition of the epiphysial junctions. Beyond a certain point the determination of age must naturally be approximate only. The eruption of the teeth and the condition of the epiphyses are trustworthy guides up to maturity, but beyond that point only a rough grouping is possible. It is in any case probable that, on the average, age changes would be earlier and more pronounced than in the skeletons of modern people. The largest number of skeletons is found in the "Mature" class—fully adult with moderate wearing of the teeth and still open cranial sutures. The "Mature +" class includes those that show commencing signs of

* The skulls discovered in 1921 are designated by letters in the Appendix Tables; those found in 1922 and 1923 by numbers.

ageing—such as the closure of the cranial sutures associated with increased wear and loss of teeth; and those in which these signs are unequivocal are placed in the "Old" class, which includes a fair proportion of the whole.

The youngest skeleton is that of an infant of about nine months. It is not possible to say what the extreme range of age may have been, but we may assume that most of the "Mature" and the "Mature +" classes would be between 25 and 40. Although the majority fall into these two classes, it is obvious that there is a considerable proportion of young persons and an appreciable number of old.

Apart from all other evidence, the conclusion that is inevitable, from a consideration of the total number of individuals represented and their age and sex distribution, is that we are dealing here with the burial place of a community. But in the absence of evidence of the length of time during which the cemetery was used and of knowledge of the probable death-rate, it is not possible to estimate and hazardous to speculate about the size of the community of which it was undoubtedly the burial place.

5. *The Mode of Burial and the Depth of the Graves.* In most of the burials it was possible to determine the position in which the body had been interred. In the majority the grave had been made sufficiently long to allow the body to be placed in it at full length, but in some the skull was found to be pressed down upon the trunk and the foot bones lay alongside the bones of the leg as if the body had been pressed into a grave too short for it. There appears to have been no exception to the rule that the bodies were buried lying on the back, sometimes with an inclination to the left side. The skull was usually found inclined to the left, though in not a few it was turned to the right. In a considerable number the legs had been crossed at the ankles, generally the right over the left; and in the male burials certainly the position of the arms was fairly constant. The right arm was found to be bent at the elbow, the hand usually folded over the chest, but in some skeletons resting on the right shoulder with the elbow fully flexed. The left arm, in contrast, was usually found extended by the side, the hand in some instances resting on the left thigh or on the lower part of the trunk. The different attitude of the two arms is to be explained by the presence of the spear on the right side, the right arm being naturally folded over it to retain it in position. It is even probable, when the right hand was found to be in the vicinity of the right shoulder, that the spear had actually been placed in the grasp of the hand. (*Archaeologia*, 1924, Figs. 2 and 3.)

The average depth of burial was found to be about 3 ft., the minimum being 2 ft. and the maximum 6 ft.

6. *Orientation of the Graves.* Full details of the orientation of the interments are to be found in the *Archaeologia* publications and in the maps accompanying them. Here it need only be mentioned that some 70 % of the bodies were found with their feet pointing between N.W. and N.E., and that the direction between S. and N.W. appeared to have been carefully avoided. In the part of the cemetery

that is judged to be of later date, there appears to have been a distinct tendency to the E. and W. position; and there is no doubt that this orientation is associated with a comparative lack, and, in some instances, a complete absence of grave furniture.

7. *Grave Furniture.* The spear head, the shield umbo, and the knife were characteristic of the male interments; and the women had been buried with their necklaces of amber and glass beads, and jewelled brooches. Occasionally a small pot was found close to the head of a female skeleton on the right side, and usually there was a small knife on the breast or at the side.

It is an interesting fact that the grave furniture was found to be much less plentiful in the later stages of the excavations; and this is believed to represent the later, possibly post-Christian, portion of the cemetery.

8. *Cinerary Urns.* The association of a considerable number of urn burials with ordinary interments is a feature usual in cemeteries of this date, and of very great interest and significance. It has often been stated that urn burial is an Anglian, inhumation a Saxon, characteristic. The proportion of urns might therefore be held to indicate the degree of Anglian admixture; but in all West Saxon cemeteries that have been investigated, some urn-burials have been found. At Bidford thirty-one more or less perfect cinerary urns were found. Like the interments, they were not arranged in any definite order, but were found scattered about irregularly among the skeletons. The urns were found at an average depth of 2 ft., and the fact that fragments of at least 120 were found, in addition to the smaller number of more or less perfect specimens, is taken to indicate that the urn-burials date from an earlier period than the inhumations. It is conjectured that the majority of the urns were then broken up in making the subsequent graves. The evidence, however, is not quite conclusive that the two methods were not in simultaneous use. Full detailed descriptions and illustrations of the cinerary urns, all of which were hand-made, are to be found in the publications in *Archaeologia.*

9. *Associated Animal Remains.* The majority of the animal bones recovered during the excavations were clearly of modern origin, but it is possible that a few of the bones of pigs, horses, sheep and oxen may be contemporary with the human remains.

10. *The State of Preservation of the Bones.* The condition of the bones is very uneven. Many of them are in excellent condition, though porcelain-like and very brittle. Others are very soft and friable, with flaking and crumbling of the surface. Many of these have had to be treated with balsam varnish to prevent them falling to pieces. Portions of the same skull or long bone are also frequently found to exhibit these extreme conditions of preservation. The excavators of the site attribute these facts to the varying nature of the subsoil. The general good preservation is thought to be due to the excellent drainage provided by the gravel deposit, 20 ft. in thickness, upon which the bodies rested. The accumulation of stagnant water was prevented, and the calcareous nature of the gravels is held to

have neutralised acidity from the atmosphere. Scattered throughout were irregular pockets of boulder clay and it is believed that the softening of many of the skulls and other bones was due to contact with these clay deposits, decalcification having been brought about by the action of sour stagnant water unable to drain away from the surface of the boulder clay.

The number of skulls and other bones available for accurate study, on account of the bad preservation of many of them, consequently falls far below the number of interments. When, in addition to this factor, there is taken into account the number of urn burials, it is found that from a cemetery that is estimated with some degree of accuracy to have contained the remains of at least 373 individuals there are available for the determination of their anthropological characters the skulls and other bones of some 70 individuals only.

The total number of skulls available for measurement, many of them incomplete in different respects, is 51, of which 32 are male and 19 female.

11. *Sex Characters.* There is no doubt that secondary sex characters are very well marked in the skulls. There are very obvious differences in the size of the mastoids and in the muscular impressions in the occipital region; and the superciliary eminences are very extensively developed in a number of male skulls. In a few there is even a filling up of the usual triangular depression between the superciliary eminence and the zygomatic process of the frontal bone. This is noticeably the case in No. 107; and in No. 36, an old male skull, much distorted, there is even the appearance of a moderate torus supraorbitalis.

12. *Age Changes.* There is little of special note to record with regard to the age changes in the skulls. It is probable, though the evidence is by no means decisive, that closure of the sutures commenced at a comparatively early age. The first sites, as a rule, appear to be the usual ones at the lower end of the coronal suture with the pterion region generally, and the hinder part of the sagittal suture, though there are some exceptions. Arthritic changes in the region of the condyles and in the glenoid fossae are almost the rule in those skulls with extensive closure of the sutures.

13. *Remarks on Cranial Anomalies.* (i) *Suture Patterns and Sutural Bones.* The patterns of the sutures in general are of average complexity with the usual scattered sutural (Wormian) bones in the lambdoid suture.

(ii) *Metopism.* There are three examples of definite metopism in the series, Nos. 17, 63, 82. In two of these the suture is closed internally but leaving definite evidence of its presence; and in both of them there are also indications of commencing external union. In both, these signs are distinctly in conformity with the age of the skulls, and union is less advanced than in the other sutures.

(iii) *Metopic Ridge.* Two of the skulls, Nos. A and 106, have slight metopic ridges and in one (D) there is a marked metopic boss occupying the position of the glabella.

(iv) *Pterion.* There is only one example of squamoso-frontal suture in the whole series—on the left side of skull B—in which the junction measures 16 mm. There are several examples of epipteric ossicles; single right in No. 92, joined to the sphenoid: double right in No. C; double left in No. 50 ii; and almost symmetrical single on both sides in No. 37.

(v) *Interparietal.* There are sutural bones of fairly large size in the region of the lambda in a number of skulls; a large one occupying the lambda in No. 93, a child; three symmetrically placed in No. 107; three on the right, one of which is sagittal, in No. D; four in No. 50 i; and a series in No. 82. In only one skull (D) are these ossicles large enough to be interpreted as portions of the interparietal bone. Two large ossicles occupy the upper half of the right portion of the squamous occipital and one of them extends across the middle line, being placed immediately behind the sagittal suture. There are in addition two imperfectly closed sutures in the occipital bone running on each side upwards and towards the middle line from below the asterion. These indicate the separation of the interparietal from the supraoccipital portion of the squamous occipital, and the whole is an example of ossification of the interparietal from several centres.

(vi) *Bregmatic Ossicles.* There are two fine examples of bregmatic ossicle (os antiepilepticum). In No. 5, a male, the ossicle lies entirely behind the coronal suture, measures 27 mm. in antero-posterior length by 15 mm. in maximum breadth, and is slightly constricted in the middle. In No. 89, female, the ossicle indents the line of the coronal suture but lies almost entirely behind it. It measures 19 mm. in antero-posterior length by 14 mm. in breadth. In neither of these examples is there any sign of the union of the ossicle. In No. 82, a metopic skull, the interfrontal suture joins the coronal to the right of the sagittal and there is evidence of the former presence of an anterior bregmatic ossicle which has united to the left half of the frontal bone. The association of metopism with the development of a bregmatic ossicle is noteworthy.

(vii) *Asterionic Ossicles.* There are also two examples of large asterionic ossicles, and it is probably more than a coincidence that these occur in the same two skulls as the bregmatic ossicles—Nos. 5 and 89. They are both on the left side and show signs of uniting with the surrounding bones.

(viii) *Obelionic Depression.* There are five examples of a groove along the sagittal suture in the region of the obelion. In three of these (Nos. 40, 75 i, and 78) the groove is well marked, in one of them (No. 75 i) it extends to within 2 cm. of the bregma and is also continued on the occipital bone as a slight depression from lambda to inion. The other two examples (Nos. 46 and 66) are slight, and in No. 46 it is associated with symmetrical depressions extending on each side along the lambdoid suture. One skull (No. C) exhibits a fairly obvious sagittal crest.

(ix) *Post-coronal Sulcus.* There is no definite example of this in the whole series.

(x) *Coronal Ossicle.* There is one example of ossicle in the coronal suture, on the right side just below the temporal line; this occurs in No. 82, already noted as a metopic skull with an anterior bregmatic ossicle.

(xi) *Carotico-Clinoid Foramen.* In No. 37, owing to damage to the skull, it may be observed that there is a complete carotico-clinoid foramen on both sides. It may be noted that this is associated with symmetrical epipteric ossicles.

(xii) *Paroccipital and Paramastoid Processes.* There is a paroccipital process in No. C and a nipple-like paramastoid process in No. D, both on the right side.

(xiii) *Ossified Trochlea.* The trochlea is ossified in both orbits of No. 60, a skull which does not show any definite signs of ageing.

(xiv) *Divided Malar Bones.* There is no example of divided malar in the series, but it may be noted that both the malars are transversely grooved in No. A.

(xv) *Nasal Aperture.* Some notes on the form of the lower border of the anterior (pyriform) aperture of the nose will be found in the remarks appended to the Table of Measurements. It is worthy of mention here that a complete sharp limiting margin is uncommon in these skulls, that prenasal fossae are present in several, and that in one (No. E) there are infranasal grooves (prenasal sulci) of anthropoid type.

(xvi) *Asymmetry.* As is the case in any collection of skulls, there is a considerable amount of asymmetry to be observed, especially in the region of the base. It is probable that the immediate causes of asymmetry are to be found not only in the well-known asymmetrical development of the occipital lobes of the cerebrum, *but also in individual habits in the "wear of the head" and in the use of the jaws.* The correlation of the different points in which a skull may be asymmetrical would be a very interesting study; one or two points as shown in this series may be indicated. The occipital region has been noted as obviously asymmetrical in 7 skulls; the left is more projecting in 6 of these, and the right in only 1. In 5 of the 6 skulls with left occipital prominence it is also noted that the right mastoid process is larger than the left. Taking the skulls as a whole the mastoids are noted as asymmetrical in 18, and of these the right is larger in 10, the left in 8. The occipital condyles are noted as asymmetrical in 6 skulls, the right condyle being anterior to the left in 2, the left anterior to the right in 4. There is, however, no obvious association of position of the condyle with either occipital prominence or larger mastoid. The glenoid cavities have been noted as asymmetrical in 13 skulls and of these the right is deeper than the left in 5, the left deeper than the right in 8. Again there is no obvious relation between these facts and the other asymmetrical points noticed. The last point in which asymmetry has been noted is in the size of the nasal bones; this is definite in 8 skulls and in all but one of these the right bone is the larger and passes over the median plane at its frontal articulation. It may be surmised that this condition is correlated with forward projection of the frontal bone occurring in skulls with a well-marked left occipital projection; but as a matter of fact in this series asymmetry of the frontal bone itself is not at all obvious, and the asymmetrical nasal bones occur,

with two exceptions, in skulls which do not show left occipital projection, and these two exceptions include the only example of a larger left nasal bone.

(xvii) *Irregularity and Malocclusion of the Teeth.* It should be recorded here that the Bidford series has been the subject of a special investigation of the incidence of irregularities of the teeth and of malocclusion. These conditions, in mild or severe degree, have been found in a surprising number of the skulls; details and discussion will be found in "The Aetiology of Irregularity and Malocclusion of the Teeth" (Dental Board of the U.K. Lectures, J. C. Brash, 1929) and "Some Notes on the Dentitions of Anglo-Saxon Skulls from Bidford-on-Avon with special reference to Malocclusion" (*Trans. B.S.S.O.*, K. C. Smyth, 1933).

14. *Brief Description and Discussion of the Principal Cranial Characters.* The main object of Part I of this paper is to place on record accurate data concerning the cranial and mandibular characters of these Bidford Saxons. The principal measurements of the individual skulls that were sufficiently well preserved for taking at least a number of the measurements included in the scheme of measurement usually followed at the Biometric Laboratory, University College, London, and in accordance with the technique practised there are tabulated in the Appendix with brief notes on any distinctive or peculiar features of particular skulls that seem worthy of mention.

Tables of the average values of these characters for the separate sexes with the numbers of observations on which the averages are based are given in the text on pp. 390—391. As the numbers of skulls on which the means are based rarely exceed and are usually considerably below 20, the standard deviations—the measures of variability—of the characters have not been considered sufficiently representative or reliable for tabulation.

A selected series of the cranial characters will be considered individually and in the aggregate later, when a comparison is made between them and the corresponding characters in the groups of Anglo-Saxon skulls from Burwell in Cambridgeshire and from the London Museums, but some of the more general features of the skulls as a group will be indicated here by reference to a few of the principal cranial characters.

(i) *Cranial Capacity.* In only 11 of the skulls, 6 male and 5 female, did the condition of the specimens warrant the risk of estimating the cubic capacity by the direct method, No. 8 shot having been used as a medium. These are entered in the table of measurements of individual skulls. For the 6 male skulls the average cubic capacity was 1553 c.c. and for the 5 female skulls 1378 c.c. The mean capacities for male and female skulls in the series were also estimated from the product of the serial means of the three dimensions, greatest length (L), maximum breadth (B) and basio-bregmatic height (H') by using Miss Hooke's formulae*, namely:

$$♂\quad C\,(\text{in c.c.}) = \cdot 000{,}366LBH' + 198{\cdot}87,$$
$$♀\quad C\,(\text{in c.c.}) = \cdot 000{,}366LBH' + 199{\cdot}43.$$

* *Biometrika*, Vol. XVIII. (1926), p. 33.

The values thus obtained, 1595·7 for the male and 1411 c.c. for the female, are rather in excess of the mean capacities obtained from the smaller numbers of specimens by the direct method*.

(ii) *Cephalic Index* (100 B/L). Omitting one or two skulls which were obviously distorted, the cephalic indices were obtained for 29 skulls, of which 10 were female. *In the males the index ranges from 69·5 to 77·8 with a mean value of 73·5*; in the females the range is from 68·4 to 76·9 with a mean of 73·8. The distribution of the skulls according to type is as follows:

Class	Range	Numbers			Per cent.
		♂	♀	♂ and ♀	♂ and ♀
Dolichocephalic	− 75	13	6	19	65·5
Mesaticephalic	75—77·4	3	4	7	24·1
	77·5—79·9	3	0	3	10·3
Brachycephalic	80 +	0	0	0	0

Nineteen of the skulls, 66 %, fall below the 75 mark and are therefore to be reckoned definitely dolichocephalic; the remainder fall within the mesaticephalic class, not a single brachycephalic skull being included in the series. It is also of interest to note that on a division of the mesaticephalic class into two subclasses at 77·5, as suggested by Sir William Turner, in order to indicate the affinities in the intermediate group, the proportion that may be considered to show a distinctly long-headed tendency is almost 90 % of the total number. We are therefore clearly dealing with a sample of a strongly dolichocephalic community.

Length-height Index (100 H'/L). The classification of the skulls according to this index is as follows:

Class	Range	Numbers			Per cent.
		♂	♀	♂ and ♀	♂ and ♀
Chamaecephalic	− 70	0	2	2	11·1
Orthocephalic	70—75	10	2	12	66·7
Hypsicephalic	75 +	1	3	4	22·2

* [It should be noted here that Miss Hooke's formulae are based on the Biometric Laboratory method, which involves the use of *mustard* seed. The results obtained by the use of shot are not comparable with those obtained by mustard seed, witness the exaggerated values obtained by Broca's use of shot. Nor is it legitimate to put into the product term in Miss Hooke's formulae the product of the serial means, for the product of the means is not the mean product. ED.]

From this point of view the skulls are predominantly of the orthocephalic or intermediate type. In the males the length-height index ranges from 70·0 to 76·2 with a mean value of 71·8, and in the females from 66·8 to 76·9 with a mean value of 72·4.

Height-breadth Index (100 B/H'). For this index we have the following distribution:

Class	Range	Numbers			Per cent.
		♂	♀	♂ and ♀	♂ and ♀
Hypsistenocephalic	−100	3	2	5	29·4
Platychamaecephalic	100+	8	4	12	70·6

The majority thus fall into the platychamaecephalic class. In the males the index ranges from 98·5 to 109·3 with a mean value of 102·2; in the females the range is from 96·3 to 112·2 with a mean value of 103·5.

The impression conveyed from a study of the foregoing cranial indices is that we are dealing with skulls in which on the whole a low mean cephalic index, i.e. dolichocephaly, is associated with a definite preponderance of breadth over height.

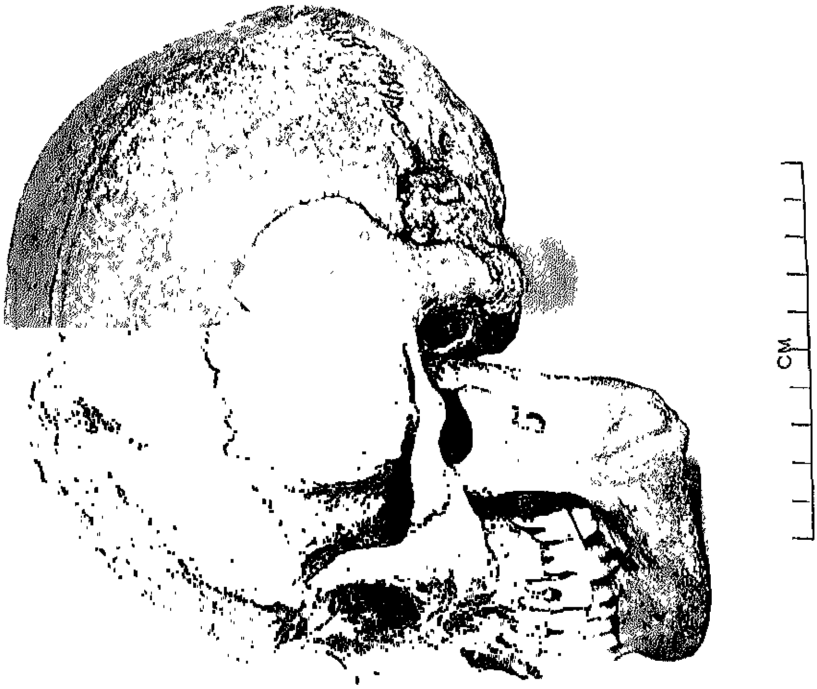
It may be recalled here that this combination of features was pointed out by Sir Wm Turner as characteristic of his series of Scottish skulls*. He then remarked: "A striking feature of the Scottish crania, therefore, was the preponderance of the cephalic index over the vertical index, notwithstanding the considerable number of dolichocephalic skulls in the series, and in this respect the crania favoured the brachycephalic rather than the dolichocephalic type. The Scottish skulls are platychamaecephalic."

Upper Facial Index (100 $G'H/GB$). According to the value of this index, the skulls may be distributed as follows:

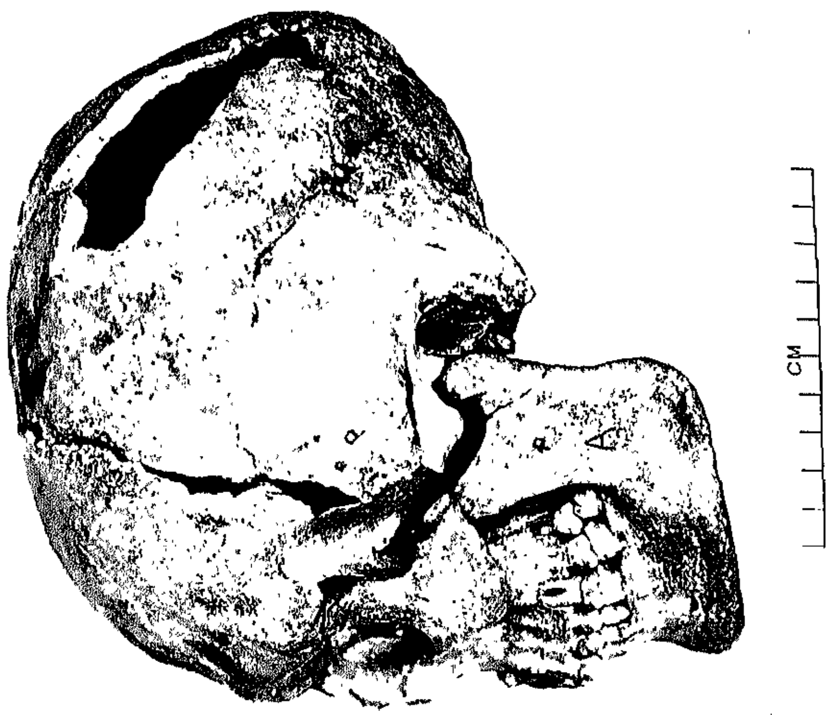
Class	Range	Numbers			Per cent.
		♂	♀	♂ and ♀	♂ and ♀
Chamaeprosopic	−75	12	7	19	76·0
Leptoprosopic	75+	5	1	6	24·0

The majority of the specimens thus show a relatively broad face. In the males the index varies from 65·3 to 80·3 with a mean value of 72·4; in the females the range is from 64·4 to 77·1 with an average of 71·5.

* *Trans. Roy. Soc. Edin.*, Vol. XL. Part III.



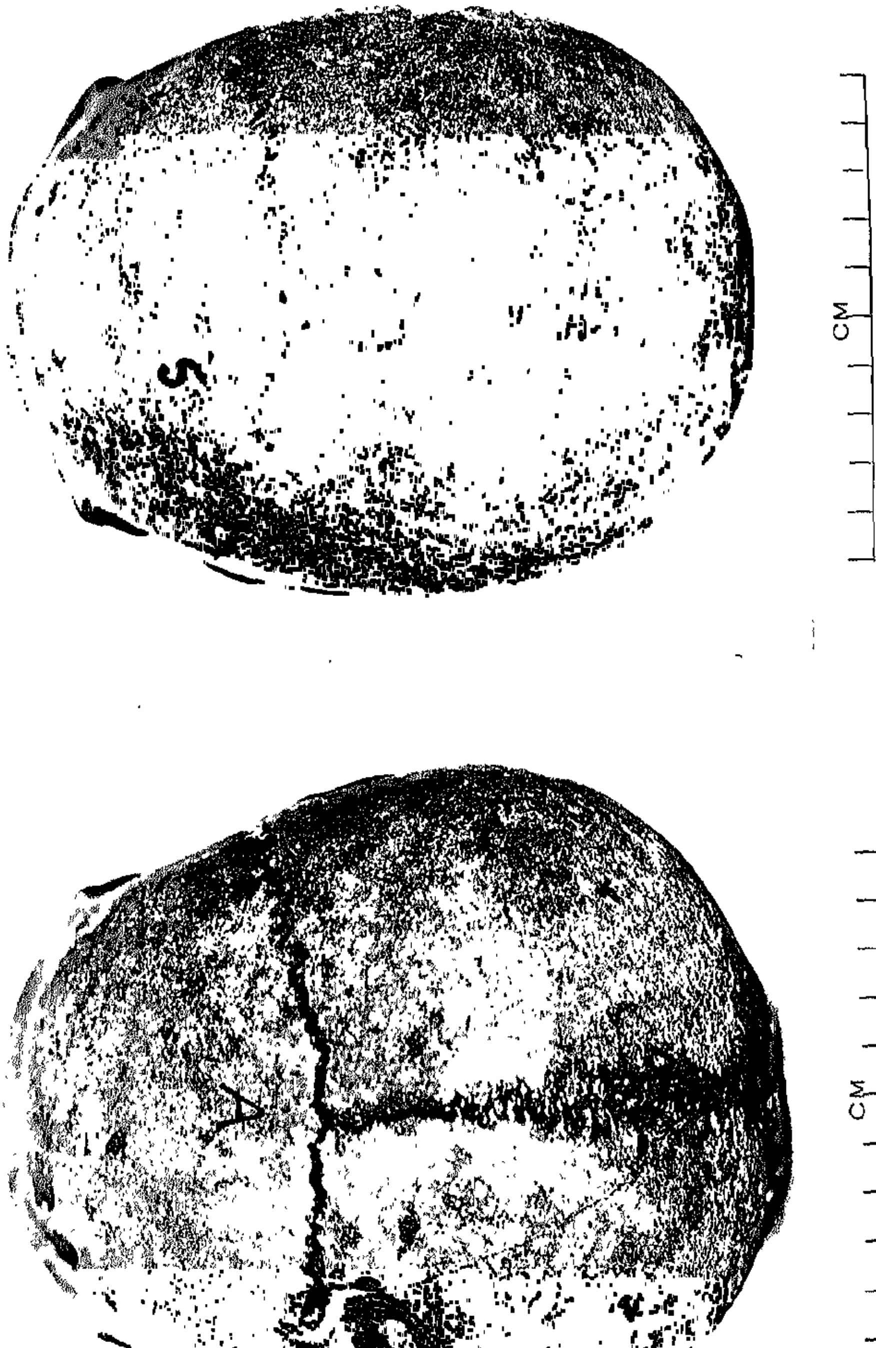
CM
No. 5



CM
No. A

Normae laterales of Male Anglo-Saxon Skulls from Bidford-on-Avon.

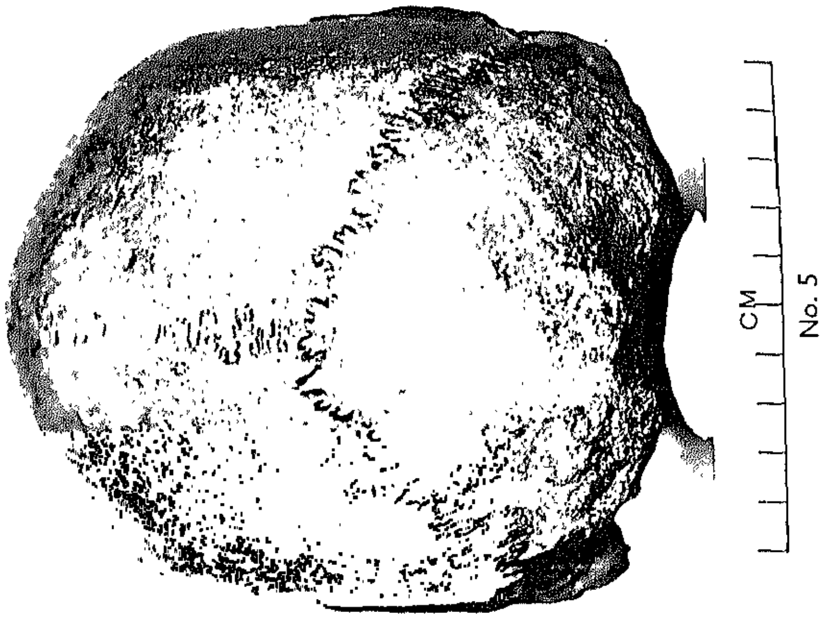
Normae faciales of Anglo-Saxon Skulls from Bidford-on-Avon.



No. 5

No. A

Normae verticales of Anglo-Saxon Skulls from Bidford-on-Avon.



Normae occipitales of Anglo-Saxon Skulls from Bidford-on-Avon.

When it was decided to take detailed measurements of the adult mandibles, 50 specimens, 31 of which were identified as ♂ and 19 as ♀, were found to be sufficiently complete to provide a number at least, if not the majority, of the usual measurements.

These mandibles were measured in accordance with the scheme of measurement now followed at the Biometric Laboratory, University College, London. As compared with the original scheme which is described in detail in *Biometrika*, Vol. XIV, pp. 253—260, the number of characters of which measurements are taken is considerably reduced. We are indebted to Dr G. M. Morant for the abbreviated list of measurements now usually taken. The measurements of the individual mandibles are given in detail in the Appendix (Tables III and IV). The mean values of the characters measured and the indices computed from these with the numbers of observations on which the averages are based are shown for males in Table VI and for females in Table VII. A detailed comparison of the mandibular characters individually and in the aggregate with the corresponding characters in the two series of Anglo-Saxon mandibles from Burwell and from the London Museums will be given later (pp. 398—404).

Serial Letter or No.	O_1, B	G_1	G_2	Remarks.
A*	·0	·4	45	…and reconstructed. Nasal bones asymmetrical. Malars transversely grooved. Base distinctly … occipital region projects. Right mastoid slightly larger than left. Slight metopic ridge.
B*		·9	42·0	…infantile type. Left squamoso-frontal suture 16 mm. long. Right spheno-parietal suture 5·5 mm. …te of irregular outline at antero-inferior angle of left parietal. Right jugular foramen very large.
C*	·0	·0	·5	…on right side. Sagittal crest. Marked asymmetry. Small right paroccipital process.
D*	·5	·5	4	…metrical. Right paramastoid process. Marked metopic boss. Three lambdoid ossicles.
F		—	—	
H		—	—	
J		—	—	
2		·5	39·5	…symmetrical nasal bones.
5		·0	·0	…al. Well-marked prenasal fossae. Bregmatic ossicle. Large left asterionic ossicle.
6		—	—	
17		·0	37·5	…nostosed at tips.
24		—	—	
36		—	—	…iary ridges uniting with supraorbital to form incipient torus.
37		47·0	34·5	…omplete carotico-clinoid foramina. Epipteric ossicles.
48		47·0	37·0	
50		—	—	…l at lambda.
60		44·5	35·0	…l bones asymmetrical. Occipital region distorted.
66		50·5	38·0	…slight obelionic depression. Teeth very much worn.
70		—	—	
78		47·0	37·0	
84		49·0	—	United anterior bregmatic ossicle. Right coronal ossicle. Small sutural bones at lambda.
84		49·5	44·5	
92	34·	—	39·5	Right epipteric ossicle.
106		47·0	43·0	…al fossae. Marked occipital asymmetry. Slight metopic ridge.
107	30·	50·5	36·0	Three lambdoid ossicles.
110	—	44·0	37·0	
11		—	—	
12		—	—	
14	35·	47·2	37·2	
15	33·	45·0	36·8	
16	31·	—	—	
18	30·	—	—	

…ographs of the cont…

…URNTS OF …

Serial Letter or No.	G_1, B'	G_1	G_2	Remarks.
	3·5	42·0	40·	…dropoid type.
	—	—	—	…ural bones in left lambdoid suture.
	2·0	—	—	
	98·0	46·0	40·	
	—	—	—	…. Lambdoid depressions.
	101·5	46·0	37·	…ing on palate.
	100·5	45·0	37·	…trical nasals. Left prenasal fossa.
	98·0	49·5	41·	
	97·0	47·5	35·	…. Prenasal fossae.
	—	—	—	
	—	—	—	…icle. Bregmatic ossicle, 19 mm. long almost entirely behind the coronal suture. Marked …l region.
	93·5	42·0	32·	

APPENDIX III. INDIVIDUAL MEASUREMENTS OF MALE ANGLO-SAXON MANDIBLES FROM BIDFORD-ON-AVON.

Serial Letter or No.	w_1	g_og_o	c_rc_r	zz	c_yl	ml	c_pl	rb'	m_2p_1	h_1	m_2h	c_rh	rl	∠ M	∠ R	∠ C'	$100\, c_rh/ml$	$100\, c_rc_r/ml$	$100\, g_og_o/c_pl$	$100\, g_og_o/c_rc_r$	$100\, rb'/rl$
A	—	108·0	104·5	48·0	22·0	102·0	82·0	32·0	29·5	32·5	31·0	81·0	67·5	111·0	89·0	72·5	79·4	102·5	131·7	103·3	47·4
F	123·5	92·0	105·0	45·5	20·5*	105·0	86·0	34·0	28·5	38·0	29·5	76·0*	71·0*	110·0	82·0	67·0	72·4	100·0	107·0	87·6	47·9
F_1	—	—	—	48·5	—	—	82·0	36·5*	28·0	31·0	32·5	69·0*	—	115·0	—	60·0	—	—	—	—	—
K	—	93·5	95·0	—	—	101·0	79·0	33·0*	27·5	32·5	25·5	63·0	—	115·0	—	67·0	62·4	94·1	118·4	98·4	—
P	125·0	95·0	106·0	45·0	24·0	107·0	79·0	33·0	29·5	35·5	30·5	75·0	70·0	116·0	79·0	73·0	70·1	99·1	120·3	89·6	47·1
2a	125·0	99·5	108·0	48·0	22·5	107·0	82·0	36·0	27·5	29·0	28·5	70·0	70·0	116·0	74·0	66·5	65·4	100·9	121·3	92·1	51·4
3	114·0	94·0	102·0	44·5	20·5	103·0	76·0	30·5	27·5	32·5	27·0	68·5	60·0	120·0	78·0	66·0	66·5	99·0	123·7	92·2	50·8
5	123·0	96·0	107·5	48·5	21·5	104·0	78·0	32·0	29·0	31·0	27·5	72·0	73·0	115·0	72·0	79·0	69·2	103·4	123·1	89·3	43·8
17	116·5	82·0	97·5	40·5	19·5	101·0	75·0	29·0	—	27·5	—	63·5	60·0	121·0	73·0	59·0	62·9	96·5	109·3	84·1	48·3
48	—	81·0	—	43·5	—	122·0	76·0	32·0*	27·5	31·5	28·5	66·0*	71·0*	135·0	60·0	—	54·1	—	106·6	—	45·1
50	115·5	85·5	87·5	44·0	—	101·0	73·5	30·5	29·5	30·0	28·0	66·0*	68·5*	118·0	68·0	70·0	65·3	86·6	116·3	97·7	44·5
58	102·0	92·0	86·0	42·5	—	110·0	74·0	29·0*	29·5	30·0	27·5	57·0*	60·0*	128·0	61·0	66·0	51·8	78·2	124·3	107·0	48·3
60	—	—	—	43·0	—	—	78·0	37·5*	28·0	31·0	28·5*	62·0*	—	120·0	—	71·0	—	—	—	—	—
66	130·5	103·0	105·0	50·5	20·0*	100·0	82·0	32·5	28·5	33·0	29·0	74·0	66·5*	110·0	88·0	62·5	74·0	105·0	125·6	98·1	48·9
70	110·5	96·0	88·0	42·5	—	92·0	61·0	26·5	28·0	26·0	23·0	60·0	60·0	125·0	69·0	60·0	54·3	95·7	157·4	109·1	44·2
78	108·5	102·5	94·0	44·0	18·0	107·0	77·0	27·5	—	27·5	—	57·0*	57·0	129·0	—	—	53·3	87·9	133·1	109·0	48·2
82	127·0	106·5	—	51·0	—	110·0	77·0	33·0	27·5*	34·0	31·0*	65·5	60·0	124·0	77·0	68·0	59·5	—	138·3	—	55·0
84	116·0	92·0	97·0	51·0	22·5	112·0	79·0	33·0	30·5	39·5	33·0	—	62·0	125·0	—	75·0	—	86·6	116·5	94·8	53·2
92	112·5	90·0	92·0	44·5	18·5	111·5	78·0	34·5	26·5	36·5	35·5	66·5	72·5	120·0	64·0	71·0	57·8	80·0	115·4	97·8	47·6
106	118·5	101·5	91·5	43·5	18·5	103·0	77·5	32·0	29·0	29·5	30·0	56·0	64·0	113·0	55·0	68·5	54·4	88·8	131·0	110·9	50·0
107	120·0	97·0	97·0	49·5	23·5	109·0	86·0	35·0	29·0	34·0	30·5	76·0	71·5	116·0	73·5	69·0	69·7	89·0	112·8	100·0	49·0
113	—	94·5	100·5	46·0	—	111·0	83·0	34·0	29·0	33·5	27·5	64·0	—	141·0	—	68·5	57·7	90·5	113·9	94·0	—
116	—	95·5	99·0	46·5	16·5	112·0	82·0	32·5	28·5	35·5	29·5	—	67·5	121·0	—	64·5	—	88·4	116·5	96·5	48·1
118	—	—	99·5	48·5	18·0*	113·5	79·0	33·0*	26·5	32·5	30·0	80·0*	71·0*	123·0	82·0	—	70·5	87·7	—	—	46·5
127	—	103·0	—	48·0	23·5	105·0	73·0	32·0	27·5	32·5	29·5	—	66·5	127·0	—	73·5	—	—	141·1	—	48·1
147	124·0	102·5	—	49·0	20·5	106·0	82·0	36·0	28·5*	32·0	29·5	66·0*	64·0	117·0	73·0	69·0	62·3	—	125·0	—	56·3
157	—	92·5	85·0	42·5	16·0*	101·0	76·0	32·0*	26·5	34·0	23·0	55·0*	52·0*	124·0	68·0	66·5	54·5	84·2	121·7	108·8	61·5
169	127·0	103·0	103·0	46·5	23·0	107·0	75·0	33·0	31·0	38·0	31·0	64·0	63·0	123·0	69·0	73·0	59·8	96·3	137·3	100·0	52·4
172	—	—	—	45·5	—	—	—	—	26·0*	30·0	28·0*	63·5*	—	—	—	73·0	—	—	—	—	—
183	102·0	80·5	83·0	44·0	19·5	101·0	74·0	32·5	30·5	28·0	24·0	57·0	64·0	117·0	60·0	68·5	56·4	82·2	108·8	97·0	50·8
185	112·0	90·0	104·0	47·5	22·0	106·0	82·0	34·5	26·5	31·5	30·0	72·0	70·0	115·0	72·0	69·0	67·9	98·1	109·8	86·5	49·3

* These measurements were taken on the right side.

APPENDIX IV. INDIVIDUAL MEASUREMENTS OF FEMALE ANGLO-SAXON MANDIBLES FROM BIDFORD-ON-AVON.

Serial Letter or No.	w_1	g_og_o	c_rc_r	zz	c_yl	ml	c_pl	rb'	m_2p_1	h_1	m_2h	c_rh	rl	∠ M	∠ R	∠ C'	$100\, c_rh/ml$	$100\, c_rc_r/ml$	$100\, g_og_o/c_pl$	$100\, g_og_o/c_rc_r$	$100\, rb'/rl$
E	—	89·0	91·5	45·5	—	—	74·0	31·0	28·0	28·5	28·0	66·0*	—	116·0	—	65·0	—	—	120·3	97·3	—
G	102·0	85·0	89·5	44·5	18·0*	103·0	72·0	30·0	29·0	27·5	25·5	63·0*	58·0*	125·0	76·0	69·5	61·2	86·9	118·1	95·0	51·7
L	—	88·5	—	41·0	23·0*	107·0	77·0	31·5	—	27·0	—	64·0	57·0*	123·0	—	62·5	59·8	—	114·9	—	55·3
2	—	—	102·0	45·5	—	—	—	31·0	27·5	31·5	26·5	—	—	—	—	71·5	—	—	—	—	—
13	112·0	91·5	92·0	41·5	—	103·0	78·0	26·0	28·0	32·5	25·5	55·0*	65·5*	120·0	70·0	63·5	53·4	89·3	117·3	99·5	39·7
27	125·5	100·5	100·0	51·0	20·0	104·0	81·0	36·0	28·5	30·0	30·5	66·0	59·0	118·0	73·5	73·0	63·5	96·2	124·1	100·5	61·0
33	—	86·0	104·0	46·0	—	94·0	73·0	32·5	28·0	30·0	27·0	65·0	60·0	116·0	78·0	77·0	69·1	110·6	117·8	82·7	54·2
46	—	98·0	95·5	44·5	20·0	105·0	79·0	34·0	26·0*	29·0	26·0	60·0	57·5	130·0	66·0	—	57·1	91·0	124·1	102·6	59·1
56 i	102·0	78·5	90·0	44·5	16·0	97·0	73·0	30·0	27·5	27·5	24·0	57·0	61·0	115·0	63·0	74·0	58·8	82·5	107·5	87·2	49·2
56 ii	125·0	91·5	94·0	49·0	21·0	107·0	81·0	32·5	28·5	30·0	30·5	62·0*	67·0	115·0	61·0	73·5	57·9	87·9	113·0	97·3	48·5
63	116·5	—	89·0	46·0	20·5	112·0	82·0	37·0*	26·5	30·5	29·5	63·0*	63·0	122·0	64·0	66·5	56·3	79·5	—	—	58·7
75	112·0	90·0	—	45·5	20·0	91·0	76·0	31·0	25·5*	26·0	26·0	66·0*	66·0	105·0	80·0	71·0	72·5	—	118·4	—	47·0
89	—	85·5	90·0	46·0	—	90·0	71·0	32·5	29·5	26·0	26·0	62·0	—	120·0	—	60·0	68·9	100·0	120·4	95·0	—
90	—	—	—	45·5	—	102·0	73·0	30·0	27·0	29·0	26·0	58·0	—	120·0	—	67·0	56·9	—	—	—	—
115	112·0	98·0	91·5	49·0	21·0	107·0	77·0	29·0	29·0	30·5	27·0	68·0	60·0	125·0	68·0	73·0	63·6	85·5	127·3	107·1	48·3
121	—	102·5	103·0	52·5	20·5*	106·0	80·0	32·5*	29·5	31·5	26·5	67·0*	67·0*	117·0	74·0	73·5	63·2	97·2	128·1	99·5	48·5
152	121·0	91·0	90·0	49·0	24·0	109·0	84·0	37·0	28·5	30·5	27·5	65·0	62·0	120·0	74·0	65·0	59·6	82·6	108·3	101·1	58·1
178	108·0	85·5	87·5	45·0	19·5	104·0	79·0	33·0	27·0	30·0	25·0	49·0	51·0	123·0	61·0	78·5	47·1	84·1	108·2	97·7	64·7
181	119·0	92·0	95·0	44·0	21·5*	106·5	72·0	25·0*?	27·5	30·5	25·0	60·0	68·0*	128·0	65·0	—	56·3	89·2	127·8	96·8	36·8?

* These measurements were taken on the right side.

PART II. THE BURWELL SKULLS, INCLUDING A COMPARISON WITH THOSE OF CERTAIN OTHER ANGLO-SAXON SERIES.

By Doris Layard and Matthew Young.

CONTENTS.

PART II.

APPENDIX TO PART II.

1. *Introduction.* Burwell lies to the north-east of Cambridge, on the borders of Cambridgeshire and Suffolk. The presence of a burial-place in this situation was first discovered in 1884, when fourteen skeletons with funerary objects were unearthed in the process of working the Victoria Lime Pits which are cut in the side of a low chalk hill that rises here out of the Fen. The land adjoining the pits is let out to allotment holders and belongs to Dr Charles Lucas, who, in 1925, requested the Cambridge Antiquarian Society to undertake excavations. These were carried out in five successive seasons under the direction of Mr T. C. Lethbridge, who published interim reports of his results in the *Cambridge Antiquarian Society's Communications*, Vols. XXVII, XXVIII, XIX and XXX, 1926—1929. A detailed account of them appears in *Recent Excavations in Anglo-Saxon Cemeteries in Cambridgeshire and Suffolk**.

* T. C. Lethbridge. *Cambridge Antiquarian Society Quarto Publications*, New Series, No. III. Cambridge, 1931.

During the excavations about 125 graves were exposed. The skeletal material was deposited in the Anatomy Museum at the University of Cambridge, and the grave-goods are preserved in the University Museum of Archaeology and Ethnology.

The graves were scattered irregularly about the site, and in no case encroached on one another; Mr Lethbridge suggests that each may have been covered formerly by a low mound or barrow, which would have served to mark its position. The graves were shallow and cut in the chalk, their depth being determined by the thickness of the soil above the chalk. In the majority, the bodies were orientated with heads towards the west. The grave-goods were poor both in quantity and quality, and are remarkable for the fact that the ornaments are all of Kentish workmanship and belong to types known to be prevalent in Kent at the end of the Pagan period.

After reviewing in detail the material from Burwell and comparing it with material from other Anglo-Saxon cemeteries in the neighbourhood, Mr Lethbridge has come to the conclusion that the Burwell cemetery is that of a Christian Anglo-Saxon community of the seventh century.

We are indebted to Professor H. A. Harris and Dr W. L. H. Duckworth for granting permission and providing facilities for the measurement of the skulls.

2. *Cranial Material available for Investigation and Measurement.* Though the skeletons that were excavated were numbered serially from 1 to 140, the number of adult crania that were sufficiently complete after reconstruction to permit of some at least of their principal measurements being taken was only 67. Of this number 45 were considered to be male and 22 female. The measurements of the juvenile skulls, some of which were considerably distorted, were all excluded from the averages. The principal measurements of the individual crania are tabulated in detail in the Appendices V and VI, and the mean values of these characters for male and female groups of crania are given in Tables I and II. For the male series, where the number of crania on which the mean values of characters are based is usually in excess of 30 and frequently in the neighbourhood of 40, the standard deviations and coefficients of variation of the different characters have been calculated; to these mean values and standard deviations their standard errors are appended. As the means of the characters in the female series are, with a few exceptions, based on less than 20 observations, the standard deviations have not been deemed sufficiently representative for insertion in the table. For comparison with the mean values of the cranial characters of these Burwell Anglo-Saxons and of the Bidford-on-Avon Anglo-Saxon skulls measured by J. C. Brash and described in Part I of this paper, the corresponding means of the Anglo-Saxon crania from the London Museums which were measured and described by G. M. Morant* have been tabulated.

The Burwell male series of crania may be considered fairly homogeneous, since the standard deviations of a selection of the principal characters are not larger than

* *Biometrika*, Vol. XVIII. (1926), p. 56.

TABLE I.

Comparative Table of Means for Male Crania

Characters	Nos.	Burwell			London Museums	Bidford-on-Avon
		Means	σ	v	Means	Means
C	40	[1539·12 ± 10·24]			[1543·3] (31)	[1505·7] (11)
F	44	187·19 ± 0·78	5·18 ± 0·55	2·77	186·3 (91)	186·3 (10)
L	45	189·62 ± 0·84	5·60 ± 0·59	2·95	190·0 (58)	192·8 (10)
B	45	141·08 ± 0·82	5·50 ± 0·58	3·88	141·7 (103)	141·7 (10)
B'	41	95·22 ± 0·51	3·40 ± 0·36	3·57	97·3 (59)	96·6 (24)
H'	40	136·29 ± 0·61	5·75 ± 0·64	4·22	134·0 (31)	136·7 (11)
OH	37	114·09 ± 0·81	4·95 ± 0·58	4·34	114·3 (17)	115·6 (9)
LB	40	102·38 ± 0·83	5·26 ± 0·59	5·14	104·1 (31)	100·2 (12)
Q	37	312·80 ± 1·86	11·32 ± 1·32	3·62	311·4 (15)	
Q'	36	317·06 ± 1·90	11·37 ± 1·34	3·59	316·1 (27)	307·3 (15)
S	42	381·64 ± 1·58	10·25 ± 1·12	2·69	379·5 (64)	382·4 (12)
S_1	42	128·38 ± 0·94	6·09 ± 0·66	4·72	129·3 (84)	128·1 (20)
S_2	42	130·86 ± 1·13	7·31 ± 0·80	5·59	129·0 (87)	131·0 (18)
S_3	42	122·55 ± 1·15	7·44 ± 0·81	6·07	121·7 (71)	121·7 (11)
S_1'	42	112·69 ± 0·73	4·76 ± 0·52	4·22		112·7 (21)
S_2'	42	117·30 ± 0·89	5·74 ± 0·63	4·89		117·1 (10)
S_3'	42	99·70 ± 0·85	5·52 ± 0·60	5·53	98·5 (37)	99·8 (11)
U	42	527·76 ± 2·01	13·05 ± 1·42	2·47	528·9 (73)	537·1 (11)
GH	34	115·84 ± 1·14	6·65 ± 0·81	5·74		118·9 (12)
$G'H$	31	69·02 ± 0·90	5·04 ± 0·64	7·30	71·7 (23)	69·5 (21)
GB	31	94·71 ± 1·02	5·68 ± 0·72	6·00	95·0 (19)	95·3 (17)
J	21	134·02 ± 1·31	6·40 ± 0·92	4·78	133·3 (34)	130·0 (12)
NH'	34	50·15 ± 0·60	3·51 ± 0·43	7·00	52·7 (28)	51·7 (13)
NH, R	34	50·12 ± 0·62	3·61 ± 0·44	7·20	52·2 (22)	
NH, L	34	50·04 ± 0·62	3·63 ± 0·44	7·25	52·3 (22)	
NB	34	24·12 ± 0·31	1·80 ± 0·22	7·46	24·5 (28)	24·1 (10)
O_1', R	12	39·67 ± 0·61	2·10 ± 0·43	5·29	40·3 (16)	
O_1, R	30	42·02 ± 0·34	1·86 ± 0·24	4·43	42·9 (19)	40·4 (14)
O_1, L	33	41·85 ± 0·30	1·71 ± 0·21	4·09	42·2 (16)	40·5 (10)
O_2, R	30	32·78 ± 0·42	2·30 ± 0·30	7·02	33·6 (29)	33·4 (14)
O_2, L	33	32·77 ± 0·35	2·02 ± 0·25	6·16	33·5 (27)	34·1 (10)
G_1	11	48·50 ± 1·05	3·49 ± 0·74	7·20	50·1 (20)	47·4 (18)
G_1'	14	44·57 ± 0·87	3·26 ± 0·62	7·31		
G_2	29	40·41 ± 0·48	2·59 ± 0·34	6·41	41·3 (27)	38·9 (18)
GL	33	94·11 ± 1·01	5·79 ± 0·71	6·15	96·0 (22)	101·9 (11)
fml	37	37·32 ± 0·60	3·63 ± 0·42	9·73	37·5 (20)	39·0 (12)
fmb	37	31·11 ± 0·39	2·38 ± 0·28	7·65	31·1 (18)	32·1 (12)
$100\ B/L$	45	74·76 ± 0·47	3·16 ± 0·33		74·7 (52)	73·5 (10)
$100\ B/F$	44	75·75 ± 0·45	3·00 ± 0·32		74·9 (112)	74·1 (18)
$100\ H'/L$	40	71·87 ± 0·51	3·22 ± 0·36		71·2 (25)	71·7 (11)
$100\ B/H'$	40	104·43 ± 0·82	5·19 ± 0·58		104·9 (61)	102·2 (11)
$100\ G'H/GB$	30	72·46 ± 1·07	5·88 ± 0·76		75·3 (43)	72·4 (17)
$100\ G'H/J$	21	51·85 ± 0·88	4·02 ± 0·62			
$100\ NB/NH, R$	34	48·32 ± 0·74	4·31 ± 0·52		48·1 (19)	
$100\ NB/NH, L$	34	48·39 ± 0·75	4·39 ± 0·53		47·9 (19)	
$100\ NB/NH'$	34	48·28 ± 0·84	4·43 ± 0·70		47·5 (58)	46·9 (10)
$100\ O_2/O_1, R$	30	78·11 ± 1·03	5·64 ± 0·73		77·9 (10)	82·8 (14)
$100\ O_2/O_1, L$	33	78·34 ± 0·71	4·06 ± 0·50		79·1 (10)	84·2 (10)
$100\ fmb/fml$	37	83·84 ± 1·29	7·85 ± 0·91	—	82·3 (18)	82·5 (12)
$100\ G_2/G_1$	11	82·95 ± 1·71	5·68 ± 1·21	—	81·5 (18)	82·3 (18)
$\angle P$	34	87°·07 ± 0·56	3°·24 ± 0·39	—	88°·1 (16)	85°·2 (10)
$\angle N$	32	63°·30 ± 0·61	3°·47 ± 0·43	—	62°·1 (16)	66°·7 (11)
$\angle A$	32	75°·80 ± 0·60	3°·37 ± 0·42	—	75°·5 (16)	73°·6 (11)
$\angle B$	32	40°·91 ± 0·49	2°·77 ± 0·35	—	42°·4 (16)	39°·7 (11)
$\angle \theta_1$	32	29°·88 ± 0·55	3°·09 ± 0·39	—	30°·6 (13)	28°·0 (7)
$\angle \theta_2$	32	11°·09 ± 0·64	3°·62 ± 0·45	—	11°·9 (13)	11°·1 (7)
$Oc.\ I.$	42	58·94 ± 0·44	2·83 ± 0·31	—	58·2 (35)	57·8 (14)

All length measurements are in millimetres and capacities in cubic centimetres.

TABLE II.

Comparative Table of Means for Female Crania.

Characters	Burwell	London Museums	Bidford-on-Avon
	Means	Means	Means
C	[1309·1] (13)	[1370·0] (28)	[1411·0] (10)
F	182·0 (21)	180·9 (62)	182·8 (13)
L	182·9 (22)	182·0 (55)	184·0 (14)
B	138·6 (20)	135·6 (67)	136·4 (10)
B'	95·9 (18)	94·3 (68)	95·5 (15)
H'	129·3 (13)	129·6 (28)	131·9 (7)
OH	111·6 (14)	110·5 (19)	111·2 (5)
LB	95·0 (13)	97·4 (26)	99·7 (7)
Q	305·1 (14)	—	—
Q'	309·7 (14)	301·7 (18)	303·3 (6)
S	360·5 (10)	360·5 (38)	367·1 (5)
S_1	124·9 (20)	124·4 (63)	123·0 (12)
S_2	120·0 (19)	123·2 (57)	127·3 (10)
S_3	116·6 (18)	115·2 (46)	119·6 (5)
S_1'	100·3 (20)	—	108·3 (17)
S_2'	114·1 (20)	—	114·9 (15)
S_3'	96·1 (18)	97·1 (33)	100·0 (6)
U	511·8 (14)	510·9 (54)	518·0 (6)
GH	109·8 (7)	—	110·3 (9)
$G'H$	65·3 (12)	65·9 (30)	65·4 (9)
GB	87·2 (13)	90·2 (20)	91·0 (8)
J	126·4 (6)	125·6 (28)	125·2 (3)
NH'	47·3 (12)	47·1 (32)	49·0 (9)
NH, R	47·3 (12)	48·4 (29)	—
NH, L	47·3 (12)	48·0 (27)	—
NB	22·8 (11)	24·2 (24)	23·8 (9)
O_1, R	41·0 (8)	41·6 (23)	40·0 (8)
O_1, L	39·8 (8)	—	39·8 (8)
O_2, R	33·3 (8)	32·8 (27)	33·5 (8)
O_2, L	33·1 (8)	32·9 (26)	33·9 (8)
G_1	46·2 (3)	51·2 (13)	45·1 (10)
G_1'	43·3 (3)	47·0 (13)	—
G_2	37·6 (10)	39·9 (23)	38·2 (12)
GL	89·8 (8)	94·0 (10)	96·2 (6)
fml	36·2 (13)	35·5 (19)	36·2 (4)
fmb	29·2 (11)	28·9 (19)	29·5 (4)
$100\ B/L$	75·8 (20)	74·4 (51)	73·8 (10)
$100\ B/F$	76·3 (19)	75·2 (77)	74·2 (10)
$100\ H'/L$	70·6 (12)	71·5 (27)	72·4 (7)
$100\ B/H'$	108·4 (12)	105·7 (42)	103·5 (6)
$100\ G'H/GB$	74·8 (12)	72·1 (41)	71·5 (8)
$100\ G'H/J$	51·4 (5)	—	—
$100\ NB/NH$, R	48·3 (11)	51·0 (20)	—
$100\ NB/NH$, L	48·4 (11)	51·1 (20)	—
$100\ NB/NH'$	48·6 (11)	50·2 (38)	48·6 (9)
$100\ O_2/O_1$, R	81·3 (8)	78·6 (23)	83·1 (8)
$100\ O_2/O_1$, L	82·2 (7)	79·4 (24)	85·2 (8)
$100\ fmb/fml$	82·5 (11)	81·7 (18)	81·8 (4)
$100\ G_2/G_1$	83·4 (3)	76·1 (12)	83·5 (10)
$\angle P$	86°·9 (13)	83°·9 (20)	83°·1 (5)
$\angle N$	65°·9 (8)	67°·2 (15)	66°·9 (6)
$\angle A$	73°·1 (8)	72°·6 (15)	74°·6 (6)
$\angle B$	41°·1 (8)	40°·1 (15)	38°·5 (6)
$\angle \theta_1$	28°·8 (8)	29°·0 (11)	31°·2 (3)
$\angle \theta_2$	12°·3 (8)	10°·2 (11)	6°·3 (3)
$Oc.\ I.$	59·5 (18)	59·7 (32)	61·0 (5)

All length measurements are in millimetres and capacities in cubic centimetres.

those shown by the corresponding characters in the Anglo-Saxon male skulls from the London Museums or those of the seventeenth-century Londoners which were excavated from a single pit in Whitechapel. A comparison of the values of the standard deviations in the three series for the characters selected is shown below:

Characters	Anglo-Saxons from Burwell	Anglo-Saxons from London Museums	17th-century* Londoners (Whitechapel)
L	5·60	-	6·27
F	5·18	5·49	-
B	5·50	5·79	5·28
B'	5·75	5·81	5·50
U	13·05	11·79	15·02
$100\ B/L$	3·16	--	3·26
$100\ B/F$	3·00	3·12	-

The Burwell female series is not only smaller but rather less homogeneous than the male. A casual inspection of the group showed that three of the skulls differed from the majority in the respect that they were definitely brachycephalic in type.

3. *Mean Values and Variabilities of the principal Cranial Characters and a Comparison with the corresponding Characters in the Anglo-Saxon Crania from Bidford-on-Avon and from the London Museums.* The mean values of the principal calvarial and facial characters in the Burwell collection of male and female skulls may be compared in detail in Tables I and II with the corresponding values in the two other groups of Anglo-Saxon skulls, but the characters in which the three cranial series diverge most notably from one another can be more readily seen from a study of Table III, in which the values of

$$\alpha = \frac{n_s n_{s'}}{n_s + n_{s'}} \left(\frac{M_s - M_{s'}}{\sigma_s} \right)^2$$

for the selected series of 31 characters, 19 absolute measurements and 12 indices and angles, which are subsequently used for the calculation of the coefficients of racial likeness, are given for each pair of male and female groups of skulls.

The general form of the skulls excavated at Burwell is illustrated by photographs in four normae of a typical male specimen (Skull No. 56). Photographs of its mandible from two aspects are also given.

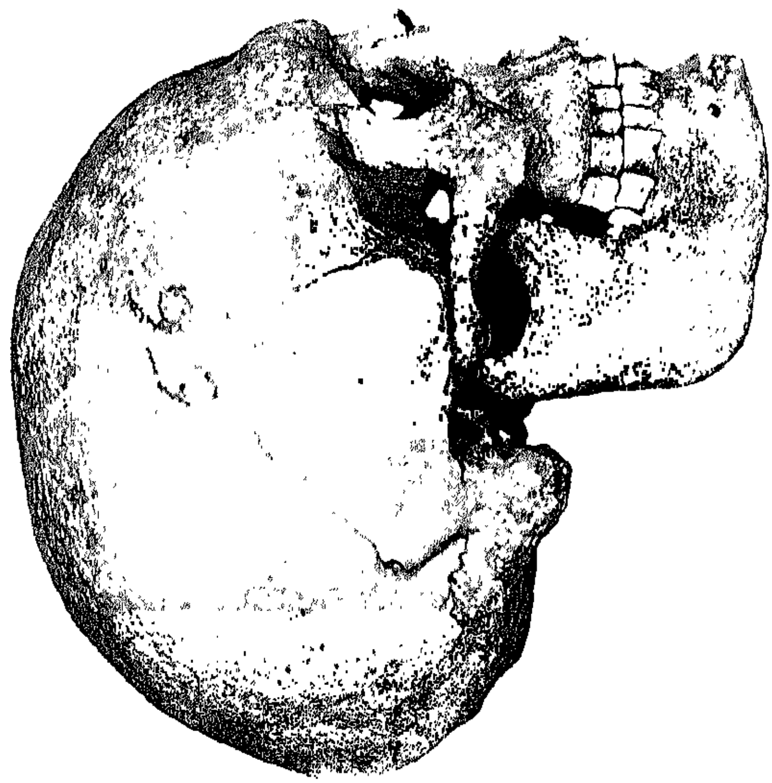
4. *Values of α for a selected Series of Characters showing the Features in which Differences in the three Cranial Series are most pronounced*†. Comparing in the first place the Burwell male series with that from the London Museums, it will be seen that in only a few of the 31 characters does the value of α exceed 2·7. These divergent characters are $G'H$, B', NH, R, O_1, R, LB and $100\ G'H/GB$. In the

* *Biometrika*, Vol. III. (1904), pp. 191—244.

† The standard deviations used for the calculation of the α's were those of the long Egyptian series of the XXVI—XXX dynasties. *Biometrika*, Vol. XVI. (1923), p. 328.

CM

Norma facialis.

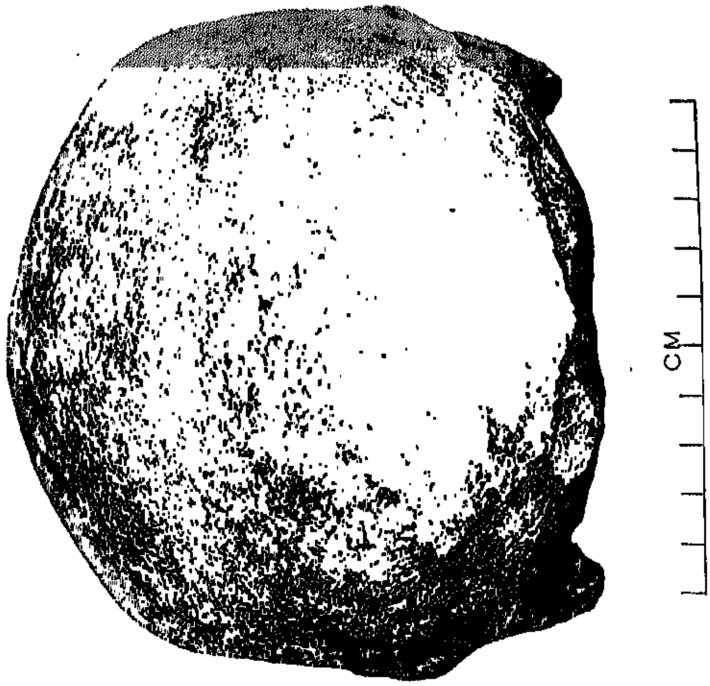


CM

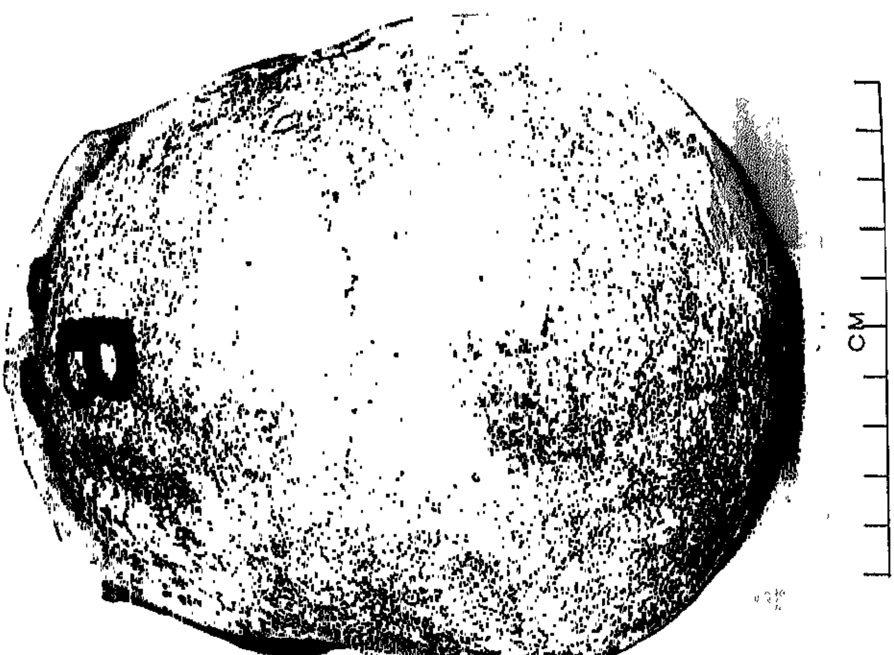
Norma lateralis (Right Profile).

Typical Anglo-Saxon Male Skull from Burwell (No. 56).

 Plate II

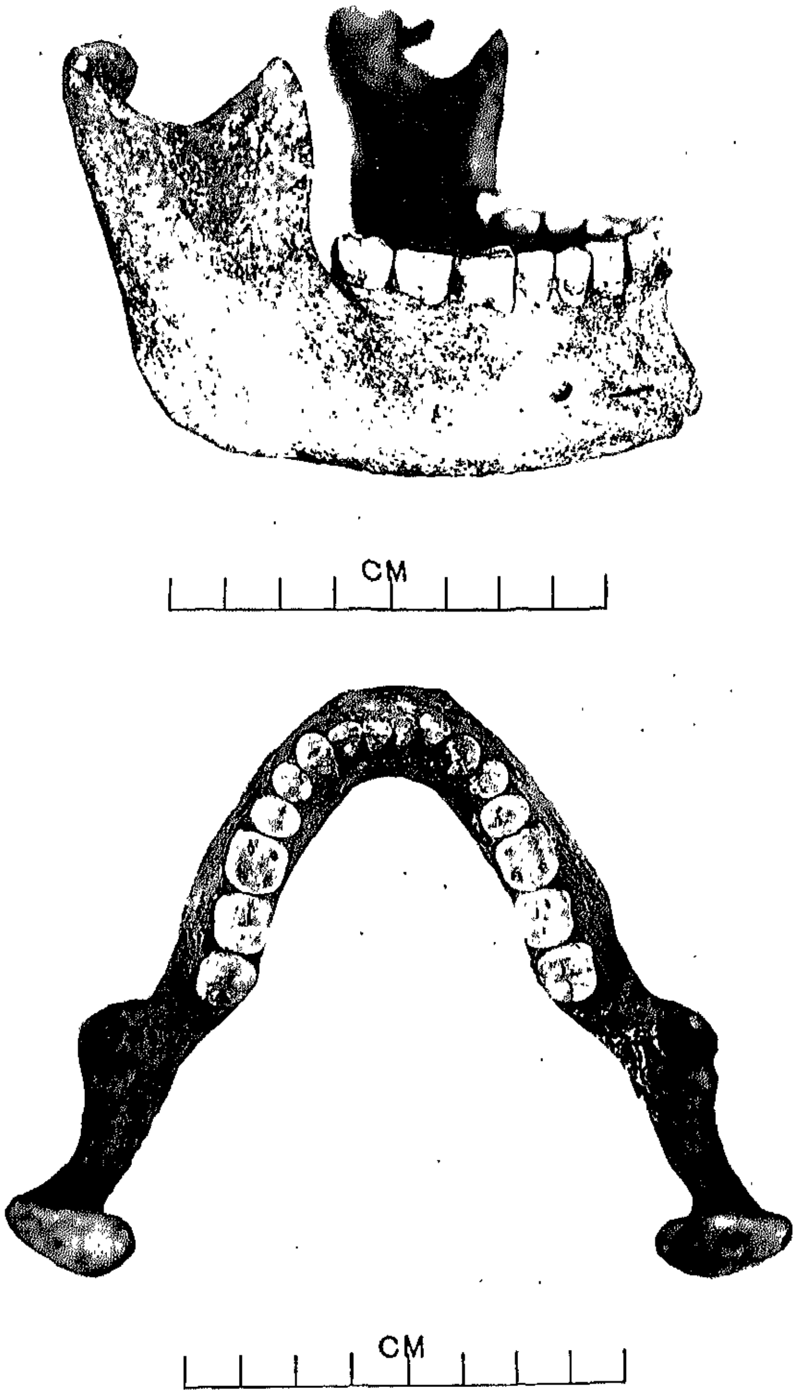


Norma occipitalis.



Norma verticalis.

Typical Anglo-Saxon Male Skull from Burwell (No. 56).



Typical Male Mandible. Anglo-Saxon Skull from Burwell (No. 56).

TABLE III.

Values of $\alpha = \dfrac{n_s n_{s'}}{n_s + n_{s'}} \left(\dfrac{M_s - M_{s'}}{\sigma_s}\right)^2$ *between the Several Cranial Series.*

Characters	Males			Females		
	Burwell and London Museums	Bidford-on-Avon and London Museums	Burwell and Bidford-on-Avon	Burwell and London Museums	Bidford-on-Avon and London Museums	Burwell and Bidford-on-Avon
$100\ B/L$	0·03	2·79	3·14	4·30	0·46	6·25
$100\ H'/L$	0·87	0·22	0·04	0·83	0·55	1·76
$100\ B/H'$	0·33	3·67	2·26	4·11	1·53	5·80
$100\ G'H/GB$	5·63	4·16	0·00	3·37	0·12	2·60
$100\ NB/NH, R$	0·03	—	—	3·64	—	—
$100\ NB/NH'$	—	0·35	1·64	—	1·31	0·00
$100\ O_2/O_1, R$	0·02	7·59	8·27	0·81	3·45	0·62
$100\ fmb/fml$	0·81	0·01	0·46	0·14	0·00	0·05
$100\ G_2/G_1$	0·33	0·12	0·07	2·95	6·90	0·00
Oc. I.	0·80	0·15	1·17	0·05	0·71	0·86
$\angle P$	1°·01	4°·93	2°·66	8°·15	0°·29	5°·99
$\angle N$	1°·40	12°·59	8°·64	0°·83	0°·04	0°·32
$\angle A$	0°·08	1°·97	3°·31	0°·11	1°·47	0°·66
C	0·02	1·86	2·15	0·89	1·11	0·08
L	0·77	2·12	4·18	0·57	2·00	0·46
B	0·00	0·00	0·00	6·79	0·27	1·58
B'	0·78	0·51	1·86	2·45	1·19	0·09
OH	—	—	—	—	—	—
H'	0·06	4·39	3·94	0·04	1·55	1·61
LB	3·20	2·42	8·46	3·31	2·35	8·11
Q'	0·14	7·19	10·40	6·43	0·15	2·20
S	0·53	0·44	0·04	0·00	0·01	0·01
U	2·48	1·31	3·66	0·07	1·97	1·17
$G'H$	5·44	3·02	0·18	0·22	0·12	0·00
J	0·33	3·10	1·53	0·14	0·02	0·14
NH, R	0·91	—	—	1·52	—	—
NH'	—	1·33	3·22	—	3·75	2·20
NB	0·78	0·58	0·00	5·50	0·39	1·84
O_1, R	3·38	18·06	8·76	0·88	6·24	1·64
O_2, R	2·50	0·10	0·94	0·40	0·89	0·05
G_1	1·64	0·23	0·75	6·96	24·00	0·32
G_2	1·64	8·09	3·61	5·76	3·50	0·31
fmb	0·00	1·56	1·96	0·16	0·30	0·07
fml	0·09	2·77	4·29	0·79	0·34	0·00

Burwell series the mean measurements of all these characters are less than in the series from the London Museums. On comparing the Bidford-on-Avon males with the London Museums males the number of cranial characters in which striking divergence is shown is distinctly greater than in the previous case. The characters in which the differences in these groups are most emphasised are O_1, R, G_1, G_2, Q', H', $100\ O_2/O_1, R$, $100\ G'H/GB$, $100\ B/H'$, and the angles N and P. O_1, R, G_1, G_2, Q' and $100\ G'H/GB$, $100\ B/H'$ and the profile angle ($\angle P$) are all smaller in the

Bidford-on-Avon than in Morant's series. The characters H', $100\,O_2/O_1, R$ and $\angle N$, on the other hand, are greater in the Bidford series. Although the length and width of the palate G_1 and G_2 are definitely less in the Bidford than in Morant's series, the relative proportion of breadth to length, $100\,G_2/G_1$, does not differ so sensibly in the two groups. While the mean heights of the right and left orbits, O_2, R and O_2, L, show closely corresponding values in the two series, the mean measurements of orbital width, O_1, R and O_1, L, are definitely smaller in the Bidford. The mean value for O_1, R in the Bidford skull, indeed, almost coincides with that tabulated for O_1', R, the breadth measured from the dacryon, in Morant's series. As a consequence, the orbital index $O_2/O_1, R$ in the Bidford group is on the average of much the same order as that tabulated for $100\,O_2/O_1', R$ in Morant's series. In measuring orbital width, J. C. Brash used the curvature method to determine the inner terminal of the diameter and not the dacryon, and the accuracy of the measurements originally taken have been verified in certain cases from the dioptographic tracings of the facial normae that were prepared. The notable difference in orbital width in the two groups is thus, so far as we are aware, not due to any difference in technique but would seem to indicate a real divergence in orbital shape. The nasal angle N is on the average nearly 5° larger in the Bidford than in the Museums series. The depth of the face, GL, is in the Bidford male almost 6 mm. greater on the average than in the Museums group, and the divergence in this measurement appears to be largely responsible for the angular difference. The mean breadth of the face, GB, is much the same in the Bidford and Morant's series, but the upper face height, $G'H$, is on the average 2 mm. less in the Bidford. As a consequence, the upper facial index, $100\,G'H/GB$, is definitely less in the Bidford than in the Museums series.

The values of α between the selected series of characters in the Burwell and Bidford-on-Avon male groups show clearly that the characters in which the Burwell crania differ from the Bidford are largely the same as those in which the crania from the London Museums differ from the latter group. The principal divergences occur in the characters O_1, R, G_2, Q', LB, H', L, $100\,O_2/O_1, R$ and $\angle N$. The mean orbital width, O_1, R, and orbital index, $100\,O_2/O_1, R$, in the Burwell group are in fairly close agreement with the corresponding values in Morant's series, and show the same characteristic difference from the Bidford group. Q' is on the average about 10 mm. greater in the Burwell than in the Bidford series and the basi-nasal length, LB, is nearly 4 mm. less. The depth of the face, GL, in the Bidford series is on the average almost 8 mm. greater than in the Burwell series, and this divergence, similar to that shown from Morant's mean value, is chiefly responsible for the large size of the $\angle N$ in the Bidford as compared with the Burwell cranium. Both the glabella-occipital length (L) and basi-bregmatic height (H') are on the average rather greater in the Bidford than in the Burwell group.

As the mean values of the characters which are tabulated for the female crania in the Burwell and Bidford-on-Avon series are usually based on relatively small numbers of observations, the values of α between the selected characters in the

female groups may be considered in less detail. Comparing the Burwell series with Morant's series, the characters in which the values of α seem to indicate the most striking divergences are B, Q', NB, G_1, G_2, $100\,G_2/G_1$, $100\,B/L$, $100\,B/H'$ and the $\angle P$, though in a few other characters α slightly exceeds 2·7. In the Burwell group the maximum parietal breadth, B, is on the average 3 mm. greater than in the Museums group. This difference in breadth is largely responsible for the divergence shown in the mean indices, $100\,B/L$ and $100\,B/H'$, in both of which α exceeds the value 4. On excluding the three definitely brachycephalic skulls, to which reference has been made, the mean cephalic index in the Burwell series is identical with that in Morant's series. Q' in the Burwell group is greater than in the Museums group, but G_1, G_2 and NB are smaller. The profile angle ($\angle P$) is on the average 3° greater in the Burwell female, though in the two male groups the mean values agree fairly closely. In the Museums series of Anglo-Saxon crania the mean value of this angle in the male is about 4° in excess of that for the female. In the Burwell series the mean value of $\angle P$ in the male just exceeds that in the female. In the long Egyptian series E*, the mean profile angle ($\angle P$) in the male is $85·81 \pm ·08$ and in the female $84·93 \pm ·09$, which would suggest that *when the number of observations is large the profile angle in the male does not* differ greatly from that in the female, probably not more than 1°. The large sexual difference in the Anglo-Saxons from the London Museums may probably be due to the small numbers on which the averages are based and yet, as will be seen later, a similar sexual difference, though of lesser degree, is found in the Bidford-on-Avon group. The difference may really be indicative of relative prognathism in the female Anglo-Saxon, though it is not evident in the Burwell type.

A scrutiny of the values of α for the selected characters in the Museums and Bidford-on-Avon females shows that the only features in which differences that may be considered significant occur are G_1, G_2, O_1, R, NH', $100\,O_2/O_1, R$ and $100\,G_2/G_1$. The mean palatal length and breadth, G_1 and G_2, in the Bidford group are apparently smaller than the corresponding values in Morant's series, and as the length is relatively more in defect than the breadth, the palatal index, $100\,G_2/G_1$, is definitely greater in the Bidford series. The orbital width, O_1, R, is less in the Bidford than in Morant's group and the orbital index apparently greater. The mean value of NH' in the Bidford group, on the other hand, is greater than in the Anglo-Saxons from the London Museums.

Comparing the Burwell and Bidford-on-Avon females, the only values of α which seem to indicate significant differences in the corresponding characters are those for LB, $100\,B/L$, $100\,B/H'$ and $\angle P$. The basi-nasal length, LB, in the Bidford female is on the average very large and exceeds the mean value for the corresponding character in the Burwell group by nearly 5 mm. A like difference was found in the male crania. The mean profile angle ($\angle P$) in the Bidford female is based on only five observations. The individual angles in this

* *Biometrika*, Vol. XVI. (1924), pp. 328—363.

group were measured on the dioptographic tracings of the normae laterales and not directly by the stationary goniometer, but this is not likely to have influenced very materially the estimation of the size of the angle, as both methods when tested on two or three male skulls gave almost identical results. The mean value of the $\angle P$, as has been mentioned, is in close agreement with that for the Anglo-Saxons from the London Museums which, though probably a smaller value than would be found if it were based on a large number of observations, may indicate a relative tendency to prognathism in the Anglo-Saxon female. In the Burwell group the mean maximum breadth is greater, and the mean maximum length and mean basi-bregmatic height less than in the Bidford. As a consequence the indices $100\ B/L$ and $100\ B/H'$ in the Burwell group both exceed the corresponding indices in the Bidford group to a greater degree than might be expected to occur fortuitously.

5. *Coefficients of Racial Likeness for the Cranial Series.* Having enumerated and discussed the characters for which the mean measurements in the groups of skulls seem to show differences of such a degree that they may probably be considered significant, we now proceed to consider the coefficients of racial likeness computed between each pair of groups of skulls for the 31 selected characters. The coefficients, crude and reduced, are shown for males in Table IV and for females in Table V. The reduced coefficients are relatively comparable as they have been brought to a common standard by making allowance for the varying number of skulls on which the crude coefficients are based.

The crude coefficients for the male groups seem to indicate quite clearly that in shape and in general characters the Burwell crania are almost identical with the Anglo-Saxon crania from the London Museums. Both these groups show a slight divergence, almost equal in degree, from the Bidford-on-Avon crania. As the mean number of skulls on which the Burwell and Museums coefficients are based is much the same, the adjustment for varying numbers of skulls does not change the relationship of the Bidford-on-Avon group to these two groups. The reduced coefficients of racial likeness between the Bidford-on-Avon male group and the Burwell and Morant's series respectively are, however, about six times the size of the coefficient between these two series.

The crude coefficients of racial likeness for the female groups, Table V, suggest that in regard to shape-characters (indices and angles) the Burwell series diverges only slightly and to an equivalent degree from the two series from Bidford-on-Avon and the London Museums. In its characters, generally, the Burwell series seems to resemble the Bidford-on-Avon series rather more closely than that from the London Museums. The London Museums cranium is almost identical in shape with the Bidford-on-Avon cranium, and diverges only slightly from it when comparison is made between all the 31 characters under review.

Though a comparison of the reduced and relatively comparable coefficients which are given in Table V suggests that in their general form the female Anglo-Saxons from the London Museums may resemble the Bidford-on-Avon crania

TABLE IV.

Coefficients of Racial Likeness between the Male Cranial Series.

	Burwell (34·8)		London Museums (35·3)		Bidford-on-Avon (14·8)	
	All Characters	Indices and Angles	All Characters	Indices and Angles	All Characters	Indices and Angles
Crude Coefficients						
Burwell (34·8) ...	—	—	0·56 ± ·17	− 0·05 ± ·28	1·96 ± ·17	1·64 ± ·28
London Museums (35·3)	0·56 ± ·17	− 0·05 ± ·28	—	—	2·27 ± ·17	2·21 ± ·28
Bidford-on-Avon (14·8)	1·96 ± ·17	1·64 ± ·28	2·27 ± ·17	2·21 ± ·28	—	—
Reduced Coefficients						
Burwell (34·8) ...	—	—	1·63 ± ·49	− 0·16 ± ·91	9·49 ± ·82	8·32 ± 1·42
London Museums (35·3)	1·63 ± ·49	− 0·16 ± ·91	—	—	10·88 ± ·82	11·74 ± 1·45
Bidford-on-Avon (14·8)	9·49 ± ·82	8·32 ± 1·42	10·88 ± ·82	11·74 ± 1·45	—	—

TABLE V.

Coefficients of Racial Likeness between the Female Cranial Series.

	Burwell (12·1)		London Museums (30·5)		Bidford-on-Avon (7·6)	
	All Characters	Indices and Angles	All Characters	Indices and Angles	All Characters	Indices and Angles
Crude Coefficients						
Burwell (12·1) ...	—	—	1·33 ± ·17	1·44 ± ·28	0·51 ± ·17	1·08 ± ·28
London Museums (30·5)	1·33 ± ·17	1·44 ± ·28	—	—	1·16 ± ·17	0·40 ± ·28
Bidford-on-Avon (7·6)	0·51 ± ·17	1·08 ± ·28	1·16 ± ·17	0·40 ± ·28	—	—
Reduced Coefficients						
Burwell (12·1) ...	—	—	7·73 ± ·99	9·11 ± 1·77	5·46 ± 1·82	12·49 ± 3·24
London Museums (30·5)	7·73 ± ·99	9·11 ± 1·77	—	—	9·54 ± 1·40	3·58 ± 2·50
Bidford-on-Avon (7·6)	5·46 ± 1·82	12·49 ± 3·24	9·54 ± 1·40	3·58 ± 2·50	—	—

more closely than they do the Burwell or than the Burwell resemble the Bidford-on-Avon group, it is obvious from the size of the probable errors of the coefficients that little emphasis can be laid on such differences as are observed. The coefficient for all characters between the Burwell and Bidford-on-Avon groups is rather less than that between the Burwell and the London Museums groups, and the latter is

less than the coefficient between the Bidford-on-Avon and the Museums groups, but it is again apparent from the size of the probable errors of the coefficients that no real differences in degree of resemblance can be postulated on such data as are available.

6. *Mandibles: Material available for Measurement.* The mandibles belonging to the Burwell skulls were also measured in accordance with the scheme of measurement now followed at the Biometric Laboratory, University College, London. 62 specimens were found to be sufficiently complete, though reconstruction was necessary in many cases, to provide measurements of the majority of the characters that are usually brought under review. Of these 42 were of the male and 20 of the female sex. The measurements of the individual mandibles are tabulated in Appendices VII and VIII.

7. *Mean Values and Variabilities of the principal Mandibular Characters and a Comparison with the corresponding Characters in the Anglo-Saxon Mandibles from Bidford-on-Avon and the London Museums, as well as the Mandibles from Dunstable.* The mean values of the characters measured and the indices computed from these for the Burwell and Bidford-on-Avon series are shown for males in Table VI and for females in Table VII. For comparison with the male series of mean values, the

TABLE VI.

Comparative Table of Means for Mandibles of Male Skulls.

Character	No.	Burwell			London Museums	Bidford-on-Avon	Dunstable
		Mean	σ	v	Mean	Mean	Mean
w_1	35	122·44 ± 1·07	6·32 ± 0·76	5·16	123·7 (25)	117·7 (20)	121·0 (20)
g_og_o	41	98·82 ± 1·02	6·56 ± 0·72	6·64	100·4 (33)	95·1 (27)	98·4 (40)
c_rc_r	37	98·81 ± 0·79	4·83 ± 0·56	4·89	100·3 (27)	97·4 (24)	99·3 (27)
zz	42	45·75 ± 0·33	2·17 ± 0·24	4·74	45·3 (57)	46·1 (30)	45·4 (42)
c_yl	39	21·19 ± 0·34	2·13 ± 0·24	10·05	21·7 (38)	20·5 (21)	21·3 (35)
ml	42	105·55 ± 0·93	6·01 ± 0·66	5·69	107·2 (31)	108·1 (28)	106·8 (34)
c_pl	42	78·12 ± 0·80	5·21 ± 0·57	6·67	77·7 (42)	78·1 (30)	76·2 (38)
rb'	42	32·07 ± 0·49	3·18 ± 0·35	9·92	33·2 (61)	32·6 (30)	32·8 (42)
m_2p_1	39	28·49 ± 0·20	1·24 ± 0·14	4·36	28·1 (59)	28·3 (29)	28·0 (39)
h_1	40	31·65 ± 0·38	2·41 ± 0·27	7·61	33·1 (40)	32·2 (31)	32·5 (39)
m_2h	39	27·42 ± 0·38	2·39 ± 0·27	8·72	27·2 (51)	28·9 (29)	26·6 (41)
c_rh	40	64·80 ± 1·03	6·51 ± 0·73	10·05	65·7 (48)	66·6 (28)	65·0 (38)
rl	42	63·64 ± 0·99	6·39 ± 0·70	10·04	64·0 (45)	65·5 (26)	64·7 (38)
$\angle M$	42	120°·11 ± 1·15	7°·43 ± 0·81	....	120°·3 (47)	119°·7 (30)	120°·9 (39)
$\angle R$	39	71°·01 ± 1·21	7°·57 ± 0·86	...	72°·0 (36)	72°·1 (22)	68°·7 (37)
$\angle C'$	40	66°·74 ± 1·02	6°·44 ± 0·72	—	68°·2 (32)	68°·4 (28)	69°·0 (30)
$100\, c_rh/ml$	40	61·66 ± 1·09	6·88 ± 0·77	—	60·9 (27)	62·9 (25)	59·3 (31)
$100\, c_rc_r/ml$	37	94·04 ± 1·15	7·02 ± 0·82	—	94·4 (15)	92·5 (24)	93·5 (25)
$100\, g_og_o/c_pl$	41	127·19 ± 1·76	11·25 ± 1·24	—	129·0 (32)	122·5 (27)	129·7 (38)
$100\, g_og_o/c_rc_r$	36	100·11 ± 1·21	7·29 ± 0·86	—	99·3 (19)	97·6 (23)	100·0 (27)
$100\, rb'/rl$	42	50·72 ± 0·88	5·73 ± 0·63	—	51·5 (45)	49·4 (26)	51·0 (38)

means of the corresponding characters in the mandibles of the Anglo-Saxon skulls in the London Museums which were measured by G. M. Morant* and of the Dunstable mandibles† which were known to be very similar to this series of Anglo-Saxon mandibles are also tabulated. For the mandibular characters of the Burwell group of males, the standard deviations and coefficients of variation are also given, as the numbers of observations on which these are based are usually about 40. The standard errors of the means and standard deviations are also included in the table. For female mandibles, the means of the characters and the numbers of observations on which these are based are only given for the three groups from Burwell, Bidford-on-Avon and the London Museums. The small number of female mandibles from Dunstable did not seem to warrant even the calculation of mean values.

The means of the corresponding characters for males in the several groups can be compared in detail in Table VI, but the characters amongst a selected series in which the male mandibles of one group differ most strikingly from those of the other groups can be seen more readily by reference to Table VIII in which the values of α between the characters for each pair of groups are given. The selected

TABLE VII.

Comparative Table of Means for Mandibles of Female Skulls.

Character	Burwell	London Museums	Bidford-on-Avon
	Mean	Mean	Mean
w_1	115·9 (16)	116·6 (22)	114·1 (11)
$g_o g_o$	91·7 (19)	92·9 (35)	90·8 (16)
$c_r c_r$	92·9 (16)	93·2 (28)	94·0 (16)
zz	44·1 (20)	44·1 (50)	46·1 (19)
$c_y l$	19·2 (19)	19·1 (35)	20·4 (13)
ml	99·9 (20)	104·2 (45)	102·8 (17)
$c_p l$	73·4 (20)	74·6 (49)	76·8 (18)
rb'	30·6 (20)	31·0 (56)	31·7 (19)
$m_2 p_1$	27·0 (18)	27·6 (57)	27·8 (18)
h_1	29·2 (18)	30·5 (31)	29·4 (19)
$m_2 h$	25·3 (18)	24·4 (52)	26·8 (18)
$c_r h$	58·4 (18)	59·2 (47)	62·0 (18)
rl	57·3 (19)	59·1 (45)	61·5 (15)
$\angle M$	122°·0 (20)	122°·5 (49)	119°·9 (18)
$\angle R$	69°·5 (16)	68°·2 (36)	69°·5 (14)
$\angle C'$	67°·8 (19)	70°·6 (32)	69°·8 (17)
$100\, c_r h/ml$	58·5 (18)	58·3 (38)	60·3 (17)
$100\, c_r c_r/ml$	93·4 (16)	91·7 (26)	90·2 (14)
$100\, g_o g_o/c_p l$	125·7 (19	126·2 (35)	118·5 (16)
$100\, g_o g_o/c_r c_r$	99·2 (15)	99·3 (23)	97·1 (14)
$100\, rb'/rl$	53·8 (19)	53·0 (43)	52·1 (15)

* *Biometrika*, Vol. XVIII. (1926), p. 96.

† *Ibid.*, Vol. XXV. (1933), pp. 147—157.

TABLE VIII.

Values of $\alpha = \frac{n_s n_{s'}}{n_s + n_{s'}}\left(\frac{M_s - M_{s'}}{\sigma_s}\right)^2$ *between the Several Male Mandibular Series.*

Character	Burwell and London Museums	Burwell and Bidford-on-Avon	London Museums and Bidford-on-Avon	Burwell and Dunstable	London Museums and Dunstable	Bidford on Avon and Dunstable
w_1	0·75	8·57	12·20	0·85	2·08	3·93
zz	0·85	0·22	1·78	0·17	0·03	1·20
rl	0·14	2·42	1·51	1·01	0·12	0·41
rb'	5·10	0·79	1·30	1·87	0·71	0·13
c_yl	1·33	1·85	5·10	0·05	0·81	2·33
ml	1·80	0·17	0·71	1·12	0·11	6·31
h_1	4·30	0·18	1·55	1·39	0·78	0·17
m_2h_1	1·26	0·22	0·26	1·63	0·08	0·50
$\angle M$	0°·28	0°·26	0°·01	1°·51	2°·49	2°·40
$100\, g_og_o/c_pl$	0·50	3·44	5·91	1·18	0·08	7·82

list of mandibular characters in Table VIII is that used by G. M. Morant for the calculation of the coefficients of racial likeness between mandibular groups and we are indebted to him for giving it to us. These characters have been selected by this author because of their relatively slight correlation with one another and because of the relative frequency with which they differ significantly in a series of racial types that were studied by him.

8. *Values of α for a selected Series of Mandibular Characters showing the Features in which Differences in the Mandibular Groups are most emphasised**. For the Burwell and London Museums series of male mandibles the only two characters of the ten tabulated in which α exceeds 2·7 are rb' and h_1. Both these characters are smaller on the average in the Burwell series than in Morant's series

Comparing the Burwell and Bidford-on-Avon groups, two characters, w_1 and $100\, g_og_o/c_pl$, alone show values of α exceeding 2·7. For the bicondylar width, w_1, the value of α exceeds 8. In the Bidford series this character is almost 5 mm. less on the average than in the Burwell series; the mean value of $100\, g_og_o/c_pl$ in the Burwell group is rather less than in the mandibles from Bidford-on-Avon.

In the London Museums and Bidford groups the values of α for three characters exceed 2·7. These are w_1, c_yl and $100\, g_og_o/c_pl$. The Bidford-on-Avon mean value of w_1 is about 6 mm. less and $100\, g_og_o/c_pl$ almost 5 units less than the corresponding values for the mandibles from the London Museums; the Bidford series thus holds in respect of these two characters a similar relationship to the Burwell and Museums groups. Reference to Table VI shows that the mean value of c_pl is of much the same order in the three groups, but that the mean value of g_og_o is 4 to

* The standard deviations used in the calculation of the α's for the mandibular characters are those of the Qau Egyptian series of the IVth dynasty which Dr G. M. Morant has kindly given to us.

5 mm. less in the Bidford group than in the other two; the difference in bigonial width is thus almost wholly responsible for the difference in the index. The average condylar length, $c_y l$, is rather less in the Bidford-on-Avon mandible than in that from the London Museums.

For the Burwell and Dunstable groups, all the values of α are under 2·7.

For the London Museums and Dunstable series only two characters, w_1 and $\angle R$, have values of α exceeding 2·7, but in neither case does the value exceed 3. The mean values of these two characters in the Dunstable mandible are rather less than the corresponding values in that from the London Museums.

In the Dunstable and Bidford-on-Avon series the values of α for two of the ten characters, namely w_1 and $100\, g_o g_o / c_p l$, are in excess of 2·7. The value for the index is almost equal to 8. The mean values of these two characters are both less in the Bidford group. The relatively narrow bicondylar width, w_1, in the Bidford group has already been commented upon. It is about 3·5 mm. less on the average than in the Dunstable group; the bigonial width, $g_o g_o$, shows a deficiency of about the same order. The mean body length, $c_p l$, on the other hand, is rather greater than in the Dunstable mandible. The relatively small bigonial width is thus again mainly responsible for the divergence in the index.

The female mandibles may be dealt with more briefly. The values of α for the selected characters in the three Anglo-Saxon series are given in Table IX.

Comparing the characters in the mandibles from Burwell with those from the London Museums, only one value of α, that for the mandibular length, ml, appreciably exceeds 2·7; the value is actually higher than 10. This character is definitely smaller in the Burwell group. The value of α for the rameal length, rl, just exceeds 2·7; in this case also the Burwell measurement is the smaller.

TABLE IX.

Values of $\alpha = \frac{n_s n_{s'}}{n_s + n_{s'}} \left(\frac{M_s - M_{s'}}{\sigma_s}\right)^2$ *between the Several Female Mandibular Series.*

Character	Burwell and London Museums	Burwell and Bidford-on-Avon	London Museums and Bidford-on-Avon
w_1	0·16	0·73	1·58
zz	0·00	5·72	8·09
rl	3·01	10·27	4·50
rb'	0·46	2·31	1·36
$c_y l$	0·07	6·18	8·90
ml	10·62	3·21	1·00
h_1	2·60	0·05	1·93
$m_2 p_1$	0·87	0·06	0·39
$\angle R$	0°·52	0°·00	0°·47
$100\, g_o g_o / c_p l$	0·03	3·81	5·51

Comparing the Burwell and Bidford-on-Avon groups, five of the ten characters have values of α exceeding 2·7. These are zz, rl, $c_y l$, ml, and $100\, g_o g_o / c_p l$. All these dimensions except the last are smaller in the Burwell mandible. The index, $100\, g_o g_o / c_p l$ is definitely greater in the Burwell group and the divergence seems to depend on a relatively smaller body length, $c_p l$.

Comparing the mandibles from the London Museums and Bidford-on-Avon, four of the ten characters under review have values of α in excess of 2·7. These are zz, rl, $c_y l$, and $100\, g_o g_o / c_p l$. All of these characters except the last are smaller in the mandible from the Museums; $100\, g_o g_o / c_p l$ is greater. The relationship of the mandible from the Museums to that from Bidford in respect of these four characters is thus the same as that of the Burwell mandible. The relatively greater index, $100\, g_o g_o / c_p l$, in the Museums mandibles than the Bidford appears to depend about equally on its two components as the mandibular body length, $c_p l$, in the Museums series is as much in defect of the Bidford-on-Avon value as the bigonial width, $g_o g_o$, is in excess of it.

9. *Coefficients of Racial Likeness for the Mandibular Series.* Having described the characters amongst those selected for computation of the coefficient of racial likeness in the mandibles in which differences between the groups were most apparent, attention may now be directed to the coefficients of racial likeness, crude and reduced, for the groups of male mandibles in Table X.

TABLE X.

Coefficients of Racial Likeness between the Several Male Mandibular Groups.

	Burwell (40·1)	London Museums (42·4)	Bidford-on-Avon (26·4)	Dunstable (37·3)
Crude Coefficients				
Burwell (40·1) ...		0·08 ± ·30	0·84 ± ·30	0·12 ± ·30
London Museums (42·4)	0·08 ± ·30		2·07 ± ·30	0·10 ± ·30
Bidford-on-Avon (26·4)	0·84 ± ·30	2·07 ± ·30		0·92 ± ·30
Dunstable (37·3) ...	0·12 ± ·30	−0·10 ± ·30	0·92 ± ·30	
Reduced Coefficients				
Burwell (40·1) ...		1·65 ± ·73	2·64 ± ·94	0·31 ± ·78
London Museums (42·4)	1·65 ± ·73		0·36 ± ·92	−0·25 ± ·76
Bidford-on-Avon (26·4)	2·64 ± ·94	0·36 ± ·92		2·98 ± ·97
Dunstable (37·3) ...	0·31 ± ·78	−0·25 ± ·76	2·98 ± ·97	

The values of the crude coefficients seem to indicate that the Burwell male mandibles, at least so far as this group of selected characters is concerned, cannot be said to differ from those in the London Museums or from Dunstable, but may show a slight divergence from the Bidford-on-Avon mandibles. The last coefficient

of racial likeness almost attains the conventional level of *significance*. The type of mandible in the London Museums is identical with that from Dunstable, but diverges appreciably from that found at Bidford-on-Avon. The Bidford-on-Avon mandibles show a sensible divergence from those excavated at Dunstable.

The reduced coefficients of racial likeness which are also given in Table X indicate the relative degrees of relationship between the different groups. The coefficient between the Burwell mandibles and those from the London Museums is almost five times that shown between the Burwell and Dunstable groups, but this difference in value cannot be said to connote any real difference on the data available; both coefficients are statistically insignificant. The coefficient between the Burwell and Bidford-on-Avon mandibles is almost nine times that shown between the Burwell and Dunstable mandibles, and, as has been mentioned, may possibly be taken as indicative of a real divergence in the two Anglo-Saxon groups.

The coefficient between the mandibles from the London Museums and those from Bidford-on-Avon is fully twice that found between the mandibles from Burwell and the latter place. The Burwell mandibles would appear to be definitely more closely related to the Bidford-on-Avon mandibles than are those in the London Museums. The relationship of the Burwell mandible to the Bidford-on-Avon mandible is of the same order as that shown between the Bidford-on-Avon and Dunstable mandibles. The Dunstable mandibles appear to be identical in type with the Anglo-Saxon mandibles from Burwell and from the London Museums.

The coefficients of racial likeness, both crude and reduced, for the female groups of mandibles are given in Table XI. As the mean numbers of mandibles on which the coefficients are based are almost the same in the Burwell and Bidford-on-Avon groups, the respective crude coefficients between these groups and the group from the London Museums are relatively comparable without adjustment.

TABLE XI.

Coefficients of Racial Likeness between the Several Female Mandibular Groups.

	Burwell (18·5)	London Museums (41·2)	Bidford-on-Avon (16·1)
Crude Coefficients			
Burwell (18·5) ...	—	0·83 ± ·30	2·23 ± ·30
London Museums (41·2)	0·83 ± ·30	—	2·37 ± ·30
Bidford-on-Avon (16·1)	2·23 ± ·30	2·37 ± ·30	—
Reduced Coefficients			
Burwell (18·5) ...	—	3·25 ± 1·17	12·95 ± 1·74
London Museums (41·2)	3·25 ± 1·17	—	10·24 ± 1·30
Bidford-on-Avon (16·1)	12·95 ± 1·74	10·24 ± 1·30	—

The coefficient between the Burwell mandibles and the mandibles from the London Museums is less than unity and statistically insignificant, although it almost attains the conventional level of significance. That between the Bidford-on-Avon and Museums mandibles is significant and nearly three times as large as the other. The crude coefficient between the Bidford-on-Avon and Burwell groups is also significant and of about the same size as that between the mandibles from Bidford-on-Avon and the London Museums.

Adjustment of the coefficients to the values they would have, if based on 100 mandibles in the groups compared, results in a reduced coefficient between the Burwell mandibles and the mandibles from the London Museums of a value exceeding 3. The reduced coefficients between the Bidford-on-Avon mandibles on the one hand and the mandibles from Burwell and the London Museums on the other have values of about 13 and 10. These do not differ from one another to a significant degree, but are of the order of magnitude of reduced coefficients of racial likeness which are occasionally found between groups of mandibles belonging to crania of distinctly divergent racial type. On the other hand, two groups of crania may show considerable divergence in type while the corresponding groups of mandibles exhibit almost identical characters. Such was indeed found to be the case on comparing the Dunstable material with the Anglo-Saxon material in the London Museums.

10. *General Discussion and Summary.* The cranial and mandibular characters of the groups of Anglo-Saxon skulls from Burwell, Cambridgeshire, and Bidford-on-Avon, Warwickshire, have been compared both individually and collectively in considerable detail and both groups have been compared in respect of the same characters with the Anglo-Saxon skulls in the London Museums which were measured and described by G. M. Morant*. The comparison of the male crania in the two first-mentioned groups is the more important as these are almost twice as numerous as the female specimens. The main inferences to be drawn from the comparison of these male groups appear to be that the Burwell skulls, both crania and mandibles, are almost identical in type with the skulls from the London Museums. If material divergence is present in any feature, it occurs in the size of the upper face and its proportions. Both these groups of Anglo-Saxons seem to show, however, some divergence, about equivalent in degree, from those excavated at Bidford-on-Avon. Where divergence in these types is found, however, it is seen to occur mainly in those characters or features, into the accurate measurement of which the personal element largely enters and in which slight variance of technique, such as identification of terminals of diameters or arcs, on the part of different observers might probably produce average differences of the order that are found in the present case. Among such characters are the orbital width O_1, R, and, as a consequence, the orbital index $100\,O_2/O_1$, R, the palatal length G_1, the palatal breadth G_2, and the transverse arc Q'. The nasal height, NH, R or NH' is also such a character although it shows no divergence in the types under review. In

* *Biometrika*, Vol. XVIII. (1926), p. 76.

only one or two of its other characters, amongst those selected for comparison, and notably the nasal angle (N), does the Bidford cranium differ sensibly from the other two types. Comment has already been made on the relatively large mean measurement of the facial depth (GL) in the Bidford cranium as compared with that shown in the other types and the influence of this chord on the size of the angle N. The differences in the recorded mean measurements of these characters may be real but some caution is advisable in accepting them as of the order of magnitude found, when, as in the present case, the measurements in the different series have been taken by different observers. Fairly substantial differences in certain characters might probably be expected in Anglo-Saxon cranial material from different localities. Reference has been made to the probable existence of two distinct types, recognizable on general inspection, amongst the male crania from the Bidford-on-Avon site. Morant* on the other hand distributed the Anglo-Saxon skulls in the London Museums according to their place of origin into the four groups: West Saxons, South Saxons, Angles and Jutes, and came to the conclusion from a comparison of the actual mean measurements of the principal characters in these groups and the coefficients of racial likeness between them "that they represent populations which are extremely similar if not absolutely identical. Only one C.R.L.—that between the Angles and West Saxons—suggests any real difference of type."

Because of the relatively small numbers of female skulls in the Burwell and Bidford-on-Avon groups, very slight reliance can be placed on the mean values of the characters which are tabulated for these being truly representative of the types, but it is of interest to note in what respects, if any, the data available corroborate the conclusions that have been drawn from a comparison of the male series. The number of female crania from the London Museums is definitely greater and should provide more reliable means. The Burwell female, unlike the male, differs appreciably in many of its characters from that in the London Museums. We have referred to the presence of a few distinctly brachycephalic skulls in the Burwell group. These are mainly responsible for the relatively great mean maximum breadth (B) in the Burwell female and its divergence in this character and in the indices of which it is a component, $100\,B/L$ and $100\,B/H'$, from the female Anglo-Saxons in the Museums. The minimum frontal breadth (B') is also significantly greater in the Burwell female skull. Significant differences are also shown in the palatal length and breadth, the palatal index, the nasal breadth, the nasal index, the transverse arc (Q') and the profile angle ($\angle P$). The mean differences are so many and so relatively large that they suggest the female crania from the two sources are really of different type.

The Bidford-on-Avon female cranium differs from that of the London Museums in its orbital, nasal and palatal measurements, the characters in which differences may most readily arise from slight differences in technique as has been mentioned in the comparison of the male skulls. In their general form, these two female

* *Biometrika*, Vol. XVIII. (1926), p. 76.

crania would appear to be almost identical, but when all (the selected) characters are considered, they seem to show a divergence in type of about the same degree as that found in the corresponding male groups.

From the Burwell female cranium, the Bidford-on-Avon cranium differs mainly in having a relatively smaller maximum parietal breadth (B) with, as a consequence, smaller length-breadth ($100\,B/L$) and height-breadth ($100\,B/H'$) indices; a greater basinasal length and a smaller profile angle ($\angle P$). When all the shape characters (indices and angles) are considered, the two crania show a divergence in type, and when the comparison embraces all the selected characters some real divergence in type still seems to be indicated. The relatively small profile angle ($\angle P$) in the Bidford-on-Avon female and also in the female Anglo-Saxon from the London Museums as compared with that in the Burwell female, and the possibility of the existence of a relative prognathism in the Anglo-Saxon female, *although it is not revealed in the specimens from Burwell*, has already been discussed.

The data for the female Anglo-Saxon crania thus seem to confirm the existence of some divergence in type between the Bidford-on-Avon and Burwell, and Bidford-on-Avon and Museums groups which was suggested by a comparison of the corresponding male groups. Comparison of the female crania from Burwell and the London Museums would also suggest some diversity in these types rather than the identity which seemed to be established by a study of the characters in the male groups, but little importance can be attached to the apparent diversity of type in the female groups in view of the heterogeneity known to exist in the female crania from Burwell and their small number.

Turning to the mandibular relationships, there seems to be evidence that, so far as the selected characters are concerned, the Burwell male mandible does not differ significantly from the Anglo-Saxon mandible in the London Museums or from that of the specimens excavated at Bidford-on-Avon, although in the latter case the coefficient of racial likeness almost attains the conventional level of significance (three times p.e.). The male mandible for the London Museums, on the other hand, differs sensibly in the selected features from that found at Bidford-on-Avon.

While the Burwell female mandible cannot be said to differ certainly from the female Anglo-Saxon mandible in the London Museums, although the coefficient of racial likeness between them is on the border line of significance, both the female Anglo-Saxon mandible from Burwell and that described by Morant appear *to differ significantly from the Bidford-on-Avon type*. The reduced coefficients of racial likeness in the last two cases are as large as are those occasionally found between samples of mandibles that belong to crania of quite diverse racial type.

In a general way it may be stated that the study of the mandibles confirms largely the evidence obtained from the crania as to the interrelationships of the three groups of Anglo-Saxon skulls.

The apparent identity of type that exists between the Dunstable mandibles, on the one hand, and the Anglo-Saxon mandibles from Burwell and from the London Museums, on the other, is of special interest as the corresponding crania when examined were found to be distinctly divergent in type.

We have provided records of a considerable amount of unpublished Anglo-Saxon cranial material and have ventured to draw some inferences from the data we have studied as to the probable relationships of the cranial samples from Bidford-on-Avon, Burwell and the London Museums, but we fully recognise, and are ready to acquiesce in the view, that a final expression of opinion on real diversity or identity of Anglo-Saxon groups from different localities will only be possible when more material is available for study and when all the evidence obtainable from this material, including that provided by cranial contours of the several normae, is utilised.

APPENDIX VII. INDIVIDUAL MEASUREMENTS OF MALE ANGLO-SAXON MANDIBLES FROM BURWELL.

Serial Letter or No.	w_1	g_og_o	c_rc_r	zz	c_yl	ml	c_pl	rh'	m_2p_1	h_1	m_1h	c_rh	rl	∠ M	∠ R	∠ G'	$\frac{100\,c_rh}{ml}$	$\frac{100\,c_rc_r}{ml}$	$\frac{100\,g_og_o}{c_pl}$	$\frac{100\,g_og_o}{c_rc_r}$	$\frac{100\,rh'}{rl}$
X2	125·0	97·5	102·5	47·5	21·0	110·0	73·5	28·0	30·5	34·0	24·5	56·0	58·0*	133·0	64·0	52·5	50·9	93·2	132·7	95·1	48·3
X8	127·0	102·0	92·5	45·5	22·0*	110·0	80·0	34·5	27·5	30·0	26·0	62·0	65·0	121·0	60·0	59·5	56·4	84·1	127·5	110·3	53·1
10	126·5	96·5	100·0	46·0	23·5	103·5	86·5	34·5	28·5	32·0	34·5	74·0	70·5	110·0	71·0	67·0	71·5	96·6	111·6	96·5	49·1
15	121·5	83·0	95·5	47·5	21·5	97·0	82·0	30·0	27·5	27·0	24·5	62·5	58·0	120·0	78·5	62·5	64·4	98·5	101·2	86·9	62·1
22	129·0	104·0	98·0	43·5	21·0	113·5	75·0	30·0	29·0	28·0	27·0	70·5	74·0	123·5	60·0	69·5	62·1	86·3	138·7	106·1	40·5
23	—	100·5	100·0	47·5	22·0	115·0	87·0	31·5	26·5	29·5	31·0		76·0	116·0	·	59·0	·	87·0	115·5	100·5	41·4
27 i	121·5	102·5	95·0	45·0	25·0	101·0	78·5	35·0	28·5*	29·0	28·5	69·0	70·5	112·0	64·0	62·0	68·3	94·1	130·6	107·9	49·6
28	122·0	91·0	96·0	46·0	—	110·0	80·0	32·5	29·5	34·5	28·0	64·0	63·0	124·0	74·0	64·0	58·2	87·3	113·8	94·8	51·6
29	—	91·5	92·5	42·5	—	91·0	68·5	30·0	31·0*	40·0	20·5	61·0	55·0	122·0	·	73·0	67·0	101·2	133·6	98·9	54·5
31	—	95·0	100·5	47·5	18·5	111·0	82·5	35·5	30·0	32·5	27·5	71·5	63·5	114·0	76·5	78·0	65·0	91·4	115·2	94·5	55·9
34	123·5	102·0	100·5	46·0	21·5	107·5	78·5	31·5	28·0	34·5	28·5	65·5	64·0	122·0	74·0	71·5	60·9	93·5	129·9	101·5	49·2
37	120·0	101·5	106·0	47·5	22·0	108·5	76·0	26·0	29·0	28·0	28·0	76·0	68·0	122·0	84·0	82·0	70·0	97·7	133·6	95·8	38·2
41	124·0	101·5	—	44·5	22·5	110·0	82·5	32·0	26·5	30·5	30·0	67·0	70·0	117·0	71·0	61·5	60·9	·	123·0	—	45·7
42	—	98·5	89·5	45·5	20·5*	107·0	77·0	32·0	29·0	35·5	26·5	50·5	68·0	122·0	55·0	67·0	47·2	83·6	127·9	110·1	47·1
43	116·5	94·0	98·0	50·5	17·5	100·5	70·5	33·0	28·5	31·0	27·0	62·0	59·0	125·0	73·0	66·5	61·7	97·5	133·3	95·9	55·9
44	121·0	103·5	—	43·5	20·0	101·0	70·0	27·0*	28·5	30·5	24·5	58·0*	52·0	130·0	73·0	78·0	52·4	—	147·9	—	51·9
47	—	101·5	107·5	49·5	23·5*	107·5	88·5	38·5	30·0	32·5	30·0*	71·5	65·0*	114·0	79·5	71·0	66·5	100·0	114·7	94·4	59·2
56	130·0	106·0	103·5	47·5	22·0	107·0	83·0	36·5	30·5	30·5	33·0	75·0	76·5	112·0	72·0	66·0	70·1	96·7	127·7	102·4	47·7
58	123·0	93·5	106·5	46·0	23·5	98·5	78·0	32·5	28·5	30·5	28·5	69·0	68·0	116·0	73·0	71·5	70·1	108·1	119·9	87·8	47·8
60	123·0	90·5	98·0	45·0	19·5	99·5	78·0	38·0	28·5	33·5	29·0	69·0	63·0	113·0	77·5	73·0	69·4	98·5	116·0	92·3	60·3
67	—	—	101·5	43·5	21·0*	108·5	83·0	30·0	—	30·5	—	61·0	69·0	114·0	60·0	67·0	56·2	93·5	·	—	43·5
68	115·5	95·5	100·5	45·5	19·0*	99·0	76·0	32·5	31·0*	31·0	24·5	66·5	61·0	115·0	80·0	84·0	67·2	101·5	125·7	95·0	53·3
70	125·0	97·5	97·0	48·0	25·5	108·5	74·5	31·5	28·5	35·0	30·5	61·5	62·0	124·0	64·5	77·0	56·7	89·4	130·9	100·5	50·8
74	125·5	91·5	97·5	44·5	20·5*	109·0	79·5	33·5	28·0	34·0	30·5	60·0	56·5	126·0	68·0	70·0	55·0	89·4	115·1	93·8	59·3
77	133·0	115·0	106·5	45·0	24·5	108·5	75·5	34·0	29·0	34·0	27·5	70·0	64·0	125·0	74·5	65·5	64·5	98·2	152·3	108·0	53·1
78	115·5	95·5	100·0	42·5	20·0	92·5	75·0	30·0	27·5	29·0	28·5	68·5	61·5	116·0	88·0	65·0	74·1	108·1	127·3	95·5	48·8
89	120·0	98·0	97·0	44·5	22·5*	107·0	79·5	30·5	27·5*	33·5	25·5	66·0*	68·0*	122·0	67·5	68·5	61·7	90·7	123·3	101·0	44·5
92	120·5	99·0	100·0	43·0	20·0	100·0	77·5	31·5	27·0	30·0	24·5	64·0	59·5	114·0	75·0	59·5	64·0	100·0	127·7	99·0	52·9
93 i	112·0	91·5	96·5	46·5	16·0	95·5	66·5	28·5	29·5*	31·0	24·0	49·0	49·0	134·0	68·0	69·0	51·3	101·0	137·6	94·8	58·2
93 ii	125·0	108·0	97·5	48·5	20·0	106·0	77·0	33·0	30·0	30·5	24·5	55·5	57·5	134·0	62·0	61·5	52·4	92·0	140·3	110·8	57·4
102	120·0	99·0	94·0	44·5	21·5	109·0	77·0	29·5	28·0	30·0	24·5	61·5	64·0	124·0	64·0	67·0	56·4	86·2	128·6	105·3	46·8
104	108·0	99·0	97·0	45·5	19·0	100·5	73·5	30·5	25·5*	—	45·5	75·0	65·5	120·0	84·0	·	74·6	96·5	134·7	92·8	46·6
107	130·5	102·0	107·5	45·5	—	100·0	74·0*	28·5	28·0*	32·5	24·0*	57·0	57·0	122·0	77·0	52·0	57·0	107·5	137·8	94·9	50·0
108	123·5	95·5	—	42·5	18·5	102·5	70·0	26·0	—	—	·	·	55·0	127·0	·				136·4	—	47·3
111	—	105·0	—	49·5	21·0	109·0	83·0	35·0	28·5	32·5	28·5	72·5	68·0	114·0	77·5	64·0	66·5		126·5	·	51·5
112	123·5	95·5	98·5	45·0	23·0*	109·5	78·0	33·5	28·5	33·0	29·5	66·5	66·0	120·0	73·5	62·0	60·7	90·0	122·4	97·0	50·8
115	125·0	110·5	95·0	46·0	21·5	110·0	72·5	30·0	27·0	35·5	27·0	55·0	63·0	143·0	53·0	64·0	46·2	79·8	152·4	116·3	47·6
121	116·0	92·5	89·5	45·5	20·5	107·5	87·5	37·0	28·5	30·5	26·5	59·0	62·0	112·0	64·0	61·5	54·9	83·4	105·7	103·4	59·7
122	127·0	106·0	92·5	43·5	22·5	105·0	86·5	28·0	27·5	29·0	27·5	72·0	67·0	116·0	76·0	70·5	68·6	88·1	141·7	114·6	41·8
123	123·0	108·5	98·5	49·5	20·5	105·0	81·0	32·5	28·5	30·5	28·5	61·0	60·0	118·0	66·0	65·5	58·1	93·8	134·0	110·2	54·2
128	106·5	92·5	—	42·5	17·5	110·0	79·0	20·5	—	30·5	—	58·0	56·5*	130·0	65·0	72·0	52·7	—	117·1	—	52·2
140	137·0	107·0	107·5	44·5	25·0	113·5	85·5	37·5	27·5	37·0	27·0	72·5	68·5	118·0	74·0	70·0	63·9	94·7	125·1	99·5	54·7

* These measurements were taken on the right side.

APPENDIX VIII. INDIVIDUAL MEASUREMENTS OF FEMALE ANGLO-SAXON MANDIBLES FROM BURWELL.

Serial Letter or No.	w_1	g_og_o	c_rc_r	zz	c_yl	ml	c_pl	rh'	m_2p_1	h_1	m_1h	c_rh	rl	∠ M	∠ R	∠ G'	$\frac{100\,c_rh}{ml}$	$\frac{100\,c_rc_r}{ml}$	$\frac{100\,g_og_o}{c_pl}$	$\frac{100\,g_og_o}{c_rc_r}$	$\frac{100\,rh'}{rl}$
2a	—	92·5	107·0	47·5	—	96·5	71·5	30·5	27·5	30·5	28·0*	68·0	·	118·0	—	72·0	70·5	110·9	129·4	86·4	—
2b	117·5	95·0	—	44·0	20·5	100·0	71·5	27·5	28·5	27·5	43·0	—	56·0	127·0	—	69·0	·	—	132·9	—	49·1
13	112·0	90·0	86·5	43·5	19·5	108·5	80·5	32·5	26·0	35·5	26·0*	55·5	66·0*	121·0	52·0	75·5	51·2	79·7	111·8	104·0	49·2
14	—	93·5	92·0	43·5	18·0*	92·0	66·5	31·5*	30·0*	27·5	24·5	54·0	57·5*	121·0	66·5	70·0	58·7	100·0	140·6	101·6	54·8
25	121·0	97·5	91·0	47·0	23·0	104·5	78·0	35·5	28·0	31·5	32·0	67·0	71·0	115·0	65·0	64·5	64·1	87·1	125·0	107·1	50·0
26	115·0	89·0	—	40·5	20·5*	100·0	80·0	29·5	28·0	25·5	26·0	65·0*	63·0*	119·0	76·5	62·5	59·0	·	111·3	—	46·8
32	110·0	88·0	91·0	45·5	18·5	94·5	66·5	29·0	28·5	29·5	25·5	58·5	60·5	126·0	67·0	73·5	61·9	96·3	132·3	96·7	47·9
38	117·0	95·5	100·5	44·5	20·0	95·5	65·0	30·0	28·5	26·0	26·5	55·5*	55·0*	130·0	70·0	71·0	58·1	105·2	146·9	95·0	54·5
39	109·0	90·5	85·5	42·5	18·0	100·0	73·0	27·5	29·5	27·0	24·5	55·0	52·0	126·0	68·0	62·0	55·0	85·5	124·0	105·8	52·9
40	120·0	94·0	92·5	42·5	18·0*	102·0	76·5	29·5	27·5*	29·0	24·0*	58·5	54·0	122·0	73·0	70·0	57·4	90·7	122·9	101·6	54·6
52	120·0	90·5	97·0	47·5	21·5	102·0	81·0	31·5	29·5*	25·5	23·5	59·0	61·5	113·0	69·0	62·0	57·8	95·1	111·7	93·3	51·2
53	106·5	81·5	93·5	46·0	19·0	94·0	69·5	32·0	26·5	27·5	25·5	60·0	55·0	123·0	78·0	68·0	63·8	99·5	117·3	87·2	58·2
59	125·0	85·0	—	42·0	22·0*	99·0	73·5	33·0	26·5*	—	28·5	59·5	63·0	117·0	62·0	69·0	60·1	—	115·6	—	52·4
61	108·0	94·5	91·0	44·5	21·0*	101·0	76·0	28·5	—	30·5	—	58·0	54·0*	125·0*	70·0	60·0	57·4	90·1	124·3	103·8	52·8
80	127·0	95·5	100·0	44·5	19·5	102·5	71·5	29·0	27·5	33·5	25·5	56·0	51·0	131·0	71·0	—	54·6	97·6	133·6	95·5	56·9
97	—	92·5	91·5	45·0	19·0	106·5	80·0	33·5	—	—	—	61·0	59·0*	125·0	75·0	60·0	57·3	85·9	115·6	101·1	56·8
119	118·0	88·0	89·5	42·0	17·0	100·0	73·5	31·5	26·5*	28·5	22·0*	58·5	60·0*	120·0	67·0	69·0	58·5	89·5	119·7	98·3	52·5
135	—	—	89·0	41·0	16·5*	99·0	75·0	32·5	30·5	31·5	27·0	—	49·5?	118·0	—	78·0	—	89·9	—	—	65·7
Y	115·0	92·0	—	46·0	18·0	95·0	69·0	28·5	28·5	29·0	24·0	50·0	47·0	129·0	68·0	60·0	52·6	—	133·3	—	60·6
Wanta No.	114·0	98·0	88·5	41·5	15·5	97·0	70·0	29·5	25·5	30·5	25·0	52·5*	53·0	126·0	65·0	68·0	54·1	91·2	140·0	110·7	55·7

* These measurements were taken on the right side.

ON THE NUMERICAL EVALUATION OF HIGH ORDER INCOMPLETE EULERIAN INTEGRALS.

By KARL AND MARGARET V. PEARSON.

I. *The Incomplete* Γ-*Function.*

The Incomplete Γ-Function is defined by

$$\Gamma_{x_0}(p) = \int_0^{x_0} e^{-x} x^{p-1}\,dx \quad \text{..............................(i)},$$

and the Incomplete Γ-Function Ratio by

$$I'(x_0, p) = \int_0^{x_0} e^{-x} x^{p-1}\,dx/\Gamma(p) \quad \text{........................(ii)}.$$

The Ratio is the probability integral of the frequency curve

$$y = y_0 e^{-x} x^{p-1} \quad \text{................................(iii)}.$$

This curve has for its characteristics

$$\text{Mode} = \tilde{x} = p - 1, \quad \text{Mean} = \bar{x} = p, \quad \text{and Standard Deviation} = \sqrt{p}\ldots\text{(iv)}.$$

In the *Tables of the Incomplete* Γ-*Function** the notation is somewhat different. Therein

$$I(u, p) = \int_0^{u\sqrt{p+1}} e^{-x} x^p\,dx/\Gamma(p+1),$$

or

$$I\left(\frac{x_0}{\sqrt{p}}, p-1\right) = \int_0^{x_0} e^{-x} x^{p-1}\,dx/\Gamma(p) = I'(x_0, p) \quad \text{..............(v)}.$$

Let us transfer (iii) to its mode and then integrate it up to a distance x' beyond the mode, where actually x' may be positive or negative.

$$I'(x_0, p)\,\Gamma(p)$$
$$= \int_0^{p-1} e^{-x} x^{p-1}\,dx + \int_0^{x_0'} e^{-(p-1)-x'} (p-1+x')^{p-1}\,dx'$$
$$= I\left(\frac{p-1}{\sqrt{p}}, p-1\right)\Gamma(p) + e^{-(p-1)}(p-1)^{p-1}\int_0^{x_0'} \left\{e^{-\frac{x'}{p-1}}\left(1+\frac{x'}{p-1}\right)\right\}^{p-1} dx'.$$

Now let us take

$$x'/(p-1) = t \quad \text{................................(vi)}.$$

Then

$$I'(x_0, p) = I\left(\frac{p-1}{\sqrt{p}}, p-1\right) + \frac{\sqrt{2\pi}\,e^{-(p-1)}(p-1)^{p-\frac{1}{2}}}{\Gamma(p)}\,\frac{\sqrt{p-1}}{\sqrt{2\pi}}\int_0^{t_0} \{e^{-t}(1+t)\}^{p-1}\,dt$$
$$= I\left(\frac{p-1}{\sqrt{p}}, p-1\right) + \frac{P_0\sqrt{p-1}}{\sqrt{2\pi}}\int_0^{t_0} W_t\,dt \quad \text{............................(vii)},$$

* Published by the *Biometrika* Office, 42*s*. net.

where $$t_0 = x_0'/(p-1) = \frac{x_0 - (p-1)}{p-1} \quad \text{.....................(viii)},$$

$$P_0 = \frac{\sqrt{2\pi}\, e^{-(p-1)}\,(p-1)^{p-\frac{1}{2}}}{\Gamma(p)} \quad \text{..........................(ix)},$$

and is, for $p = 50$ and upwards, only slightly less than unity;

$$W_t = \{e^{-t}(1+t)\}^{p-1},$$

$$\log W_t = (p-1)\{-t + \log(1+t)\} = (p-1)\left(-\frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots\right),$$

or $$W_t = e^{-\frac{1}{2}(p-1)t^2} \times e^{(p-1)\left(\frac{t^3}{3} - \frac{t^4}{4} + \frac{t^5}{5} - \dots\right)}$$

$$= e^{-\frac{1}{2}(p-1)t^2}\,(1 + T + \tfrac{1}{2}T^2 + \tfrac{1}{6}T^3 + \tfrac{1}{24}T^4 + \dots) \quad \text{..................(x)},$$

where $$T = (p-1)\frac{t^3}{3}(1 - \tfrac{3}{4}t + \tfrac{3}{5}t^2 - \dots) \quad \text{.....................(xi)}.$$

The part of W_t independent of T will give us

$$\frac{P_0\sqrt{p-1}}{\sqrt{2\pi}}\int_0^{t_0} e^{-\frac{1}{2}(p-1)t^2}\,dt.$$

Taking $\zeta^2 = (p-1)t^2$, this becomes

$$P_0\int_0^{\zeta_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\zeta^2}\,d\zeta \quad \text{.............................(xii)},$$

where $$\zeta_0 = \sqrt{p-1}\,t_0 = \frac{x_0 - (p-1)}{\sqrt{p-1}} \quad \text{........................(xiii)}.$$

This can be found at once from the Tables of the Normal Probability Integral

$$= P_0\,\tfrac{1}{2}\alpha_{\zeta_0} \quad \text{...................................(xiv)},$$

where $\frac{1}{2}\alpha_{\zeta_0}$ is always less than (or equal to) $\frac{1}{2}$.

We now turn to the remainder of W_t. This gives

$$W_T = e^{-\frac{1}{2}(p-1)t^2}\left(T + \frac{1}{2}T^2 + \frac{1}{6}T^3 + \frac{1}{24}T^4 + \dots\right)$$

$$\begin{aligned} &= e^{-\frac{1}{2}(p-1)t^2}\Big\{\frac{1}{3}(p-1)t^3\frac{1}{4}(p-1)t^4 + \frac{1}{5}(p-1)t^5 - \left(\frac{1}{6} - \frac{1}{18}(p-1)\right)(p-1)t^6 \\ &\quad + \left(\frac{1}{7} - \frac{1}{12}(p-1)\right)(p-1)t^7 - \left(\frac{1}{8} - \frac{47}{480}(p-1)\right)(p-1)t^8 \\ &\quad + \left(\frac{1}{9} - \frac{19}{180}(p-1) + \frac{1}{162}(p-1)^2\right)(p-1)t^9 \\ &\quad - \left(\frac{1}{10} - \frac{153}{1400}(p-1) + \frac{1}{72}(p-1)^2\right)(p-1)t^{10} \\ &\quad + \left(\frac{1}{11} - \frac{31}{280}(p-1) + \frac{31}{1440}(p-1)^2\right)(p-1)t^{11} \\ &\quad - \left(\frac{1}{12} - \frac{3349}{30240}(p-1) + \frac{493}{17280}(p-1)^2 - \frac{1}{1944}(p-1)^3\right)(p-1)t^{12} + \dots\Big\} \end{aligned}$$

$$\text{......(xv)},$$

and we need

$$R_T = \frac{e^{-(p-1)}(p-1)^{p-1}}{\Gamma(p)} \int_0^{x_0'} W_T dx' = \frac{e^{-(p-1)}(p-1)^p}{\Gamma(p)} \int_0^{t_0} W_T dt \quad \text{...(xvi)},$$

where $$t_0 = x_0'/(p-1) = (x_0-(p-1))/(p-1).$$

We will now change our variable to z, where

$$z = \tfrac{1}{2}(p-1)t^2 \text{ and } z_0 = \frac{\tfrac{1}{2}(p-1)(x_0-(p-1))^2}{(p-1)^2} = \tfrac{1}{2}\frac{x_0'^2}{p-1} \quad \text{...(xvii)},$$

with x_0' representing the distance from the mode to the boundary of our original Γ-function.

Since $$t = \left(\frac{2z}{p-1}\right)^{\frac{1}{2}}, \text{ we find } dt = \tfrac{1}{2}\sqrt{\frac{2}{p-1}}\, z^{-\frac{1}{2}} dz,$$

and thus we have $$R_T = \frac{1}{\sqrt{2}} \frac{e^{-(p-1)}(p-1)^{p-\frac{1}{2}}}{\Gamma(p)} \int_0^{z_0} z^{-\frac{1}{2}} W_T dz$$

$$= \frac{1}{\sqrt{2\pi}} P_0 \frac{1}{\sqrt{2}} \int_0^{z_0} z^{-\frac{1}{2}} W_T dz.$$

Substituting z for t in (xv), we find

$$\begin{aligned} I'(x_0, p) &= \int_0^{x_0} e^{-x} x^{p-1} dx/\Gamma(p) \\ &= I\left(\frac{p-1}{\sqrt{p}}, p-1\right) + P_0(\tfrac{1}{2}\alpha_{\zeta_0}) + a_1 I(\cdot 7071{,}0678 z_0, 1) \\ &\quad + a_{1\cdot 5} I(\cdot 6324{,}5553 z_0, 1\cdot 5) + a_2 I(\cdot 5773{,}5027 z_0, 2) \\ &\quad + a_{2\cdot 5} I(\cdot 5345{,}2248 z_0, 2\cdot 5) + a_3 I(\cdot 5000{,}0000 z_0, 3) \\ &\quad + a_{3\cdot 5} I(\cdot 4714{,}0452 z_0, 3\cdot 5) + a_4 I(\cdot 4472{,}1360 z_0, 4) \\ &\quad + a_{4\cdot 5} I(\cdot 4264{,}0143 z_0, 4\cdot 5) + a_5 I(\cdot 4082{,}4828 z_0, 5) \\ &\quad + a_{5\cdot 5} I(\cdot 3922{,}3227 z_0, 5\cdot 5) + a_6 I(\cdot 3779{,}6447 z_0, 6) \quad \text{...(xviii)}, \end{aligned}$$

where $$\zeta_0 = \frac{x_0'}{\sqrt{p-1}}, \quad z_0 = \tfrac{1}{2}\frac{x_0'^2}{p-1} \text{ as in (xvii)},$$

$$P_0 = \sqrt{2\pi(p-1)}\,(p-1)^{p-1} e^{-(p-1)}/\Gamma(p),$$

and equals by Stirling's Theorem

$$\left(1 + \frac{1}{12}\frac{1}{(p-1)} + \frac{1}{288}\frac{1}{(p-1)^2} - \frac{139}{51840}\frac{1}{(p-1)^3} + \dots\right)^{-1},$$

which will generally be adequate for computing P_0.

Lastly we have

$$a_1 = \cdot 26596{,}15203 \frac{P_0}{\sqrt{p-1}}, \qquad a_{1\cdot 5} = -\cdot 37500{,}00000 \frac{P_0}{(p-1)},$$

$$a_2 = \cdot 63830{,}76486 \frac{P_0}{(p-1)^1}, \qquad a_{2\cdot 5} = \cdot 41666{,}66667 \left(1 - \frac{3}{p-1}\right) \frac{P_0}{p-1},$$

$$a_3 = -1 \cdot 59576{,}91216 \left(1 - \frac{12}{7} \frac{1}{p-1}\right) \frac{P_0}{(p-1)^{1\cdot 5}},$$

$$a_{3\cdot 5} = 5 \cdot 140625 \left(1 - \frac{60}{47} \frac{1}{p-1}\right) \frac{P_0}{(p-1)^2},$$

$$a_4 = \cdot 9456{,}40961 \left(1 - \frac{171}{10} \frac{1}{p-1} + 18 \frac{1}{(p-1)^2}\right) \frac{P_0}{(p-1)^{1\cdot 5}},$$

$$a_{4\cdot 5} = -6 \cdot 5625 \left(1 - \frac{1377}{175} \frac{1}{p-1} + \frac{36}{5} \frac{1}{(p-1)^2}\right) \frac{P_0}{(p-1)^2},$$

$$a_5 = 32 \cdot 9792{,}2852 \left(1 - \frac{36}{7} \frac{1}{(p-1)} + \frac{1440}{341} \frac{1}{(p-1)^2}\right) \frac{P_0}{(p-1)^{2\cdot 5}},$$

$$a_{5\cdot 5} = 2 \cdot 67361{,}11111 \left(1 - \frac{4437}{80(p-1)} + \frac{30141}{140(p-1)^2} - \frac{162}{(p-1)^3}\right) \frac{P_0}{(p-1)^2},$$

$$a_6 = -28 \cdot 3692{,}2883 \left(1 - \frac{15759}{700(p-1)} + \frac{2493}{35(p-1)^2} - \frac{648}{13(p-1)^3}\right) \frac{P_0}{(p-1)^{2\cdot 5}} \qquad \ldots\ldots\text{(xix)}.$$

Illustration I. Let us now test the above results on a sample case; we will take the worst possible case, namely $p = 50$ on the border of the Γ-Function Tables, and find from those tables the value of the required Γ-function ratio by which to test our expansion. Let us take x_0' considerable, $= 2\sqrt{p-1}$, i.e. $= 14$. Accordingly

$$\zeta_0 = \frac{x_0'}{\sqrt{p-1}} = 2, \text{ and } z_0 = \tfrac{1}{2}\frac{x_0'^2}{p-1} = 2,$$

also. Further, $x_0 = (p-1) + x_0' = 63$, and the actual value

$$= I'(63, 50) = I'\left(\frac{63}{\sqrt{p}}, p-1\right) = I(\cdot 8909{,}54543, 49) = \cdot 959{,}5825$$

by interpolation from the *Incomplete Γ-Function Tables.*

Again, $I\left(\frac{p-1}{\sqrt{p}}, p-1\right)$, the area up to the mode,

$$= I\left(\frac{49}{\sqrt{50}}, 49\right) = I(6 \cdot 9296{,}4646, 49),$$

and from the same tables, $\qquad = \cdot 462{,}1044.$

Calculating P_0, we find for its value, whether from the full value or Stirling's approximation, $\cdot 9983{,}00788$. The normal term is

$$P_0 \times \int_0^{\zeta_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\zeta^2} d\zeta = P_0 \times \cdot 477{,}2499,$$

since $\zeta_0 = 2$. Thus this term $= \cdot 476{,}4390$. Calculating the a_s's and $I'\left(\frac{z_0}{\sqrt{s+1}}, s\right)$'s, we find

	a_s	I_s	$a_s \times I_s$
a_1	$+\cdot 0379{,}2994$	$I(1\cdot 4142{,}1356,\ 1) \quad = \cdot 593{,}9947$	$+\cdot 022{,}5302$
$a_{1\cdot 5}$	$-\cdot 0076{,}4006$	$I(1\cdot 2649{,}1106,\ 1\cdot 5) = \cdot 450{,}5816$	$-\cdot 003{,}4425$
a_2	$+\cdot 0018{,}5779$	$I(1\cdot 1547{,}0054,\ 2) \quad = \cdot 323{,}3226$	$+\cdot 000{,}6007$
$a_{2\cdot 5}$	$+\cdot 0079{,}6922$	$I(1\cdot 0690{,}4497,\ 2\cdot 5) = \cdot 220{,}2225$	$+\cdot 001{,}7550$
a_3	$-\cdot 0044{,}8199$	$I(1\cdot 0000{,}0000,\ 3) \quad = \cdot 142{,}8765$	$-\cdot 000{,}6404$
$a_{3\cdot 5}$	$+\cdot 0020{,}8171$	$I(0\cdot 9428{,}0909,\ 3\cdot 5) = \cdot 088{,}5874$	$+\cdot 000{,}1844$
a_4	$+\cdot 0018{,}1243$	$I(0\cdot 8944{,}2719,\ 4) \quad = \cdot 052{,}6530$	$+\cdot 000{,}0954$
$a_{4\cdot 5}$	$-\cdot 0022{,}9861$	$I(0\cdot 8528{,}0287,\ 4\cdot 5) = \cdot 030{,}0831$	$-\cdot 000{,}0691$
a_5	$+\cdot 0017{,}5674$	$I(0\cdot 8164{,}9658,\ 5) \quad = \cdot 016{,}5637$	$+\cdot 000{,}0291$
$a_{5\cdot 5}$	$-\cdot 0000{,}4846$	$I(0\cdot 7844{,}6454,\ 5\cdot 5) = \cdot 008{,}8087$	$-\cdot 000{,}0004$
a_6	$-\cdot 0009{,}6015$	$I(0\cdot 7559{,}2895,\ 6) \quad = \cdot 004{,}5338$	$-\cdot 000{,}0045$

These include all the terms up to the order $\frac{1}{(p-1)^{2\cdot 5}}$.

We will now sum these terms. The sum of the $I\left(\frac{p-1}{\sqrt{p}}, p-1\right)$ and the normal term is $\cdot 938{,}5434$. Adding in the a_1 term, we have $\cdot 961{,}0736$, whence we proceed as follows:

Terms included	Approximate value	Size and place of error	Terms included	Approximate value	Size and place of error
Up to a_1	·961,0736	+1 in 3rd	Up to a_4	·959,6262	+4 in 5th
„ $a_{1\cdot 5}$	·957,6311	−2 in 3rd	„ $a_{4\cdot 5}$	·959,5571	−2 in 5th
„ a_2	·958,2318	−1 in 3rd	„ a_5	·959,5862	+4 in 6th
„ $a_{2\cdot 5}$	·959,9868	+4 in 4th	„ $a_{5\cdot 5}$	·959,5858	+3 in 6th
„ a_3	·959,3464	−2 in 4th	„ a_6	·959,5813	−1 in 6th
„ $a_{3\cdot 5}$	·959,5308	−5 in 5th	True value	·959,5825	0 in 7th

Thus with 13 terms of the expansion (xviii), we have only an error of a unit in the sixth decimal place. Accordingly if a table* could be computed of P_0 and the $I\left(\frac{p-1}{\sqrt{p}}, p-1\right)$, together with the first eleven a's, all values of the Incomplete Γ-function could be thrown back on the existing tables for values of $p > 50$, with an accuracy of the order ·000,001, or thereabouts.

Of course one fully admits that this expansion of high order Γ-functions in terms of low order Γ-functions suffers from the same defects as expansions in incomplete normal moment functions, or in tetrachoric functions. But such a table as we have spoken of would probably provide all the accuracy needful in the majority of statistical problems, and would be less laborious than the use of quadrature formulae.

* It might be needful to form the table for every unit value of p from 50 to 100, but later it would probably be adequate to take the intervals at several units distance.

The table above indicates that with every additional a_s term included the approximation becomes closer. Further it indicates that for examples such as our first illustration, we shall have accuracy to the third decimal place if we include terms up to $a_{2\cdot5}$, to the fourth decimal place if we proceed to $a_{3\cdot5}$, while up to a_5 we have accuracy to the fifth decimal place.

Illustration II. We must work another example to illustrate the process, namely $p = 101$, and we will take our $x_0 = 120$, i.e. $= p - 1 + 2\sqrt{p-1}$, so that we are finding

$$\int_0^{120} e^{-x} x^{100}\, dx / \Gamma(101).$$

By choosing $x_0' = 20 = 2\sqrt{p-1}$, our entire series of $I\left(\frac{x_0}{\sqrt{x+1}}, s\right)$ remain the same and need not be recalculated, all the a_s's, however, are changed as well as P_0.

We find $P_0 = \cdot9991{,}6702$, and since $\frac{1}{2}\alpha_{\zeta_0}$ remains the same, the normal term contribution is

$$\cdot9991{,}6702 \times \cdot477{,}2499 = \cdot476{,}8524.$$

Further*
$$\int_0^{p-1} e^{-x} x^{p-1}\, dx / \Gamma(p) = \cdot473{,}4402.$$

Together, these give us $\cdot950{,}2926$.

We will now form a table as before:

	Value of a_s	Value of I_s	$a_s I_s$	Value up to a_s	Exact value less value up to a_s
a_1	$+\cdot0265{,}73008$	$\cdot593{,}9947$	$+\cdot015{,}7848$	$\cdot966{,}0774$	$\cdot000{,}7453$
$a_{1\cdot5}$	$-\cdot0037{,}46876$	$\cdot450{,}5816$	$-\cdot001{,}6883$	$\cdot964{,}3891$	$+\cdot000{,}9430$
a_2	$+\cdot0006{,}37776$	$\cdot323{,}3226$	$+\cdot000{,}2062$	$\cdot964{,}5953$	$+\cdot000{,}7368$
$a_{2\cdot5}$	$+\cdot0040{,}38300$	$\cdot220{,}2225$	$+\cdot000{,}8893$	$\cdot965{,}4846$	$-\cdot000{,}1525$
a_3	$-\cdot0015{,}67107$	$\cdot142{,}8765$	$-\cdot000{,}2239$	$\cdot965{,}2607$	$+\cdot000{,}0714$
$a_{3\cdot5}$	$+\cdot0005{,}07077$	$\cdot088{,}5874$	$+\cdot000{,}0449$	$\cdot965{,}3056$	$+\cdot000{,}0265$
a_4	$+\cdot0007{,}84984$	$\cdot052{,}6530$	$+\cdot000{,}0412$	$\cdot965{,}3468$	$\cdot000{,}0147$
$a_{4\cdot5}$	$-\cdot0006{,}04681$	$\cdot030{,}0831$	$-\cdot000{,}0182$	$\cdot965{,}3286$	$+\cdot000{,}0035$
a_5	$+\cdot0003{,}12710$	$\cdot016{,}6037$	$+\cdot000{,}0052$	$\cdot965{,}3338$	$-\cdot000{,}0017$
$a_{5\cdot5}$	$-\cdot0001{,}19509$	$\cdot008{,}8087$	$-\cdot000{,}0011$	$\cdot965{,}3327$	$\cdot000{,}0006$
* a_6	$-\cdot0002{,}21647$	$\cdot004{,}5338$	$-\cdot000{,}0010$	$\cdot965{,}3317$	$+\cdot000{,}0004$

The exact value was found by quadratures from $x_0 = 120$ to the end of the curve to be $\cdot034{,}6679$, and $1 - \cdot034{,}6679 = \cdot965{,}3321$. It will be seen that our final result, using up to the a_6 term, is only four units out in the seventh decimal place. Up to $a_{3\cdot5}$ we are only out two units in the fifth decimal place, and up to $a_{2\cdot5}$ only between one and two units in the fourth decimal place.

It will be clear to the reader that to replace an incomplete Γ-function by a normal curve will not give a correct result even in the *second* decimal place, when $p = 101$, if we want an area reaching to about twice the standard deviation from

* Found by quadrature from 45 ordinates at unit distance apart.

the mode. It would work better, of course, if we knew the area up to the mode, and were only seeking areas up to $\pm\sqrt{p}$ on either side of it. But clearly the safest way to reach high order incomplete Γ-functions is to form a table of $\int_0^{p-1}\frac{e^{-x}x^{p-1}\,dx}{\Gamma(p)}$ and of the functions of p, i.e. $P_0, a_1, a_{1\cdot 5}\ldots a_6$, and so throw such incomplete Γ-functions back on the already published table.

Will any trained computer volunteer for the work?

II. *The Incomplete* B-*Function.*

Let the required integral be the B-function ratio

$$I_{x_0}(p+1, q+1)=\int_0^{x_0}\frac{x^p(1-x)^q\,dx}{\mathrm{B}(p+1, q+1)}\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

where $\mathrm{B}(p+1, q+1)=\Gamma(p+1)\Gamma(q+1)/\Gamma(p+q+2)$.

We will suppose $q>p$, which it is always feasible to arrange, if needful, by aid of the transformation $x=1-x'$, and finding the area from the other end of the curve.

We have

$$\begin{aligned}I_x(p+1, q+1)&=\frac{\Gamma(p+q+2)}{\Gamma(p+1)\Gamma(q+1)}\int_0^{x_0}x^p e^{q\log(1-x)}\,dx\\&=\frac{\Gamma(p+q+2)}{\Gamma(p+1)\Gamma(q+1)}\int_0^{x_0}x^p e^{-q(x+\frac{1}{2}x^2+\frac{1}{3}x^3+\ldots)}\,dx\\&=\frac{\Gamma(p+q+2)}{\Gamma(p+1)\Gamma(q+1)}\int_0^{x_0}x^p e^{-qx}e^{-q\frac{x^2}{2}-q\frac{x^3}{3}-q\frac{x^4}{4}-\ldots}\,dx.\end{aligned}$$

We may now expand the exponential

$$e^{-q\frac{x^2}{2}-q\frac{x^3}{3}-q\frac{x^4}{4}-\ldots}$$

as equal to

$$1+c_2x^2+c_3x^3+\ldots+c_sx^s\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

and our integral becomes

$$\begin{aligned}I_{x_0}(p+1, q+1)&=\frac{\Gamma(p+q+2)}{\Gamma(p+1)\Gamma(q+1)}\int_0^{x_0}x^p e^{-qx}(1+c_2x^2+c_3x^3+\ldots+c_sx^s+\ldots)\,dx\\&=\frac{\Gamma(p+q+2)}{\Gamma(q+1)}\left\{\int_0^{x_0}\frac{x^p e^{-qx}\,dx}{\Gamma(p+1)}+c_2\int_0^{x_0}\frac{x^{p+2}e^{-qx}\,dx}{\Gamma(p+1)}+\ldots\right.\\&\qquad\left.+c_s\int_0^{x_0}\frac{x^{p+s}e^{-qx}\,dx}{\Gamma(p+1)}+\ldots\right\}\\&=\frac{\Gamma(p+q+2)}{\Gamma(q+1)}\left\{\frac{I'(z_0, p+1)}{q^{p+1}}+c_2\frac{I'(z_0, p+3)}{q^{p+3}}(p+2)(p+1)\right.\\&\qquad+c_3\frac{I'(z_0, p+4)}{q^{p+4}}(p+3)(p+2)(p+1)\ldots\\&\qquad\left.+c_s\frac{I'(z_0, p+s+1)}{q^{p+s+1}}(p+s)(p+s-1)\ldots(p+1)+\ldots\right\}\\&\qquad\qquad\ldots\ldots\text{(iii)},\end{aligned}$$

where we have taken $z=qx$ and $z_0=qx_0$.

If we now use the notation of the *Tables of the Incomplete Γ-Function* we have, on putting $p-1$ for p, and $q-1$ for q,

$$I_{x_0}(p,q)=\int_0^{x_0} x^{p-1}(1-x)^{q-1}\,dx/B(p,q)$$

$$=\frac{\Gamma(p+q)}{\Gamma(q)}\left\{\frac{I\left(\frac{x_0(q-1)}{\sqrt{p}},\,p-1\right)}{(q-1)^p}+c_2 I\left(\frac{x_0(q-1)}{\sqrt{p+2}},\,p+1\right)\frac{(p+1)p}{(q-1)^{p+2}}\right.$$

$$+c_3 I\left(\frac{x_0(q-1)}{\sqrt{p+3}},\,p+2\right)\frac{(p+2)(p+1)p}{(q-1)^{p+3}}+\dots$$

$$\left.+c_s I\left(\frac{x_0(q-1)}{\sqrt{p+3}},\,p+s-1\right)\frac{(p+s-1)\dots(p+1)}{(q-1)^{p+s}}+\dots\right\}\quad\dots\dots\text{(iv)}$$

Thus if $p<50$ and $q>50$, we have reduced our integral to a series of incomplete Γ-function ratios, which can be taken out of the above-mentioned tables. We have to determine the c's and to consider the convergency of the series.

Turning back to (ii) and taking logarithmic differentials, we have

$$(-qx-qx^2-qx^3-\dots-qx^s-\dots)(1+c_2x^2+c_3x^3+\dots+c_sx^s+\dots)$$
$$=2c_2x+3c_3x^2+\dots+sc_sx^{s-1}+\dots.$$

Hence we easily find $c_2=-\frac{1}{2}q$, $c_3=-\frac{1}{3}q$, and generally

$$c_s=\frac{s-1}{s}c_{s-1}-\frac{q}{s}c_{s-2}\quad\dots\dots\dots\dots\dots\dots\dots\dots\text{(v)}.$$

To suit the notation of (iv), we must, however, put $q-1$ for q, and we then have

$$c_2=-\frac{q-1}{2},\quad c_3=-\frac{q-1}{3},\quad c_4=\frac{1}{8}(q-1)^2\left(1-\frac{2}{q-1}\right),$$

$$c_5=\frac{1}{6}(q-1)^2\left(1-\frac{6}{5(q-1)}\right),\quad c_6=-\frac{1}{48}(q-1)^3\left(1-\frac{26}{3(q-1)}+\frac{8}{(q-1)^2}\right),$$

$$c_7=-\frac{1}{24}(q-1)^3\left(1-\frac{22}{5(q-1)}+\frac{24}{7(q-1)^2}\right),$$

$$c_8=\frac{1}{384}(q-1)^4\left(1-\frac{68}{3(q-1)}+\frac{696}{10(q-1)^2}-\frac{48}{(q-1)^3}\right),$$

$$c_9=\frac{1}{144}(q-1)^4\left(1-\frac{472}{45(q-1)}+\frac{892}{35(q-1)^2}-\frac{16}{(q-1)^3}\right),$$

$$c_{10}=-\frac{1}{3840}(q-1)^5\left(1-\frac{140}{3(q-1)}+\frac{904}{3(q-1)^2}-\frac{23088}{35(q-1)^3}+\frac{384}{(q-1)^4}\right),$$

$$c_{11}=-\frac{1}{1152}(q-1)^5\left(1-\frac{916}{45(q-1)}+\frac{3716}{35(q-1)^2}-\frac{6704}{35(q-1)^3}+\frac{1152}{11(q-1)^4}\right),$$

$$c_{12}=\frac{1}{46080}(q-1)^6\left(1-\frac{250}{3(q-1)}+\frac{28828}{27(q-1)^2}-\frac{478{,}024}{105(q-1)^3}+\frac{155{,}552}{21(q-1)^4}-\frac{3840}{(q-1)^5}\right)$$

$$\dots\dots\text{(vi)}.$$

Now let us write (iv) in the form

$$I_{x_0}(p, q) = a_0 I\left(x_0(q-1)/\sqrt{p},\, p-1\right) + a_2 I\left(x_0(q-1)/\sqrt{p+2},\, p+1\right)$$
$$+ a_3 I\left(x_0(q-1)/\sqrt{p+3},\, p+2\right) + \ldots + a_s I\left(x_0(q-1)/\sqrt{p+s},\, p+s-1\right) + \ldots \quad \ldots\ldots\text{(vii)}.$$

Then we have

$$a_0 = \frac{\Gamma(p+q)}{\Gamma(q)}\frac{1}{(q-1)^p}, \quad a_2 = -\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+1)p}{2(q-1)^{p+1}},$$

$$a_3 = -\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+2)(p+1)p}{3(q-1)^{p+2}},$$

$$a_4 = +\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+3)(p+2)(p+1)p}{8(q-1)^{p+2}}\left(1-\frac{2}{q-1}\right),$$

$$a_5 = +\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+4)(p+3)(p+2)(p+1)p}{6(q-1)^{p+3}}\left(1-\frac{6}{5(q-1)}\right),$$

$$a_6 = -\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+5)(p+4)(p+3)(p+2)(p+1)p}{48(q-1)^{p+3}}\left(1-\frac{26}{3(q-1)}+\frac{8}{(q-1)^2}\right),$$

$$a_7 = -\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+6)(p+5)\ldots(p+1)p}{24(q-1)^{p+4}}\left(1-\frac{22}{5(q-1)}+\frac{24}{7(q-1)^2}\right),$$

$$a_8 = +\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+7)(p+6)\ldots(p+1)p}{384(q-1)^{p+4}}\left(1-\frac{68}{3(q-1)}+\frac{696}{10(q-1)^2}-\frac{48}{(q-1)^3}\right),$$

$$a_9 = +\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+8)(p+7)\ldots(p+1)p}{144(q-1)^{p+5}}\left(1-\frac{472}{45(q-1)}+\frac{892}{35(q-1)^2}-\frac{16}{(q-1)^3}\right),$$

$$a_{10} = -\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+9)(p+8)\ldots(p+1)p}{3840(q-1)^{p+5}}$$
$$\times\left(1-\frac{140}{3(q-1)}+\frac{964}{3(q-1)^2}-\frac{23088}{35(q-1)^3}+\frac{384}{(q-1)^4}\right),$$

$$a_{11} = -\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+10)(p+9)\ldots(p+1)p}{1152(q-1)^{p+6}}$$
$$\times\left(1-\frac{916}{45(q-1)}+\frac{3716}{35(q-1)^2}-\frac{6704}{35(q-1)^3}+\frac{1152}{11(q-1)^4}\right),$$

$$a_{12} = +\frac{\Gamma(p+q)}{\Gamma(q)}\frac{(p+11)(p+10)\ldots(p+1)p}{46080(q-1)^{p+6}}$$
$$\times\left(1-\frac{250}{3(q-1)}+\frac{28828}{27(q-1)^2}-\frac{478{,}024}{105(q-1)^3}+\frac{155{,}552}{21(q-1)^4}-\frac{3840}{(q-1)^5}\right) \quad \ldots\ldots\text{(viii)}.$$

Illustration (i). As a first example, we will find the incomplete B-function ratio

$$I_{\cdot 024}(4, 101) = \int_0^{\cdot 024} x^3(1-x)^{100}\,dx/\mathrm{B}(4, 101),$$

or, if $x' = (1-x)$,

$$= \int_{\cdot 976}^{1} (1-x')^3\, x'^{100}\,dx'/\mathrm{B}(4, 101).$$

This integral is fairly easy to find by the expansion of $(1-x')^p$ and direct integration, using many figure logarithms to determine $(\cdot 976)^{101}$, which equals $\cdot 0859{,}8675{,}90$. Thus we obtain

$$I_{\cdot 024}(4,\ 101) = \cdot 2402{,}41262.$$

We now proceed to compute our a's from (viii) and determine the I's by interpolation. The results are given in the table below:

	a_s	I_s	Values from Γ-Tables	$a_s \times I_s$	Value of $S(a_s I_s)$ up to a_s
a_0	$+1\cdot 1035{,}5021$	$I_0(1\cdot 2,\ 3)$	$\cdot 2212{,}7710$	$\cdot 2441{,}90397$	$\cdot 2441{,}90397$
a_2	$-\ \cdot 1103{,}55021$	$I_2(\cdot 9797{,}95896,\ 5)$	$\cdot 0356{,}7257$	$-\cdot 0039{,}36647$	$\cdot 2402{,}53750$
a_3	$-\ \cdot 0044{,}4201$	$I_3(\cdot 9071{,}14735,\ 6)$	$\cdot 0115{,}9372$	$-\cdot 0000{,}51180$	$\cdot 2402{,}02570$
a_4	$+\ \cdot 0113{,}55532$	$I_4(\cdot 8485{,}28136,\ 7)$	$\cdot 0333{,}3734$	$+\cdot 0000{,}37897$	$\cdot 2402{,}40467$
a_5	$+\ \cdot 0012{,}21145$	$I_5(\cdot 8,\ 8)$	$\cdot 0008{,}6210$	$+\cdot 0000{,}01053$	$\cdot 2402{,}41529$
a_6	$-\ \cdot 0012{,}71078$	$I_6(\cdot 7589{,}46618,\ 9)$	$\cdot 0002{,}0019$	$-\cdot 0000{,}00254$	$\cdot 2402{,}41268$
a_7	$-\ \cdot 0002{,}65953$	$I_7(\cdot 7230{,}27229,\ 10)$	$\cdot 0000{,}4249$	$+\cdot 0000{,}00011$	$\cdot 2402{,}41277$
				True value	$=\cdot 2402{,}41262$

Now since our *Table of the Incomplete* Γ-*Function Ratios* is only computed to seven decimal places, we cannot trust the last column to more than seven decimals. It is thus not worth while including a_8. At a_6 we are correct to seven decimals. Thus when q is large compared to p, six terms are adequate to obtain seven-figure accuracy. When four-figure accuracy is adequate, a_4 will suffice. To obtain adequate values for the I_s's we used δ^4 interpolation for a_2 and a_3, but found that with a_3 the inclusion of δ^4, although modifying I_3 in the seventh decimal place, gave no change in the value of $a_3 I_3$ up to the ninth decimal place. It is accordingly sufficient to interpolate the I_s values, using only δ^2, except in the case of the values $s=2$, or perhaps $s=3$, as p increases relative to q.

It would thus seem that the method is satisfactory if p be as low as 4. Further, it can be a good deal shortened, if we do not use formulae (viii), but proceed in the following manner. We have seen by (v) that

$$c_s = \frac{s-1}{s} c_{s-1} - \frac{q}{s} c_{s-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(v)},$$

and further, by (iv), that

$$a_s = c_s \frac{(p+s-1)(p+s-2)\ldots(p+1)p}{(q-1)^s} \frac{\Gamma(p+q)}{\Gamma(q)(q-1)^p}$$

$$= C_s \times f_s = C c_s \frac{(p+s-1)(p+s-2)\ldots(p+1)p}{(q-1)^s} \quad \ldots\ldots\ldots\ldots\text{(ix)},$$

where $C = \dfrac{\Gamma(p+q)}{\Gamma(q)(q-1)^p}$, and $f_s = c_s \dfrac{(p+s-1)(p+s-2)\ldots(p+1)p}{(q-1)^s}$.

Hence in (v), replacing c_s by a_s, we find

$$a_s = \frac{p+s-1}{(q-1)^s}\{(s-1)a_{s-1} - (p+s-2)a_{s-2}\} \ldots\ldots\ldots\ldots\ldots\ldots\text{(x)},$$

or, again,

$$f_s = \frac{p+s-1}{(q-1)^s}\{(s-1)f_{s-1} - (p+s-2)f_{s-2}\} \ldots\ldots\ldots\ldots\ldots\text{(x } bis\text{)}.$$

The latter formula has some advantages if we wish to deal with more moderate numbers, as C may be large. We then multiply by C at the end of our desired series of f_s's.

Now the series of a_s's or f_s's can be *most rapidly ascertained in succession* by the machine from (x) or (x *bis*). For example, $a_0 = C$, $a_1 = 0$, or $f_0 = 1$, $f_1 = 0$, and for $p = 10$, $q = 101$, say, we have the series

$$a_2 = \tfrac{11}{200}(a_1 - 10a_0), \quad a_3 = \tfrac{12}{300}(2a_2 - 11a_1), \quad a_4 = \tfrac{13}{400}(3a_3 - 12a_2),$$
$$a_5 = \tfrac{14}{500}(4a_4 - 13a_3), \quad a_6 = \tfrac{15}{600}(5a_5 - 14a_4), \quad \text{etc.}$$

The rapidity of this method is great, but it has the disadvantage that, if a slip be made in one a_s, it is carried on to later a's. Hence it is advisable to check the last a by means of the corresponding value of a in (viii). Thus there is no difficulty in finding the a's to the s needed to give the number of places necessary in the required incomplete B-function. In the example worked out above, we obtain seven decimal-place accuracy for $s = 6$.

Actually at the very best we cannot hope for greater accuracy, for the *Tables of the Incomplete Γ-Function* only reach to seven decimals, and accordingly, as C grows greater than unity, we shall not be correct in the sixth or seventh figure when we multiply by C. We need a Γ-function table to some 10 to 12 figures, if C lies between 300 and 400. We will illustrate these points on a couple of examples.

Illustration (ii). Let us take $p = 10$, $q = 101$ and $x_0 = {\cdot}075$. Here

$$C = 1{\cdot}7018{,}2143{,}78,$$

and the general form for

$$I_s = I\left(\frac{x_0(q-1)}{\sqrt{p+s}}, p+s-1\right) = I\left(\frac{7{\cdot}5}{\sqrt{p+s}}, q+s\right),$$

and further

$$f_s = \frac{q+s}{100s}\{(s-1)f_{s-1} - (8+s)f_{s-2}\}.$$

Working the f's out in succession from this, we obtained the values given in the second column of the table below.

The true value of the function found by "brute force" integration and multiplying out was ·3109,71212, and checked by quadrature ·3109,7120. The value ·3109,712 will be correct to seven places and serve for comparison.

Table for $p = 10$, $q = 101$ and $x_0 = {\cdot}075$.

	f_s	$a_s = C \times f_s$	I_s From Γ-Function Tables	$a_s I_s$	$S(a_s I_s)$ up to a_s
0	1·0000,0000	1·7018,2143,78	I_0(2·3717,0824,5, *9*) = ·2235,0236	+ ·3805,1427	+ ·3805,1427
2	− ·55	− ·9360,0179,08	I_2(2·1650,0350,9, *11*) = ·0792,4131	− ·0741,7001	+ ·3063,4426
3	− ·044	− ·0748,8014,33	I_3(2·0801,2573,6, *12*) = ·0426,6583	− ·0031,9482	+ ·3031,4944
4	+ ·21021	+ ·3577,3988,44	I_1(2·0044,5931,4, *13*) = ·0215,6463	+ ·0077,1453	+ ·3108,6397
5	+ ·0395,5952	+ ·0673,2323,92	I_5(1·9364,9167,3, *14*) = ·0102,6042	+ ·0006,9076	+ ·3115,5473
6	− ·0686,2856,00	− ·1167,9355,47	I_6(1·875, *15*) = ·0046,0831	− ·0005,3822	+ ·3110,1651
7	− ·0229,7518,08	− ·0390,9965,52	I_7(1·8190,1718,8, *16*) = ·0019,5890	− ·0000,7659	+ ·3109,3992
8	+ ·0199,1615,23	+ ·0338,9373,49	I_8(1·7677,6695,3, *17*) = ·0007,9001	+ ·0000,2678	+ ·3109,6670
9	+ ·0109,9814,58	+ ·0187,1688,03	I_9(1·7206,1800,4, *18*) = ·0003,0301	+ ·0000,0567	+ ·3109,7237
10	− ·0049,3004,11	− ·0083,9107,07	I_{10}(1·6770,5098,3, *19*) = ·0001,1074	− ·0000,0093	+ ·3109,7144
11	− ·0046,9583,97	− ·0079,9148,07	I_{11}(1·6366,3417,7, *20*) = ·0000,3864	− ·0000,0031	+ ·3109,7113
12	+ ·0008,2177,53	+ ·0013,9851,48	I_{12}(1·5990,0537,3, *21*) = ·0000,1288	+ ·0000,0002	+ ·3109,7115

The a's are correct to their ten decimals and there is no difficulty in tabling more, in fact a_{13}, a_{14}, a_{15} and a_{16} were found. But to make use of these a_s's it would be needful to have the I_s's to more than seven figures. The I_s's found by interpolation have been used to *eight* figures, but the last figure is, of course, unreliable. The value of C being >1, the eighth and ninth figures of I for the lower values of s would be needful to obtain the value of the incomplete B-function exact to the seventh decimal place. *Thus the value obtained from the* Γ-*Function* Tables is ·3109,7115 or six units in error in the eighth figure. This is what we might have anticipated. On the basis of this example we can assert that for $p=10$ the true value can be found to some six units in the eighth decimal.

Application of Dr Müller's Method.

In order to test the labour required in working various processes for finding the incomplete B-function, we worked our $I_{.075}(10, 101)$ by Müller's Continued Fraction Method*. The results are given in the table below. It may be noted that when two successive convergents agree to the number of decimal places we require, we may stop our computing, and take the required value as that common to the two convergents.

We shall use practically Dr Müller's notation*. The true value of

$$I_{.075}(10, 101) = \cdot 3109{,}7121\dagger,$$

$$k = p + q - 1 = 110, \quad y_0 = 1 - x_0 = \cdot 925, \quad C = \frac{x_0^p\, y_0^{q-1}\,\Gamma(k+1)}{\Gamma(p+1)\,\Gamma(q)} = \cdot 1086{,}2676{,}9752,$$

$$a_r = \frac{101 - r}{10 + r}, \quad t = \frac{x_0}{y_0} = \frac{3}{37}, \quad p_r = p_{r-1} + b_r p_{r-2}, \quad q_r = q_{r-1} + b_r q_{r-2}.$$

r	b_r	p_r	q_r	Cp_r/q_r convergent to $I_{.075}(10, 101)$
1	1·0000,0000,00	1·0000,0000,00	1·0000,0000,00	
2	− ·7371,0073,71	1·0000,0000,00	·2028,9920,29	
3	+ ·0081,8181,82	1·0081,8181,82	·3310,8108,11	
4	− ·5660,0831,60	·5021,7350,22	·1822,7791,20	
5	+ ·0097,9209,98	·6087,6060,88	·2153,1718,83	
6	− ·4640,5405,41	·3807,5500,43	·1325,5310,34	
7	+ ·1145,2702,70	·4504,7020,77	·1572,1280,08	·3112,58254
8	− ·3758,9427,00	·3073,5238,14	·1073,8082,53	·3108,01233
9	+ ·1208,2070,01	·3617,8194,64	·1203,8233,08	·3109,54894
10	− ·3186,3442,39	·2638,4880,74	·0821,0519,14	·3109,74810
11	+ ·1226,8847,80	·3082,3537,38	·1070,7084,72	·3109,71047
12	− ·2750,9052,51	·2350,5145,90	·0823,1052,33	·3109,71063
13	+ ·1221,4812,21	·2733,0183,11	·0954,0831,51	·3109,71185
14	− ·2409,9980,32	·2105,0985,40	·0750,3004,18	·3109,71217

It will be seen that convergents 7 and 8 give three-figure accuracy, 9 and 10 four-figure accuracy, 10 and 11 five-figure accuracy, 12 and 13 six-figure accuracy,

* *Biometrika*, Vol. XXII. pp. 291 *et seq.* See also our first footnote, p. 423 below.

† Value obtained by "brute force," by quadrature and by the Γ-Function Table.

and 13 and 14 the required seven-figure accuracy. Müller's method is certainly shorter than the "brute force" or quadrature methods, and about on a par with the Γ-Function Table method at $p = 10$.

Illustration (iii). A still more extreme case was now taken, namely,

$$p = 36, \quad q = 101 \quad \text{and} \quad x_0 = {\cdot}20.$$

Here $C = 392{\cdot}0702{,}0930$, and it is clear that with a seven-figure table only of the incomplete Γ-function we cannot hope after multiplying by such a C to reach more than about four-figure accuracy for an incomplete B-function.

We found the following values:

Table for $p = 36$, $q = 101$ and $x_0 = {\cdot}20$.

	f_s	$a_s = Cf_s$	I_s Values from Γ-Tables	$a_s I_s$	$S(a_s I_s)$ up to $a_s I_s$
f_0	1·0000,0000	392·0702,0930	I_0 (3·3333,3333,3, *35*) = ·0008,0363	+ ·3150,7938	+ ·3150,7938
f_2	− 6·60	− 2611·1875,0394	I_2 (3·2444,2842,1, *37*) = ·0002,1708	− ·5668,3660	− ·2517,5722
f_3	− 1·6872	− 661·5008,5713	I_3 (3·2025,6307,7, *38*) = ·0001,0808	− ·0718,9191	− ·3236,4913
f_4	+ 24·181,701	+ 9480·9010,3483	I_4 (3·1622,7766,0, *39*) = ·0000,5323	+ ·5046,7156	+ ·1810,2243
f_5	+ 13·0022,3808	+ 5097·7902,0539	I_5 (3·1234,7523,9, *40*) = ·0000,2547	+ ·1298,4072	+ ·3108,6315
f_6	− 61·6514,7226	− 24172·8818,4320	I_6 (3·0800,6699,9, *41*) = ·0000,1192	− ·2881,4075	+ ·0227,2240
f_7	− 54·1811,1569	− 21242·8013,0869	I_7 (3·0499,7140,9, *42*) = ·0000,0541	− ·1149,2356	− ·0922,0116
f_8	+ 118·7993,2634	+ 46577·0767,4282	I_8 (3·0151,1344,5, *43*) = ·0000,0239	+ ·1113,2065	+ ·0191,1949
f_9	+ 160·3044,8195	+ 62874·1360,0242	I_9 (2·9814,2306,9, *44*) = ·0000,0102	+ ·0641,3162	+ ·0832,5111
f_{10}	− 170·2750,5090	− 66759·7748,6840	I_{10} (2·9485,3012,2, *45*) = ·0000,0045	− ·0300,4190	+ ·0532,0921
f_{11}	− 372·9827,2825	− 146235·4163,3026	I_{11} (2·9172,9982,6, *46*) = ·0000,0022	− ·0321,7179	+ ·0210,3742
f_{12}	+ 146·0854,9140	+ 57275·7691,8889	I_{12} (2·8867,5134,6, *47*) = ·0000,00105	+ ·0060,1396	+ ·0270,5138
f_{13}	+ 711·9055,9844	+ 279152·2633,0105	I_{13} (2·8571,4285,7, *48*) = ·0000,00050	+ ·0139,5761	+ ·0410,0899

The true value is ·0400,4983, or the excess at this point is roughly six units in the fifth figure. But even this agreement is illusory. It depends on taking the value of I_s to eight or nine figures in the last two cases, where our tables only give seven figures! Had we taken $I_{12} = {\cdot}0000{,}00104$ instead of ·0000,00105 we should have been only two units in error in the sixth figure instead of six in the fifth figure. But the reader will ask how we found I_{12} and I_{13}. If he will turn up $p = 47$ and the argument lying between 2·8 and 2·9 in the *Incomplete Γ-Function Tables*, he will find that he has to interpolate between

$$u_0 = {\cdot}0000{,}000, \quad \delta^2 u_0 = {\cdot}0000{,}000, \quad \delta^4 u_0 = {\cdot}0000{,}000, \text{ and}$$

$$u_1 = {\cdot}0000{,}001, \quad \delta^2 u_1 = {\cdot}0000{,}002, \quad \delta^4 u_1 = {\cdot}0000{,}001,$$

and matters are still worse for I_{13}. These units in the last figure may be anything from just over ·05 to just under ·15, and when we have to multiply by over 392, what they are is essential. Actually interpolating from the *Table*, the value found for I_{12} is ·0000,00080, which can only be taken as ·0000,001, but in multiplying by over 392, it is essential to know whether 80 or 100 is the better value to adopt. Actually, when the a_s's are as large as they are in this example, a table of the incomplete Γ-function ratio would have to exist to at least *ten* decimal places.

It is not the large number of a_s's required which really distresses us—they are not hard to run off on the machine. The difficulty is that the I_s's cannot be obtained to an adequate number of places. How then did we select I_{12} to be ·0000,00105 and I_{13} to be ·0000,00050? Simply by looking at the I_s column and noting that I_{s+1} is always somewhat less than half I_s, the preceding value. A round number *under* half 22 is 105, and a round number under half 105 is 50! It is clear that such a process is by no means to be commended, and taken together with the erratic values of the a_s's, so obvious in the products, $a_s I_s$, we must conclude that the reduction of an incomplete B-function to the *Tables of the Incomplete Γ-Function*, while satisfactory for $p=10$ and $q=101$, and probably some way beyond this value of p, fails to be a practically useful method when p gets to 36, or some third or more of q. We need, therefore, a more satisfactory method when p and q approach more nearly to equality, and one or both lie outside the B-Function Table.

Various methods have been applied, such as expanding the incomplete B-function in terms of incomplete B-functions lying inside the published *Incomplete B-Function Table*, but none of these seemed for p and q of the order of 36 and 101 as short as Dr Müller's continued fraction method. That method has peculiar advantages, when we only need the incomplete B-function ratio to four or five decimal places; it converges steadily from the early $C \times p_s/q_s$, while the $S(a_s I_s)$ may be most erratic before it approaches the true value.

In all the methods there are many chances of even a good computer making a slip. But systematic arrangement of the material and experience in the use of the methods makes them far shorter than appears at first sight. The reader must also remember that it is solely the mathematician who, for special purposes, will aim at seven-figure accuracy. For the statistician, four- or even three-figure accuracy is often adequate.

We give in the table below the data requisite to find $I_{.2}(36, 101)$ to seven-figure accuracy by continued fractions.

$$I_{.2}(36,\ 101) = \cdot 0409,4983 \text{ (by quadrature).}$$

$$p=36, \quad q=101, \quad k=p+q-1=136, \quad x_0=\cdot 2, \quad y_0=1-x_0=\cdot 8,$$

$$C=\frac{x_0^p y_0^{q-1}\,\Gamma(k+1)}{\Gamma(p+1)\,\Gamma(q)}=\cdot 0147,53914, \quad u_r=\frac{101-r}{36+r}, \quad t=\frac{x_0}{y_0}=\frac{1}{4},$$

$$p_r=p_{r-1}+b_r p_{r-2}, \quad q_r=q_{r-1}+b_r q_{r-2}\,*.$$

Thus the eighth and ninth convergents give the value correct to four figures, the eleventh and twelfth correct to five figures, the thirteenth and fourteenth to seven figures as required.

* The general expressions for the b's are

$$b_{2r}=-\frac{(p+r-1)(p+r)}{(p+2r-2)(p+2r-1)}u_r t, \quad b_{2r+1}=\frac{r(k+r)}{(p+2r-1)(p+2r)}t,$$

while

$$p_1=q_1=1, \text{ and } p_2=1.$$

r	h_r	p_r	q_r	Continued fraction p_r/q_r	$C \times p_r/q_r$, converging to $I_{\cdot 2}(36, 101)$
1	+ 1·0000,0000,00	1·0000,0000,00	1·0000,0000,00	—	—
2	− ·0756,7567,57	1·0000,0000,00	·3243,2432,43	—	—
3	+ ·0243,5988,62	1·0243,5988,62	·3486,8421,05	—	—
4	− ·6179,1497,98	·4064,4490,64	·1482,7935,22	—	—
5	+ ·0442,3076,92	·4517,5313,21	·1637,0192,30	—	—
6	− ·5676,8292,68	·2210,2129,81	·0795,2626,64	2·7792,2386,45	·0410,0443
7	+ ·0605,4006,97	·2483,7046,42	·0894,3679,22	2·7770,5022,83	·0409,7236
8	− ·5236,7109,63	·1326,2799,87	·0477,9118,51	2·7751,5609,67	·0409,4441
9	+ ·0739,9577,17	·1510,0636,29	·0544,0912,96	2·7753,8648,40	·0409,4789
10	− ·4848,4848,48	·0867,0187,87	·0312,3764,59	2·7755,5738,28	·0409,5033
11	+ ·0851,4492,75	·0995,5930,45	·0358,7030,73	2·7755,3531,02	·0409,5001
12	− ·4503,9316,46	·0605,0937,19	·0218,0108,54	2·7755,2807,11	·0409,5127
13	+ ·0944,1489,36	·0699,0925,30	·0251,8777,66	2·7755,2298,92	·0409,4983
14	− ·4196,4285,71	·0455,1692,73	·0160,3910,68	2·7755,2408,97	·0409,4984

Conclusions. Methods have been indicated of throwing back values of the incomplete Γ-function ratio and the incomplete B-function ratio lying outside the ranges of the published tables on the *Tables of the Incomplete* Γ-*Function*. These methods work adequately for the Γ-function and also adequately for the B-function, if p be small as compared with q. But in the latter case, if p be comparable with q, the constant $C = \dfrac{\Gamma(p+q)}{\Gamma(q)\,(q-1)^p}$ becomes very large, and the existing tables of the Γ-function ratio only giving seven decimal places are inadequate for the method, which would require at least ten decimal places. After considering various methods for determining an incomplete B-function ratio with comparable p and q, we believe the most efficient to be Müller's method of the continued fraction, illustrations of which are given in the paper.

The authors of this paper have not overlooked the value of the late Mr Soper's tract, *Numerical Evaluations of the Incomplete* B-*Function* *, but have endeavoured to make a somewhat more direct answer to the question often put: What is to be done when we need the numerical value of incomplete Eulerian integrals lying outside the existing tables?

In conclusion we desire cordially to thank Miss F. N. David for kindly working out certain values for us, as, being away from home, we had no machine, and no table of more than seven-figure logarithms.

* *Tracts for Computers*, No. VII. Cambridge University Press.

FURTHER INVESTIGATION OF THE MORPHOMETRIC CHARACTERS OF THE INDIVIDUAL BONES OF THE HUMAN SKULL.

By KARL PEARSON AND T. L. WOO, PH.D.

(1) *Introduction.*

IN 1930 the then Director of the Biometric Laboratory determined that the time was ripe to undertake a wide system of measurements on the individual bones of the human skull. A series of 63 measurements was selected, which could be taken on the non-disjointed crania, and the term morphometric was introduced for such measurements to distinguish them from the existing ethnometric measurements, which may cover several bones at once. It was noted, however, that morphometric measurements may be found to have first-class ethnographical value when, in the future, they shall have been taken on a number of racial groups. It was proposed to start the morphometric measurements on an adequately long series of crania, and discuss in the first place the results which flowed from the measurement of homologous bones. The well-known E series of Egyptian crania of the 26th to the 30th dynasties, numbering about 800 male crania, were chosen, and an indefatigable worker, both from the measuring and computing sides, Dr T. L. Woo, was selected to undertake the work in 1930. After a year's labour in measuring and computing, Dr Woo published, in 1931*, the first report on these morphometric measurements, with some assistance from the Director of the Laboratory in the preparation of his text. This dealt with the 50 characters out of the 63, which belonged to homologous bones, i.e. 25 pairs of corresponding measurements on the right and left sides of the skull. The whole report dealt with absolute sizes, and not with the shapes of individual bones. The discussion was chiefly concerned with asymmetry in size. The following conclusions were reached:

(i) The mean or type skull of a race is definitely and markedly asymmetrical; asymmetry is not only a peculiarity of the individual skull; it is also a racial character.

(ii) The frontal and parietal bones are with marked significance greater on the right side, so also are the approximately antero-posterior lengths on the temporal. Both malar measurements are in excess on the left, but these measurements have little to do with the brain. On the other hand while the arcs from asterion to sagittal plane†, and from asterion to basion were significantly greater on the right, that from the lambda to the asterion along the lambdoid suture was *markedly* dominant on the left.

* *Biometrika*, Vol. XXII. pp. 324—352.

† For definition of this (and other) measurements see *ibid.* pp. 325—329.

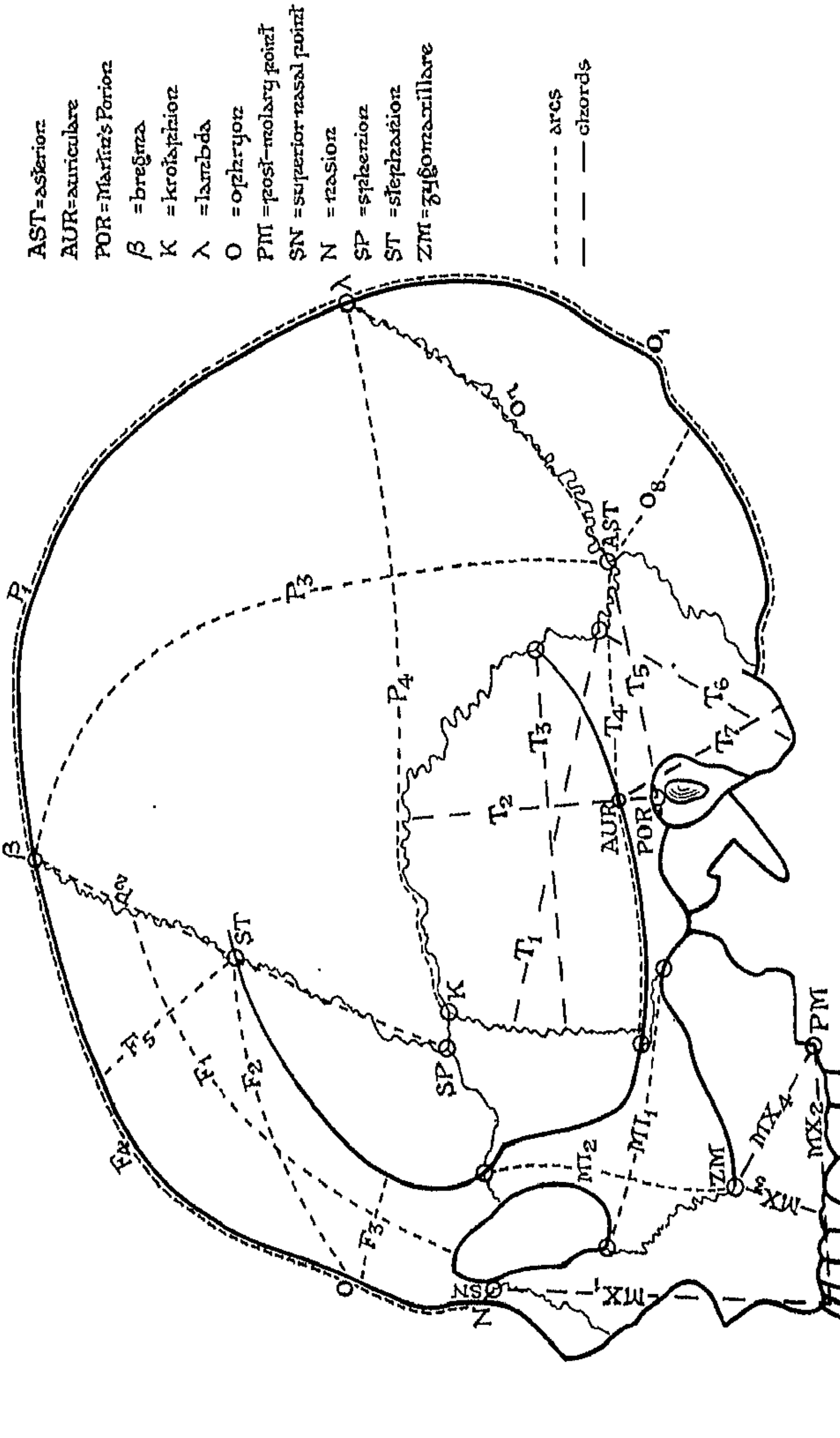

AST = asterion
AUR = auriculare
POR = Martin's Porion
β = bregma
K = krotaphion
λ = lambda
O = ophryon
PM = post-molary point
SN = superior nasal point
N = nasion
SP = sphenion
ST = stephanion
ZM = zygomaxillare
arcs
chords

Diagram 1

(iii) The variability, either absolute or relative, showed no marked lateral trend, and the greater lateral variability was not significantly associated with the greater lateral characters.

The report showed the large part played by homology in the correlation of morphometric characters. It also indicated that facial and frontal homologous characters were more highly correlated (·86–·98) while those of the temporal, parietal and occipital regions were less so (·54–·85). The face, possibly owing to sexual selection, is the most symmetrical region of the skull. This report, however, left much for future consideration and the present paper will deal with many problems left unanswered in the first report.

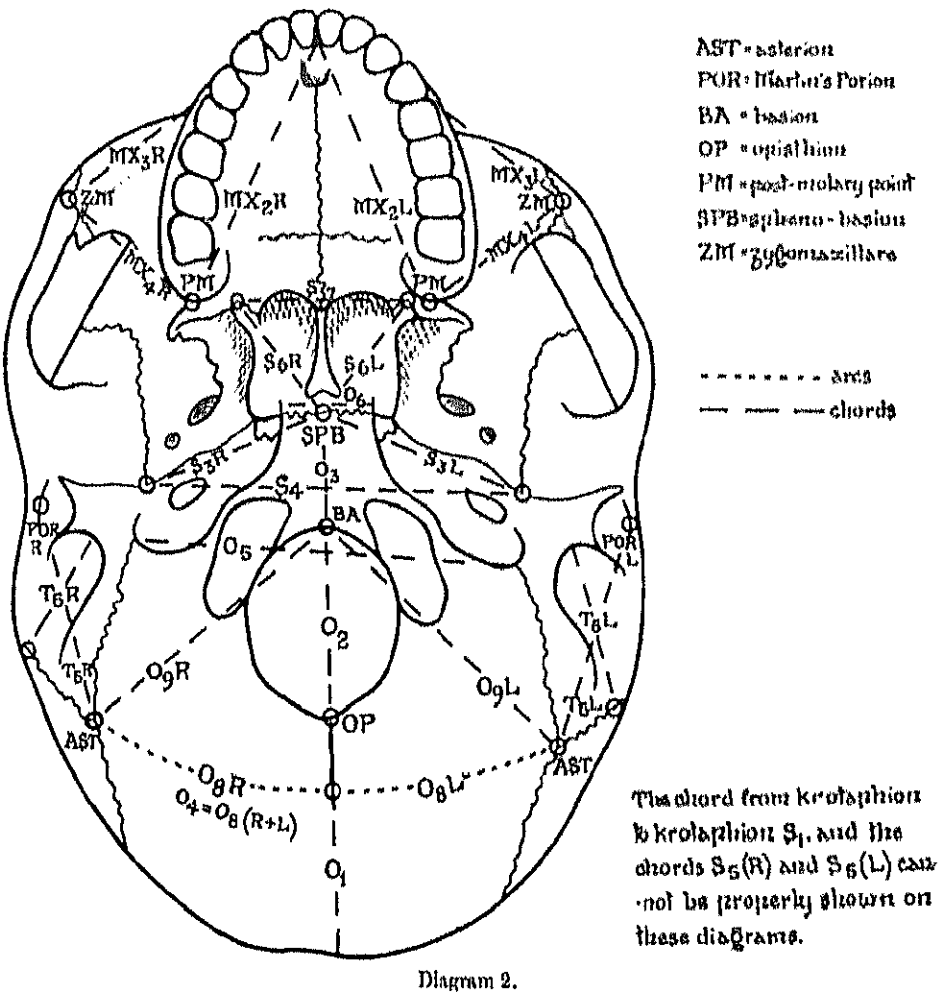


Diagram 2.

Diagrams 1 and 2, giving respectively the norma lateralis and norma basalis of a skull, indicate approximately the measurements taken during the whole of this investigation.

(2) *Constants of the Thirteen Additional Characters.*

We place first on record the remaining thirteen characters dealt with in Dr Woo's measurements; these were principally but not entirely taken along or perpendicular to the median sagittal plane. We shall not repeat here the definitions of the 50 bilateral characters discussed in the first report, but we reproduce here the diagrams indicating those characters with addition of the thirteen further characters.

List of the Thirteen Additional Characters.

Frontal	F_3, Minimum arc between the terminals of the minimum frontal diameter. {B'}.
	F_4, Minimum arc from nasion to bregma. {S_1}.
	F_5, Minimum arc from right stephanion to left stephanion.
Parietal	P_1, Minimum arc from bregma to lambda. {S_2}.
Occipital	O_1, Minimum arc from lambda to opisthion. {S_3}.
	O_2, Chord from opisthion to basion. {*fml*}.
	O_3, Chord from basion to spheno-basion.
	O_4, Minimum arc from right asterion to left asterion.
	O_5, Chord between the points, right and left, where the occipito-mastoid suture meets the posterior border of the jugular foramen.
	O_6, Breadth of the pars basilaris, taken between the legs of small calipers as far forward as possible.
Sphenoid	S_1, Chord between right and left krotaphions.
	S_4, Chord between the extreme posterior points on the right and left of the sphenoid exposed on the base of the skull.
	S_7, Chord between the lowest points, right and left on the suture between the medial pterygoid plate and the palate bone.

Four of these are familiar ethnometric characters and we have placed against them in curled brackets the symbols used to denote them in the Biometric Laboratory memoirs. They are the sole cranial characters common to the present morphometric and the earlier ethnometric researches of that Laboratory.

We have tabled the constants of these thirteen additional characters in Table I, p. 428.

(3) *Variation of the Sixty-three Characters.*

We are now able to consider the variation of our 63 characters taken as a single group. In Table II (p. 429), we have collected the variation constants: in column (a) the bone; in column (b) its character; in column (c) the absolute variation of the character; in column (d) the relative variation, or percentage variation on size of character; in column (e) the characters of the bone in order of intensity of absolute variation; in column (f) the order of intensity of relative variation, are provided. In the last two columns the order of absolute sizes of the characters are placed for each bone character in square brackets. When we examine column (e) we see that

TABLE I. *Constants of the Non-Bilateral Characters used in this Paper.*

(This Table must be taken in conjunction with Table I of the former Memoir*.)

Bone	Character (cases)	Means	Standard Deviations	Coefficients of Variation
Frontal	F_1 (891) F_4 (887) F_5 (888)	101·5839 ± ·1417 127·2100 ± ·1413 139·5221 ± ·2181	6·3047 ± ·1023 6·2372 ± ·0999 9·6355 ± ·1542	6·2040 ± ·0982 4·9031 ± ·0787 6·9061 ± ·1110
Parietal	P_1 (897)	128·3726 ± ·1651	7·2175 ± ·1167	5·6150 ± ·0912
Occipital	O_1 (864) O_2 (882) O_3 (870) O_4 (875) O_5 (874) O_6 (861)	115·0093 ± ·1574 35·7330 ± ·0502 26·3397 ± ·0534 128·0791 ± ·1378 73·2151 ± ·0818 21·8501 ± ·0418	6·8601 ± ·1113 2·6047 ± ·0418 2·3343 ± ·0377 6·0422 ± ·0974 3·5874 ± ·0579 1·8190 ± ·0295	5·9651 ± ·0972 7·2894 ± ·1177 8·8625 ± ·1444 4·7175 ± ·0762 4·8998 ± ·0792 8·3250 ± ·1362
Sphenoid	S_1 (732) S_4 (881) S_7 (822)	114·5027 ± ·1332 70·3871 ± ·0817 35·4720 ± ·0687	5·3447 ± ·0942 3·5943 ± ·0578 2·9187 ± ·0486	4·6677 ± ·0825 5·1065 ± ·0823 8·2281 ± ·1378

the absolute variations in the characters of each individual bone, even allowing for random sampling, are not proportional to the absolute size of the organs measured. Thus it is demonstrated that the coefficients of variation will not be constant—any more than the standard deviations for a single cranial bone, much less for all the bones, i.e. the coefficients of variation must depend on the size of the character measured.

At the same time it will be clear that there is considerable positive correlation between the intensity of the absolute variation and the size of the varying character.

This is shown by our Table II where the intensities of the variation are arranged in decreasing order, and against them are put the sizes of the corresponding organs, [1] marking the largest measurement, [2] the second and so on for each bone. If we now average the bracket numbers for the upper and lower halves of the group of characters for each bone, we find that for the six bones—frontal, temporal, sphenoid, malar, maxillary and occipital—these averages are much lower for the greater intensities of variation, only in the case of the parietal are they equal. In other words, the absolute variabilities of the larger-sized characters are the larger.

The last column (f) shows, however, that the coefficient of variation overcorrects this defect; for if we except the frontal where equality exists, and the short malar series of only four characters, where the two larger coefficients of variation belong to the two large-sized characters, then in the case of all the remaining five bones

* *Biometrika*, Vol. XXII. pp. 330—331.

TABLE II. *Variability, Absolute and Relative, of the 63 Morphometric Characters.*

(a) Bone	(b) Character	(c) Standard Deviation	(d) Coefficient of Variation	(e) Order of Absolute Variation			(f) Order of Relative Variation		
Frontal	F_1 R	5·10	5·18	F_5	[1]	Mean First	F_5	[1]	Mean First
	F_1 L	5·06	5·16	F_3	[3]	Four *2·50*	F_3	[3]	Four *4·25*
	F_2 R	4·78	5·43	F_4	[2]		F_2 (L)	[7]	
	F_2 L	4·08	5·69	F_1 (R)	[4]		F_2 (R)	[6]	
	F_3	6·40	6·12	F_1 (L)	[5]	Mean Last	F_1 (R)	[4]	Mean Last
	F_4	6·24	4·00	F_2 (L)	[7]	Four *5·50*	F_1 (L)	[5]	Four *4·25*
	F_5	9·64	6·91	F_2 (R)	[6]		F_4	[2]	
Parietal	P_1	7·25	5·05	P_1	[5]	Mean First	P_1	[5]	Mean First
	P_2 R	5·93	5·27	P_3 (L)	[4]	Four *4·25*	P_2 (R)	[6]	Four *5·50*
	P_2 L	5·70	5·13	P_4 (L)	[2]		P_2 (L)	[7]	
	P_3 R	5·92	3·58	P_2 (R)	[6]		P_3 (L)	[4]	
	P_3 L	6·04	3·70	P_3 (R)	[3]	Mean Last	P_3 (R)	[3]	Mean Last
	P_4 R	6·84	3·30	P_4 (R)	[1]	Four *4·25*	P_4 (L)	[2]	Four *2·50*
	P_4 L	5·96	3·39	P_2 (L)	[7]		P_4 (R)	[1]	
Temporal	T_1 R	3·89	4·48	T_4 (L)	[2]	Mean First	T_6 (R)	[12]	Mean First
	T_1 L	3·83	4·46	T_4 (R)	[1]	Seven *4·71*	T_6 (L)	[11]	Seven *10·14*
	T_2 R	3·78	8·11	T_3 (R)	[5]		T_7 (R)	[13]	
	T_2 L	3·71	7·93	T_3 (L)	[6]		T_2 (R)	[8]	
	T_3 R	4·27	6·45	T_1 (R)	[3]		T_7 (L)	[14]	
	T_3 L	4·25	6·47	T_1 (L)	[4]		T_2 (L)	[7]	
	T_4 R	4·30	4·31	T_6 (R)	[12]		T_3 (L)	[6]	
	T_4 L	4·48	4·50	T_6 (L)	[11]	Mean Last	T_3 (R)	[5]	Mean Last
	T_5 R	2·75	5·08	T_2 (R)	[8]	Seven *10·29*	T_5 (L)	[9]	Seven *4·71*
	T_5 L	2·77	6·01	T_2 (L)	[7]		T_5 (R)	[10]	
	T_6 R	3·79	8·33	T_7 (R)	[13]		T_4 (L)	[2]	
	T_6 L	3·78	8·24	T_7 (L)	[14]		T_1 (R)	[3]	
	T_7 R	2·94	8·14	T_5 (L)	[9]		T_1 (L)	[4]	
	T_7 L	2·86	7·94	T_5 (R)	[10]		T_4 (R)	[1]	
Sphenoid	S_1	5·34	4·67	S_1	[1]	Mean First	S_7	[11]	Mean First
	S_2 R	3·43	4·57	S_4	[4]	Six *3·50*	S_6 (L)	[10]	Six *8·00*
	S_2 L	3·50	4·68	S_2 (L)	[3]		S_6 (R)	[9]	
	S_3 R	1·93	5·33	S_5 (L)	[6]		S_5 (L)	[6]	
	S_3 L	2·01	5·51	S_5 (R)	[5]		S_5 (R)	[5]	
	S_4	3·59	5·11	S_2 (R)	[2]		S_3 (L)	[7]	
	S_5 R	3·44	6·07	S_7	[11]	Mean Last	S_3 (R)	[8]	Mean Last
	S_5 L	3·45	6·12	S_6 (L)	[10]	Six *7·83*	S_4	[4]	Six *4·86*
	S_6 R	2·40	6·74	S_6 (R)	[9]		S_2 (L)	[3]	
	S_6 L	2·44	6·87	S_3 (L)	[7]		S_1	[1]	
	S_7	2·92	8·23	S_3 (R)	[8]		S_2 (R)	[2]	
Malar	Ml_1* R	4·38	7·37	Ml_1 (L)	[1]	Mean First	Ml_1 (L)	[1]	Mean First
	Ml_1* L	4·55	7·64	Ml_1 (R)	[2]	Two *1·50*	Ml_1 (R)	[2]	Two *1·50*
	Ml_2* R	3·11	6·30	Ml_2 (L)	[3]	Mean Last	Ml_2 (L)	[3]	Mean Last
	Ml_2* L	3·21	6·42	Ml_2 (R)	[4]	Two *3·50*	Ml_2 (R)	[4]	Two *3·50*
Maxillary	Mx_1 R	3·93	5·90	Mx_1 (R)	[1]	Mean First	Mx_3 (L)	[5]	Mean First
	Mx_1 L	3·90	5·68	Mx_1 (L)	[2]	Four *3·50*	Mx_3 (R)	[6]	Four *6·50*
	Mx_2 R	2·87	5·15	Mx_3 (L)	[5]		Mx_4 (L)	[8]	
	Mx_2 L	2·80	5·17	Mx_3 (R)	[6]		Mx_4 (R)	[7]	
	Mx_3 R	3·14	7·09	Mx_2 (L)	[3]	Mean Last	Mx_1 (R)	[1]	Mean Last
	Mx_3 L	3·25	7·94	Mx_2 (R)	[4]	Four *5·50*	Mx_1 (L)	[2]	Four *2·50*
	Mx_4 R	2·59	6·75	Mx_4 (L)	[8]		Mx_2 (L)	[3]	
	Mx_4 L	2·68	6·99	Mx_4 (R)	[7]		Mx_2 (R)	[4]	
Occipital	O_1	6·80	5·07	O_1	[2]	Mean First	O_3	[11]	Mean First
	O_2	2·60	7·20	O_4	[1]	Six *4·17*	O_6	[12]	Six *7·67*
	O_3	2·33	8·86	O_7 (L)	[3]		O_2	[10]	
	O_4	6·04	4·72	O_7 (R)	[4]		O_1	[2]	
	O_5	3·59	4·90	O_8 (R)	[8]		O_8 (R)	[8]	
	O_6	1·82	8·33	O_5	[7]		O_7 (L)	[3]	
	O_7 R	5·34	5·49	O_9 (R)	[5]	Mean Last	O_7 (R)	[4]	Mean Last
	O_7 L	5·54	5·61	O_9 (L)	[6]	Six *8·83*	O_8 (L)	[9]	Six *5·33*
	O_8 R	3·69	5·78	O_8 (L)	[9]		O_5	[7]	
	O_8 L	3·29	5·20	O_2	[10]		O_4	[1]	
	O_9 R	3·43	4·60	O_3	[11]		O_9 (R)	[5]	
	O_9 L	3·30	4·43	O_6	[12]		O_9 (L)	[6]	

* Unfortunately in Dr Woo's paper (*Biometrika*, Vol. XXII. p. 330, Table I, Ml_1 and Ml_2 should be interchanged; all the constants of the former are larger than those of the latter.

the moiety of the coefficients which are larger give a larger average of the square bracket numbers or correspond to characters of lesser size. In other words, the coefficient of variation of a character is not the same for all characters but roughly varies inversely as its size.

These results appear to indicate that if variation in the rates of bone growth be different from individual to individual, these rates cannot be the same for all directions of growth in the same bone. In other words, a bone in its growth must change not only its size but its shape. If for an individual, the individual bone changed its size only, not its shape, then it seems reasonable to suppose that the coefficients of variation would be the same for all measurements on the same bone. It seems then highly improbable that bones grow isotropically, or if growth from a centre of ossification be isotropic, the presence of more than one centre causes the growth of the bone as a whole to be heterotropic.

We can obtain some rough appreciation of the influence of absolute size on absolute variation by calculating the correlation coefficient ρ of order of size and order of variation for each bone and pooling these orders. We find that $\rho = \cdot 2087$ and, by the formula $r = 2 \sin (\frac{1}{6} \pi \rho)$, deduce

$$r = \cdot 2804.$$

This average correlation for the seven bones is a rough measure of the influence of the size of a character on its absolute variability; it is not so large as we might have anticipated. It would have been greater had we pooled all the place measurements together, but it seemed desirable to free ourselves of such great differences of size as occur, for example, between our sphenoidal and parietal measurements.

If the same method of correlation be applied to the absolute sizes and the coefficients of variation we find $\rho = -\cdot 1138$ and $r = -\cdot 1191$. Thus the relative variation exhibits over 50 % less correlation than the absolute variation does with absolute size, but it has overcorrected.

To sum up: The correlations of absolute size with both absolute and relative variability as represented by the standard deviation and the coefficient of variation respectively are small but significant in the case of measurements taken on the individual bones of the human skull. The correlation of absolute size with the standard deviation is positive, and with the coefficient of variation is negative; the latter is considerably smaller than the former, but the distribution of the total variation over the mean length overcorrects. So far no one has suggested a constant which will give a better measure of relative variation than the coefficient of variation*. It may be that a better exists, but it is more probable that variation is so peculiar to each character that it is not possible by the examination of a single

* It might be thought that a better coefficient could be found by taking the ratio of each deviation to its own length. Let c be the character, $\bar{c}$ its mean, σ_c its standard deviation, $v = \sigma_c/\bar{c}$ and $V = 100v$ its coefficient of variation. Then instead of $V = 100\sigma_c/\bar{c}$ let us take

$$F = 100 \sqrt{S \frac{(c - \bar{c})^2}{Nc^2}},$$

constant to assert whether A is more variable than B. Only when we compare the coefficients of variation of the same character in two sexes or two races can we get a partial answer to the question.

It remains to be considered whether it is possible to discover whether there are any special directions of measurement on the bones of the skull, transverse, sagittal or vertical, which are associated with larger relative variabilities. The difficulty here is that some of the measurements, especially some of the arcs, have very mixed directions.

The following classes were made:

Sagittal, *s*, i.e. practically parallel to the Frankfurt horizontal and the median sagittal plane. *st*, sagittal transverse, in which were included $t(s)$, transverse, with a smaller amount of sagittal in it, and $s(t)$, sagittal, with a smaller amount of transverse in it; *sv*, sagittal vertical; with which were included $s(v)$ and $v(s)$, sagittal with a less amount of vertical and vertical with some small amount of sagittal; and finally *svt*, a measurement combining all three directions. Thus for sagittal measurements we have *s*, *st*, *sv*, *stv*. In a similar manner we have for transverse measurements *t*, *ts*, *tv* and *tvs*, and for vertical measurements *v*, *vs*, *vt* and *vst*. The order of the letters in these classes is naturally indifferent. One or other of these classes was to the best of the writer's judgment affixed to every one of the 63 characters; and then the position of the coefficient of variation in each bone class taken and placed under the direction class of its measurement. These

where N is the total number of observations. Then

$$F^2 = \frac{100^2}{N} S \frac{(\Delta c)^2}{(\bar{c}+\Delta c)^2}, \text{ where } \Delta c = c - \bar{c}$$

$$= \frac{100^2}{N} S \left(\frac{\Delta c}{\bar{c}}\right)^2 \left(1 + \frac{\Delta c}{\bar{c}}\right)^{-2}$$

$$= \frac{100^2}{N} S \frac{(\Delta c)^2}{\bar{c}^2} \left(1 - 2\frac{\Delta c}{\bar{c}} + 3\frac{(\Delta c)^2}{\bar{c}^2} - 4\frac{(\Delta c)^3}{\bar{c}^3} + 5\frac{(\Delta c)^4}{\bar{c}^4} - \dots\right)$$

$$= 100^2 \left(\frac{\sigma_c^2}{\bar{c}^2} - 2\sqrt{\beta_1}\frac{\sigma_c^3}{\bar{c}^3} + 3\beta_2\frac{\sigma_c^4}{\bar{c}^4} - 4\frac{\beta_5}{\sqrt{\beta_1}}\frac{\sigma_c^5}{\bar{c}^5} + 5\beta_4\frac{\sigma_c^6}{\bar{c}^6} - \dots\right),$$

where the β's have the customary statistical significance. For a symmetrical system $\sqrt{\beta_1}$ and $\beta_5/\sqrt{\beta_1}$ etc. $=0$, and for a normal distribution $\beta_2 = 3$, $\beta_4 = 15$, and then

$$F = 100v\,(1 + 9v^2 + 75v^4)^{\frac{1}{2}},$$

if we neglect terms in v^6. But to the same degree of approximation

$$1 + 9v^2 + 75v^4 = \left(1 - \frac{23}{3}v^2\right)^{-\frac{27}{23}},$$

or, finally,

$$F = V\frac{1}{\left(1 - \frac{23}{30,000}V^2\right)^{\frac{27}{46}}}.$$

Thus, though F may be theoretically more satisfactory than V, it is clear that F can only be constant as V becomes constant, and V is easier to compute than F. Indeed it is difficult to think of any other lengths than $\bar{c}$ and σ_c for a normal frequency which could give a ratio possibly independent of absolute size. It may be noted that when V is small, F will be almost identical with V in value, and that when V is even as high as six, F will only differ from V by about 1 to 2 per cent.

positions were then averaged for each class. As the positions denote descending order of intensity of relative variation, a low average position (a great number for the mean position) marks *less* variation. There resulted the following scheme:

(i) *Sagittal or in part sagittal measurements.*

Class ...	*s*	*st*	*sv*	*stv*	All
Mean Position	7·333	5·692	5·471	6·111	5·909

(ii) *Transverse or in part transverse measurements.*

Class ...	*t*	*ts*	*tv*	*tvs*	All
Mean Position	6·000	5·692	4·833	6·111	5·688

(iii) *Vertical or in part vertical measurements.*

Class ...	*v*	*vt*	*vs*	*vst*	All
Mean Position	3·875	4·833	5·471	6·111	5·179

This tabulation gives a uniformity of results far more decisive than anything which might have been anticipated. Whether we judge by the pure directions, *s*, *t*, *v*, or the total of measurements in which one of these directions appears, the order is the same; sagittal measurements are on the average least variable, vertical measurements most variable, and the transverse intermediate 7·333—6·000—3·875, or 5·909—5·688—5·179. And again when we have a bi-directional class *st* is less variable than *sv*, and *ts* less variable than *tv*. Again in group (iii) *v*, the most variable direction, is more variable when combined with *t* than when combined with *s*, because *t* is more variable than *s*.

These results seem undoubtedly suggestive; the individual bones of the skull tend to vary most in the vertical, less in the transverse and least in the sagittal direction.

If we venture to attribute individual deviations from the type to individual rates of growth, then the want of constancy in the coefficient of variation may be due to the fact that the growth of a cranial bone is not the same in the three fundamental directions, sagittal, transverse and vertical. Thus we may conclude that the coefficient of variation is likely to be most useful when we are comparing the *same* measurement in two races or in the two sexes.

If we attempt the same method on the absolute variations, even after "placing" by the separate bones and not by the skull as a whole, we do not get the same clear-cut result, the correlation between absolute size and absolute variation is too great. We find:

(i) *Transverse or in part transverse measurements.*

Class ...	*t*	*ts*	*tv*	*tvs*	All
Mean Position	5·600	6·000	4·667	3·833	5·233

(ii) *Sagittal or in part sagittal measurements.*

Class ...	*s*	*st*	*sv*	*stv*	All
Mean Position	5·167	6·000	6·684	3·833	5·886

(iii) *Vertical or in part vertical measurements.*

Class ...	*v*	*vt*	*vs*	*vst*	All
Mean Position	4·500	4·667	6·684	3·833	5·487

Thus, if judged by pure vertical lengths, the absolute variation is greater in the vertical than in the other two directions, but stands only second if we judge by all measurements which have a vertical element. Again the sagittal length if judged by the pure sagittal lengths is more variable than the transverse, but less variable if judged by all the measurements which have a sagittal element. In neither case do we get the same uniformity in the bi-directional groups as when we deal with the relative variabilities. Thus there appears to be a more systematic relationship among the relative than among the absolute variabilities. There is no full clue at present to the origin of the differences among the latter.

Thus judged by uni-directional averages the order of increasing absolute variability is transverse, sagittal and vertical, but judged by all variabilities in *which these directions occur it is sagittal, vertical and transverse. Nor are the bi-directional* place averages arranged in any satisfactory system. Accordingly, while the relative variabilities do fit a more or less consistent arrangement, the absolute variabilities beyond a moderate correlation with absolute size do not provide us with any clue at present to the source of the differences, and the apparent (if possibly not real) irregularities in their values.

(4) *On the Indices of Individual Cranial Bones.*

The large number of indices which could be formed, namely: 21 for the frontal, 21 for the parietal, 91 for the temporal, 55 for the sphenoid, 6 for the malar, 28 for the maxillary and 66 for the occipital bones, or a total of 288, rendered some selection needful. First, some 29 indices were chosen and the correlation tables formed. Among these were twelve pairs of indices for homologous characters and five for non-bilateral indices. These indices, their means and standard deviations, are given in Table III, p. 434, while in the last column of the table are placed the correlations of the corresponding indices of homologous bones.

We may first inquire whether the size of the index has any influence on its variation. In Table IV, p. 435, these indices are arranged in order of size, and we shall discuss their variabilities from this standpoint.

If we take the first 15, or greater indices, we find an average standard deviation of 5·96, and the last 15, or lesser indices, one of 5·58, on the basis of which we can make no definite statement. This view is confirmed if we remove the five non-bilateral indices which give a mean of 7·26, and average the first 12 homologous indices giving 5·68 and the last 12 giving 5·30. Again, if we compare the 12 pairs of homologous indices, the variations in the pairs are very close; but in six cases the larger and in six cases the less are associated with the greater variation. In view of these facts it is impossible to suppose that size of index is at all associated with intensity of variation of index. The coefficient of

TABLE III. *Constants of the Individual Bone Indices.*

Bone	Index and Number		Mean	Standard Deviation	Correlation of corresponding indices of bilateral bones
Frontal	$100\,F_3/F_1$ (860)		82·4187 ± ·1236	5·3639 ± ·0874	
	$100\,F_2/F_1$ (866)		101·7822 ± ·1791	7·1297 ± ·1125	
	$100\,F_2/F_1$ (863)	R	80·4779 ± ·0993	3·9757 ± ·0630	·8171 ± ·0075
		L	80·3985 ± ·0913	4·1531 ± ·0667	
Parietal	$100\,P_2/P_1$ (743)	R	88·1682 ± ·1472	5·9607 ± ·1041	·7892 ± ·0093
		L	87·0263 ± ·1389	5·0125 ± ·0982	
	$100\,P_3/P_1$ (737)	R	93·3989 ± ·0750	3·0181 ± ·0530	·4077 ± ·0207
		L	92·0588 ± ·0750	3·0555 ± ·0538	
Temporal	$100\,T_2/T_1$ (813)	R	70·5635 ± ·1416	6·2574 ± ·1023	·7231 ± ·0086
		L	71·3737 ± ·1432	6·2958 ± ·1020	
	$100\,T_3/T_1$ (825)	R	99·3648 ± ·1913	8·1484 ± ·1353	·6185 ± ·0145
		L	100·1015 ± ·1902	8·1360 ± ·1350	
Sphenoid	$100\,S_2/S_1$ (723)	R	89·7510 ± ·1604	6·3966 ± ·1134	·5922 ± ·0056
		L	90·3105 ± ·1621	6·4643 ± ·1147	
	$100\,S_6/S_5$ (812)	R	100·6178 ± ·1834	7·7478 ± ·1297	·3018 ± ·0043
		L	100·5061 ± ·1858	7·8510 ± ·1314	
Malar	$100\,Ml_2/Ml_1$ (716)	R	83·1522 ± ·1175	5·8201 ± ·1012	·8846 ± ·0057
		L	83·9902 ± ·1518	6·0232 ± ·1073	
Maxillary	$100\,Mx_2/Mx_1$ (508)	R	83·6063 ± ·1614	5·3867 ± ·1141	·9311 ± ·0068
		L	84·2008 ± ·1622	5·4163 ± ·1147	
	$100\,Mx_3/Mx_1$ (414)	R	61·1908 ± ·1101	3·3215 ± ·0779	·7894 ± ·0125
		L	61·3768 ± ·1133	3·4180 ± ·0801	
Occipital	$100\,O_6/O_5$ (857)		83·4959 ± ·2209	9·5893 ± ·1562	
	$100\,O_4/O_1$ (850)		110·8271 ± ·1623	7·0150 ± ·1148	
	$100\,O_5/(O_2+O_4)$ (847)		117·8042 ± ·1071	7·2210 ± ·1184	
	$100\,O_8/O_7$ (856)	R	65·6917 ± ·1080	4·5020 ± ·0749	·6014 ± ·0140
		L	64·4655 ± ·1000	4·3312 ± ·0707	
	$100\,O_8/O_9$ (840)	R	85·6060 ± ·1242	5·3090 ± ·0878	·4769 ± ·0179
		L	85·0042 ± ·1187	5·1271 ± ·0830	

variation, if indeed it had meaning in the case of indices, would thus not be steadier than the absolute variation as measured by the standard deviation. The cause or causes of such diversity of variation ranging from 3·0 to 9·6 units must be sought elsewhere than in size of index*.

* Dr G. M. Morant suggests that the order of laying down the bones and the period during which they grow—depending to some extent on their size—may provide a partial explanation of the diversity in variability and in the intensity of the effect of homology in the case of these indices. He points out that Mx_2, which is an indirect measure of the dental arcade, is probably determined at an early age,

Let us look into two further questions which can be answered from Table III.

The mean of the indices in Table III on the right side of the skull is 83·47, and that of the indices on the left side is 83·42, whence it is clear that there is no lateral dominance of the indices as a whole. Pursuing the same simple method, we find

Mean standard deviation on the right side = 5·499,

and Mean standard deviation on the left side = 5·483.

There is thus clearly no lateral dominance in variation as a whole. Thus by the use of indices we have largely freed ourselves from lateral dominance, either in size of index or magnitude of its variation.

TABLE IV. *Twenty-nine Indices of Cranial Bones in order of Size.*

Order	Index	Standard Deviation	Order	Index	Standard Deviation
1	$100\,O_6/(O_2+O_3)$	7·22	16	$100\,Mx_2/Mx_1\,(L)$	5·42
2	$100\,O_4/O_1$	7·02	17	$100\,Mx_2/Mx_1\,(R)$	5·39
3	$100\,F_5/F_4$	7·02	18	$100\,Ml_2/Ml_1\,(L)$	6·02
4	$100\,S_6/S_7\,(R)$	7·75	19	$100\,O_6/O_3$	9·59
5	$100\,S_6/S_7\,(L)$	7·85	20	$100\,Ml_2/Ml_1\,(R)$	5·85
6	$100\,T_6/T_5\,(L)$	8·13	21	$100\,F_3/F_4$	5·36
7	$100\,T_6/T_5\,(R)$	8·15	22	$100\,S_5/S_4\,(R)$	6·39
8	$100\,P_3/P_4\,(R)$	3·02	23	$100\,S_5/S_4\,(L)$	6·46
9	$100\,P_3/P_4\,(L)$	3·06	24	$100\,T_2/T_3\,(L)$	6·21
10	$100\,F_2/F_1\,(R)$	3·98	25	$100\,T_2/T_3\,(R)$	6·23
11	$100\,F_2/F_1\,(L)$	4·15	26	$100\,O_8/O_7\,(R)$	4·59
12	$100\,P_2/P_1\,(R)$	5·95	27	$100\,O_8/O_7\,(L)$	4·33
13	$100\,P_2/P_1\,(L)$	5·61	28	$100\,Mx_3/Mx_1\,(R)$	3·32
14	$100\,O_8/O_9\,(R)$	5·37	29	$100\,Mx_3/Mx_1\,(L)$	3·42
15	$100\,O_8/O_9\,(L)$	5·13			

We will now arrange our homologous indices in the order of intensity of their correlation coefficients, see Table V, p. 436.

These give an average correlation of homologous indices of ·7414, marking a high relationship due to homology after we have removed the lateral dominance. The sources of the differences in the intensity of homologous influence on the various bone indices is not easy to discover*. It is true that all the indices do not belong to the same class; in the second, third and eighth indices in Table V,

and that P_3 and P_4 are the greatest lengths measured. The index of one stands at the top of the homology order and variations of the maxillary indices are on the average small. The index P_3/P_4 stands at the bottom of the homology order. It may, however, be noted that its variation is the least in Table IV. The sphenoid is one of the earliest bones to be laid down, and its indices stand second and third in the homology order. There will obviously be discordances in the interpretation of Tables IV and V on this basis, but investigations might prove profitable when the authors have devoted more time to a study of what is known regarding the development of the skull.

* See however the preceding footnote.

S_4 = distance from krotaphion to krotaphion, and S_5 = distance between the lowest points right and left on the suture between the medial-pterygoid plate and the palate bone, are transverse lengths crossing the median sagittal plane: and again P_1 is the common arc between the two parietal bones. Hence these three indices differ from the remaining nine as not being composed of homologous lengths on opposite sides of the skull, but have the denominators in the pair of correlated indices the same. But this cannot be said to throw much light on their relative positions in the above table. In choosing the ratios of bone lengths some attempts were made to take them so that we should obtain a rough ratio of length to breadth, this accounts for indices like Mx_2/Mx_1, Ml_2/Ml_1, F_2/F_1, T_2/T_3, P_2/P_1, T_6/T_5, O_8/O_7 and O_8/O_9. But these eight indices show great differences in their correlations, and are in no way classed in position together. P_2, P_1, T_6, T_5, O_8, O_7, O_9 end chiefly at cranial points, but we are far from certain that that gives them a reduced homologous correlation, and as a matter of fact the lengths of the first six indices are in very much the same condition as to their terminals.

TABLE V. *Order of Correlation Coefficients for Homologous Indices.*

Order	Index	Correlation Coefficient	Order	Index	Correlation Coefficient
1	$100\,Mx_2/Mx_1$	$+\cdot 9344 \pm \cdot 0038$	7	$100\,Mx_3/Mx_1$	$+\cdot 7911 \pm \cdot 0125$
2	$100\,S_6/S_7$	$+\cdot 9048 \pm \cdot 0043$	8	$100\,P_2/P_1$	$+\cdot 7602 \pm \cdot 0203$
3	$100\,S_5/S_4$	$+\cdot 8822 \pm \cdot 0056$	9	$100\,T_6/T_5$	$+\cdot 6185 \pm \cdot 0145$
4	$100\,Ml_2/Ml_1$	$+\cdot 8806 \pm \cdot 0057$	10	$100\,O_8/O_7$	$+\cdot 6044 \pm \cdot 0148$
5	$100\,F_2/F_1$	$+\cdot 8171 \pm \cdot 0075$	11	$100\,O_8/O_9$	$+\cdot 4789 \pm \cdot 0173$
6	$100\,T_2/T_3$	$+\cdot 7931 \pm \cdot 0086$	12	$100\,P_3/P_1$	$+\cdot 4057 \pm \cdot 0207$

All, we think, that at present we are justified in concluding is, that homology plays a large part in determining the shape as well as the size of homologous bones, but that while this influence has very significant differences as between one bone and a second, we cannot so far state why homology should be much more active in the maxillary or sphenoid bones than in the parietal or occipital. Far more correlations of homologous indices must be worked out before light can be thrown on this point.

We have seen by the mere process of averaging that there is no dominance of either side of the skull in variation *as a whole*. But this does not exclude dominance in the case of individual bones. Table VI indicates that such dominance most probably exists in the case of two indices, although the ratio Δ_{R-L} to its probable error is not "markedly" significant.

In the case of both bones, where dominance of indicial variability is probably significant, that dominance is on the right side. But before we consider this point, it is worth while to inquire whether there is significance in the series of twelve ratios as a *whole*; for this purpose we apply the P_{λ_n} test. Multiplying the last column by ·67449, we shall have the ratio of the deviation to the standard deviation

TABLE VI. *Difference of Absolute Variability for Homologous Indices.*

Category of Significance of Difference	Index	Difference and its Probable Error $\Delta_{R-L} \pm$ p.e. of Δ_{R-L}	Ratio of Difference to its Probable Error Δ_{R-L}/p.e. of Δ_{R-L}
Probably significant	$100 P_2/P_1$	+ ·3382 ± ·0880	+ 3·84
	$100 O_8/O_7$	+ ·2587 ± ·0821	+ 3·15
Possibly, but far from definitely, significant	$100 Ml_2/Ml_1$	− ·1741 ± ·0708	− 2·46
	$100 O_8/O_9$	+ ·2389 ± ·1066	+ 2·24
	$100 F_2/F_1$	− ·1734 ± ·0855	− 2·03
Most probably insignificant	$100 Mx_3/Mx_1$	− ·0965 ± ·0686	− 1·41
	$100 S_6/S_7$	− ·1030 ± ·0786	− 1·31
	$100 S_6/S_4$	− ·0747 ± ·0759	− 0·98
	$100 P_3/P_4$	− ·0414 ± ·0690	− 0·60
	$100 Mx_2/Mx_1$	− ·0256 ± ·0576	− 0·44
	$100 T_2/T_3$	+ ·0197 ± ·0879	+ 0·22
	$100 T_6/T_5$	+ ·0178 ± ·1501	+ 0·12

of that deviation, and the probability integral table of the normal curve* provides us at once with the probability integrals of the twelve values. In Table VII the corresponding logarithms and their sums are provided.

TABLE VII.

Criterion for the General Significance of the Twelve Ratios in Table VI.

Order	Deviation / S.D. of Deviation	p	$\log p$	$p' = 1 - p$	$\log p$
1	+ 2·590	·99520	$\bar{1}$·997,8231	·00480	$\bar{3}$·681,2412
2	+ 2·125	·98321	1·992,6463	·01679	$\bar{2}$·225,0507
3	− 1·659	·04856	2·686,2787	·95144	$\bar{1}$·978,3814
4	+ 1·511	·93460	$\bar{1}$·970,6258	·06540	$\bar{2}$·815,5777
5	− 1·369	·08550	2·931,9661	·91450	$\bar{1}$·961,1837
6	− 0·951	·17080	$\bar{1}$·232,4870	·82920	$\bar{1}$·918,6593
7	− 0·884	·18835	$\bar{1}$·274,9656	·81165	$\bar{1}$·909,3688
8	− 0·661	·25431	1·405,3631	·74569	$\bar{1}$·872,5583
9	− 0·405	·34274	$\bar{1}$·534,9648	·65726	$\bar{1}$·817,7372
10	− 0·297	·38323	$\bar{1}$·583,4596	·61677	$\bar{1}$·790,1232
11	+ 0·148	·55883	1·747,2797	·44117	$\bar{1}$·644,6060
12	+ 0·081	·53228	$\bar{1}$·726,1401	·46772	$\bar{1}$·669,9859
		$\log_{10}\lambda_n = -5$·915,9989		$\log_{10}\lambda'_n = -6$·715,5266	

$$\sqrt{n}\log_{10} e = \sqrt{12} \times {\cdot}434{,}2945 = 1{\cdot}504{,}4403.$$

$$u = -\log_{10}\lambda_n/(\sqrt{n}\log_{10} e) = 3{\cdot}9324, \quad u' = -\log_{10}\lambda'_n/(\sqrt{n}\log_{10} e) = 4{\cdot}4638.$$

* The true curve of distribution of the standard deviations of samples from a normal curve is given by Helmert's equation—a Type III curve, but in the case of a large sample this passes into a normal

Hence $\qquad Q_{\chi_n} = 1 - I\,(11,\ 3{\cdot}9324), \quad Q_{\chi'_n} = 1 - I\,(11,\ 4{\cdot}4638),$

where $I\,(p,\ u)$ is the gamma integral ratio as tabled in the *Incomplete Γ-Function Tables.* Since $Q_{\chi'_n}$ is smaller than Q_{χ_n}, it suffices to consider its value.

Thus we have $\qquad Q_{\chi'_n} = {\cdot}1562,$

and this does not indicate any great improbability of the twelve differences being due to random sampling from a population of no difference between *R* and *L*.

It is possible therefore when treating the twelve differences as a whole to consider them as random samples from populations having equal variation, notwithstanding that treated individually the first two differences might be considered as probably significant. This point—i.e. of the difference between a single $\Delta/(\text{p.e. of } \Delta)$ and a whole series of such values—has too often been neglected in craniometric memoirs.

(5) *Comparison of the Influence of Homology on Index Correlations and Absolute Lengths.*

Returning to Table V, we recall the fact that the average correlation of the twelve homologous indices is ·7414. For the twenty-five absolute homologous lengths, measured on the separate cranial bones, it is ·8198. Thus on the average the absolute homologous lengths were more closely related than the indices, i.e. the sizes of the bones were more closely related than their shapes.

We may analyse this result a little further in the following table:

Correlation of Homologous Characters.

Bone	Mean Index Correlation	Mean Absolute Lengths Correlation
Sphenoid	·8935	·8293
Malar	·8806	·9309
Maxillary	·8019	·9108
Frontal	·8171	·9221
Temporal	·7059	·7894
Parietal	·5974	·7253
Occipital	·5407	·7087
All cases	·7414	·8198

It will be noticed that in both cases, index and absolute lengths, the temporal, parietal and occipital bones are the least highly correlated. The facial bones,

curve, with mean at the modal value of the S.D. But the difference of two quantities each following a normal distribution, if divided by the standard deviation of that difference, follows Student's *z* curve, which again passes into a normal curve for large samples. Since each pair of homologous bones has the same number of individuals, we may take a series of probability integrals, each corresponding to a normal curve based upon different-sized samples for each pair of different homologous indices.

malar, maxillary and frontal, form also, in both cases, a group with high correlations. The sphenoid stands at the top in the index column, and intermediate between the two groups in the absolute lengths column. With this one exception the influence of homology is more intense in the case of every bone in regard to size than to shape; also, apart from the sphenoid, the order of intensity of the influence of homology is much the same for indices and lengths. The greater symmetry of the face over the brain box is a noteworthy fact. Possibly it has been due to sexual selection; we notice failure of homology in the face far more than in the case of the retroceding calvaria.

(6) *Method of studying the Index Correlations of Individual Cranial Bones.*

In order to study these index correlations, we shall require to classify our cranial bones. Such bones may be either on the same or opposite sides of the median sagittal plane. If on the same side, they may be either contiguous or non-contiguous. If on opposite sides, they may be homologous or non-homologous. If non-homologous, they may be contiguous or non-contiguous with their homologous bone. The consideration of contiguity is *a priori* of great importance because it is not unreasonable to consider that contiguity would have considerable influence on measurements of *shape**.

In Table VIII we have classified the cranial bones with regard to their "contiguity." On the basis of these measures of contiguity, and taking into account that the components of some of the indices do not possess "laterality," i.e. are measured across the median sagittal plane or along it, we can divide up our correlations for purposes of study into the following classes:

(i) Correlation of Indices of Contiguous Bones (same side).

(ii) Correlation of Indices of slightly Contiguous Bones (same side).

(iii) Correlation of Indices of Non-contiguous Bones (same side).

(iv) Cross-Correlations. Index of a Bone with the Index of a Bone contiguous with the homologue of the first (on the opposite side).

(v) Cross-Correlations. Index of a Bone with the Index of a Bone non-contiguous with the homologue of the first (on the opposite side).

(vi) Correlation of two Indices of the same Bone.

(vii) Correlation of the same Indices on Homologous Bones†.

(viii) Correlation of different Indices on Homologous Bones.

* Where the union of two bones is an "overlap" (e.g. temporal and parietal bones) and not a serrated suture, the measurements have been taken to the edge of the overlap (e.g. squamous suture), the skulls not having been separated into their individual bones. This may have some influence on the correlation of the indices of contiguous bones, but it has not been possible to trace it. Nor have we been able to find in the index correlations differences associated with ossification in cartilage and development in membrane.

† This has already been discussed: see pp. 436—438 above.

TABLE VIII. *Contiguity of Cranial Bones.*

Bone	Frontal	Parietal	Temporal	Sphenoid	Occipital	Maxillary	Malar
Frontal	—	Contiguous	Non-contiguous	Contiguous	Non-contiguous	Slightly (*a*) contiguous	Contiguous
Parietal	Contiguous	...	Contiguous	Slightly (*b*) contiguous	Contiguous	Non-contiguous	Non-contiguous
Temporal	Non-contiguous	Contiguous	-	Contiguous	Contiguous	Non-contiguous	Slightly (*c*) contiguous
Sphenoid	Contiguous	Slightly (*b*) contiguous	Contiguous		Contiguous	Contiguous	Contiguous
Occipital	Non-contiguous	Contiguous	Contiguous	Contiguous	-	Non-contiguous	Non-contiguous
Maxillary	Slightly (*a*) contiguous	Non-contiguous	Non-contiguous	Contiguous	Non-contiguous		Contiguous
Malar	Contiguous	Non-contiguous	Slightly (*c*) contiguous	Contiguous	Non-contiguous	Contiguous	-

(*a*) Superior nasal point to Dacryon.
(*b*) From Sphenion to Krotaphion.
(*c*) On the temporal process.

There are many degrees of "Contiguity," but the above notes will best explain what we mean by "Slight Contiguity."

(ix) Correlation of Indices on Contiguous Bones, one Index of the pair having no laterality.

(x) Correlation of Indices on Non-contiguous Bones, one or both Indices of the pair having no laterality.

Under these classes we shall study our index correlations.

(7) *Correlation of Indices of Non-contiguous Bones (on same side).*

We will start with Group (iii) as most convenient for our present purposes.

Table IX indicates that we have 40 index correlations in our series of 29 indices—of which 27 have been worked out—when we confine our attention to non-contiguous bones of the same side. In this series those of highest value, as a rule positive, are between indices of the occipital and parietal bones and the maxillary index Mx_2/Mx_1. There seems little obvious reason why a breadth-height index of the maxillary should be less closely associated with a height-length index of the parietal, and in a *positive* sense, than with such indices as O_8/O_7 and O_8/O_9 of the distant parts of the occipital. Sometimes, as in the cases of F_2/F_1 with T_2/T_3 and O_8/O_9 with Ml_2/Ml_1, the right and left sides give correlations of opposite sign. Again, the correlations on the two sides, while differing within the limits of random sampling, differ fairly considerably from each other. Without being able to give any explanation of why certain correlations were higher than others, we note that all the correlations are less than 0·1, and thus perfectly useless for the purpose of predicting the shape of one bone from that of a second non-contiguous bone on the same side.

TABLE IX. *Correlation of Indices of Non-contiguous Bones* (*on same side*).

Index Pair	Correlation and p.e.	Index Pair	Correlation and p.e.
$O_8/O_9\,(R)$, $Mx_2/Mx_1\,(R)$	$+\cdot0674\pm\cdot0284$	$F_2/F_1\,(R)$, $T_2/T_3\,(R)$	$+\cdot0387\pm\cdot0230$
$O_8/O_9\,(L)$, $Mx_2/Mx_1\,(L)$	$+\cdot0819\pm\cdot0283$	$F_2/F_1\,(L)$, $T_2/T_3\,(L)$	$-\cdot0071\pm\cdot0230$
$P_3/P_4\,(R)$, $Mx_2/Mx_1\,(R)$	$+\cdot0551\pm\cdot0297$	$O_8/O_9\,(R)$, $Ml_2/Ml_1\,(R)$	$+\cdot0131\pm\cdot0249$
$P_3/P_4\,(L)$, $Mx_2/Mx_1\,(L)$	$+\cdot0782\pm\cdot0248$	$O_8/O_9\,(L)$, $Ml_2/Ml_1\,(L)$	$-\cdot0361\pm\cdot0247$
$O_8/O_7\,(R)$, $Mx_2/Mx_1\,(R)$	$+\cdot0744\pm\cdot0284$	$T_6/T_5\,(R)$, $Mx_3/Mx_1\,(R)$	$-\cdot0308\pm\cdot0311$
$O_8/O_7\,(L)$, $Mx_2/Mx_1\,(L)$	—	$T_6/T_5\,(L)$, $Mx_3/Mx_1\,(L)$	—
$T_2/T_3\,(R)$, $Mx_3/Mx_1\,(R)$	$+\cdot0711\pm\cdot0307$	$F_2/F_1\,(R)$, $T_6/T_5\,(R)$	$+\cdot0245\pm\cdot0231$
$T_2/T_3\,(L)$, $Mx_3/Mx_1\,(L)$	—	$F_2/F_1\,(L)$, $T_6/T_5\,(L)$	—
$F_2/F_1\,(R)$, $O_8/O_7\,(R)$	$+\cdot0677\pm\cdot0230$	$O_8/O_9\,(R)$, $Mx_3/Mx_1\,(R)$	$-\cdot0191\pm\cdot0310$
$F_2/F_1\,(L)$, $O_8/O_7\,(L)$	—	$O_8/O_9\,(L)$, $Mx_3/Mx_1\,(L)$	—
$P_3/P_4\,(R)$, $Ml_2/Ml_1\,(R)$	$-\cdot0406\pm\cdot0258$	$P_2/P_1\,(R)$, $Ml_2/Ml_1\,(R)$	$-\cdot0133\pm\cdot0258$
$P_3/P_4\,(L)$, $Ml_2/Ml_1\,(L)$	$-\cdot0652\pm\cdot0258$	$P_2/P_1\,(L)$, $Ml_2/Ml_1\,(L)$	—
$T_2/T_3\,(R)$, $Mx_2/Mx_1\,(R)$	$+\cdot0375\pm\cdot0282$	$P_2/P_1\,(R)$, $Mx_3/Mx_1\,(R)$	$+\cdot0092\pm\cdot0323$
$T_2/T_3\,(L)$, $Mx_2/Mx_1\,(L)$	$+\cdot0632\pm\cdot0282$	$P_2/P_1\,(L)$, $Mx_3/Mx_1\,(L)$	—
$P_2/P_1\,(R)$, $Mx_2/Mx_1\,(R)$	$-\cdot0535\pm\cdot0296$	$F_2/F_1\,(R)$, $O_8/O_9\,(R)$	$+\cdot0044\pm\cdot0232$
$P_2/P_1\,(L)$, $Mx_2/Mx_1\,(L)$	—	$F_2/F_1\,(L)$, $O_8/O_9\,(L)$	$+\cdot0073\pm\cdot0231$
$P_3/P_4\,(R)$, $Mx_3/Mx_1\,(R)$	$+\cdot0532\pm\cdot0322$	$O_8/O_7\,(R)$, $Mx_3/Mx_1\,(R)$	$-\cdot0046\pm\cdot0310$
$P_3/P_4\,(L)$, $Mx_3/Mx_1\,(L)$	—	$O_8/O_7\,(L)$, $Mx_3/Mx_1\,(L)$	—
$T_6/T_5\,(R)$, $Mx_2/Mx_1\,(R)$	$+\cdot0529\pm\cdot0285$	$O_8/O_7\,(R)$, $Ml_2/Ml_1\,(R)$	$+\cdot0019\pm\cdot0249$
$T_6/T_5\,(L)$, $Mx_2/Mx_1\,(L)$	—	$O_8/O_7\,(L)$, $Ml_2/Ml_1\,(L)$	—

As there were only 9 negative as against 18 positive correlations, we doubted whether the whole series could be treated as a random sample from a set of populations with zero correlation in each. Yet it seemed worth while examining this. We applied the $P_{\lambda n}$ test, treating samples (in all cases over 430) as having their r's distributed normally. We found that the more stringent set of the probability integrals gave

$$Q_{\lambda' n} = \cdot0066,$$

or, only in about 7 in 1000 trials would so poor a result arise in such large samples from populations having zero correlation between the indices*. Thus we feel bound to discard the hypothesis of these indices even in the case of non-contiguous bones being uncorrelated. Yet in the bulk of cases the indices are so small and irregular that we do not feel able to give much scope to a "general control factor" regulating the shapes of non-adjacent as well as adjacent bones.

If we leave out of account the correlations of O_8/O_9, O_8/O_7 and P_3/P_4 with Mx_2/Mx_1, then we find

$$Q_{\lambda' n} = \cdot1790,$$

or, it would not be an unreasonable assumption to suppose the remaining 22 index correlations to be really zero. This is a somewhat arbitrary procedure, yet it suffices to indicate how slender the correlations are. Indeed, if we applied a test

* Besides the hypothesis made in the text of zero r for very large samples being distributed normally, the r must be that of material which more or less closely follows a normal surface. If we adopt the assumption, not unreasonable, that the absolute measurements of the cranial bones do so, the indices will fail to do so with *theoretical* accuracy, but all the same with adequate *practical* sufficiency.

of thrice the probable error to each result individually, none of them would pass as certainly significant. Such a method of testing would not, however, be convincing when we have to deal, not with a single value, but with a large series of correlation coefficients.

Accordingly, to get a measure of the "general factor" controlling development of shape of the individual bones, we may find the mean correlation of the 27 non-contiguous bones = +·0197, and ask whether samples from 27 populations having this correlation, but their own standard deviations, could reasonably have given the 27 observed values; we find

$$Q_{\chi^2_n} = \cdot 3452.$$

Accordingly, we may conclude that the general factor controlling the shapes of non-contiguous bones may amount to about +·02, while the factor of homology gives an average correlation of ·74, or has thirty-seven times the intensity of the former.

The startling conclusion here is the smallness of the relationship between the shapes of the skull at non-contiguous parts.

(8) *Correlation of Indices of slightly Contiguous Bones (on same side).* Group (ii).

The coefficients are given in Table X.

TABLE X. *Correlation of Indices of slightly Contiguous Bones (on same side).*

Index Pair	Correlation and p.e.	Index Pair	Correlation and p.e.
$P_3/P_4\,(R)$, $S_5/S_4\,(R)$	$+\cdot 1205 \pm \cdot 0247$	$F_2/F_1\,(R)$, $Mx_3/Mx_1\,(R)$	$-\cdot 0109 \pm \cdot 0308$
$P_3/P_4\,(L)$, $S_5/S_4\,(L)$	—	$F_2/F_1\,(L)$, $Mx_3/Mx_1\,(L)$	
$P_2/P_1\,(R)$, $S_5/S_4\,(R)$	$-\cdot 0074 \pm \cdot 0250$	$F_2/F_1\,(R)$, $Mx_2/Mx_1\,(R)$	$+\cdot 0107 \pm \cdot 0281$
$P_2/P_1\,(L)$, $S_5/S_4\,(L)$	—	$F_2/F_1\,(L)$, $Mx_2/Mx_1\,(L)$	$+\cdot 0068 \pm \cdot 0283$
$T_6/T_5\,(R)$, $Ml_2/Ml_1\,(R)$	$+\cdot 0598 \pm \cdot 0247$	$P_3/P_4\,(R)$, $S_6/S_7\,(R)$	$-\cdot 0339 \pm \cdot 0257$
$T_6/T_5\,(L)$, $Ml_2/Ml_1\,(L)$	—	$P_3/P_4\,(L)$, $S_6/S_7\,(L)$	
$P_2/P_1\,(R)$, $S_6/S_7\,(R)$	$+\cdot 0482 \pm \cdot 0251$	$T_2/T_3\,(R)$, $Ml_2/Ml_1\,(R)$	$+\cdot 0113 \pm \cdot 0246$
$P_2/P_1\,(L)$, $S_6/S_7\,(L)$	—	$T_2/T_1\,(L)$, $Ml_2/Ml_1\,(L)$	$-\cdot 0001 \pm \cdot 0245$

The only value which seems exceptional here is the correlation of the parietal P_3/P_4 with sphenoidal S_5/S_4, especially when we note the negative correlation of P_2/P_1 with the same sphenoidal index. Assuming that there was no arithmetical error in these values, the system of ten correlations was tested as a random sample from a zero population. Assuming normal distributions for the indices, their correlation coefficients r in the sampling would follow the curve $z = z_0\,(1 - r^2)^{\frac{1}{2}(n-4)}$. As no n is less than 434, and n can take values up to between 800 and 900, it is sufficient to use a normal distribution in each case for r. Calculating the probability integrals and applying the P_{λ_n} test, we found $Q_{\chi^2_n} = \cdot 0086$. The ten values can scarcely therefore be looked upon as random samples from parent populations with

zero correlation. Omitting the first parietal-sphenoid correlation, we found for the nine remaining correlations, $Q_{\lambda' n} = \cdot 3266$, or the slightly contiguous group may in the great bulk be treated as without correlation.

The mean of the correlations of the indices for these slightly contiguous bones is $+\cdot 0145$. We next inquired whether these correlations could be considered as a random sample from ten populations having this value for their index correlations, but standard errors varying with the sizes of the samples. These sizes being large, we again supposed normal distribution of the coefficients and applied the $P_{\lambda n}$ test: we found $Q_{\lambda n} = \cdot 2312$ and $Q_{\lambda' n} = \cdot 1422$; the latter is the more stringent, but does not indicate that a random distribution of correlations from populations with a correlation of ·0145 is very improbable.

It would seem, therefore, that the factor for "slight contiguity," giving a standard correlation of ·0145, is not definitely greater than that for non-contiguity with a correlation of ·0197. There appears, accordingly, no reason for separating the slightly contiguous from the non-contiguous bones, and we may say that the general factor linking the shapes of bones which are not to a considerable extent contiguous does not produce a general correlation exceeding ·02.

(9) *Correlation of Indices of Contiguous Bones (on same side).* Group (i).

We now come to the study of our first group. From this group we shall be able to appreciate the maximum influence of contiguity on measurements of cranial shape. We thus reach Table XI, containing the index correlations of bones on the same side having fairly appreciable parts in contact. It will be seen at once that the condition of things is changed, 13 correlations out of 44 have numerically a value in excess of 0·1 and they range nearly up to ·16. Their average is $-\cdot 01638$, or, say, $-\cdot 0164$*. It is difficult to lay much stress on negative as against positive *index* correlations, for the formation of the indices, after the choice of lengths, must be more or less arbitrary, and, as a rule, but not absolutely, the choice was directed to obtaining an index under 100. However, for the indices selected, the indices of contiguous bones give on the whole a negative correlation, or the effect of increasing contiguity is to reduce correlation (·0197, ·0145, $-\cdot 0164$). If we consider the indices regardless of sign, then we find:

Non-contiguous bones	·0397
Slightly contiguous bones	·0434
Contiguous bones	·0655

These results confirm the view that the slightly contiguous may be classed with the non-contiguous, but indicate that contiguity gives an association somewhat less than double non-contiguity, if we neglect the sign of the correlations.

We will now test whether it is possible to consider our index correlations of contiguous bones as a random sample from 44 populations with the correlation

* The mean of the negative index correlations is $-\cdot 0788$, of the positive $+\cdot 0514^5$, and of the whole system without regard to sign ·06549.

TABLE XI. *Correlation of Cranial Indices on Contiguous Bones (on same side).*

Index Pair	Correlation and p.e.	Index Pair	Correlation and p.e.
P_3/P_4 (R), O_8/O_7 (R)	− ·1578 ± ·0238	S_6/S_7 (L), Mx_3/Mx_1 (L)	
P_3/P_4 (L), O_8/O_7 (L)		P_2/P_1 (R), O_8/O_9 (R)	− ·0540 ± ·0244
Mx_3/Mx_1 (R), Ml_2/Ml_1 (R)	− ·1217 ± ·0298	P_2/P_1 (L), O_8/O_9 (L)	
Mx_3/Mx_1 (L), Ml_2/Ml_1 (L)	·1472 ± ·0287	S_6/S_7 (R), O_8/O_9 (R)	+ ·0137 ± ·0241
T_2/T_3 (R), S_5/S_4 (R)	+ ·1107 ± ·0246	S_6/S_7 (L), O_8/O_9 (L)	
T_2/T_3 (L), S_5/S_4 (L)	+ ·1318 ± ·0239	S_5/S_4 (R), Ml_2/Ml_1 (R)	+ ·0135 ± ·0261
P_2/P_1 (R), O_8/O_7 (R)	− ·1313 ± ·0240	S_5/S_4 (L), Ml_2/Ml_1 (L)	+ ·0011 ± ·0257
P_2/P_1 (L), O_8/O_7 (L)		T_6/T_5 (R), O_8/O_7 (R)	+ ·0420 ± ·0234
P_3/P_4 (R), O_8/O_9 (R)	− ·1237 ± ·0240	T_6/T_5 (L), O_8/O_7 (L)	
P_3/P_4 (L), O_8/O_9 (L)	− ·1256 ± ·0243	S_5/S_4 (R), Mx_3/Mx_1 (R)	+ ·0386 ± ·0325
Mx_3/Mx_1 (R), Ml_2/Ml_1 (R)	− ·1206 ± ·0313	S_5/S_4 (L), Mx_3/Mx_1 (L)	
Mx_3/Mx_1 (L), Ml_2/Ml_1 (L)		F_2/F_1 (R), Ml_2/Ml_1 (R)	+ ·0308 ± ·0245
P_3/P_4 (R), T_2/T_3 (R)	+ ·1270 ± ·0240	F_2/F_1 (L), Ml_2/Ml_1 (L)	+ ·0140 ± ·0243
P_3/P_4 (L), T_2/T_3 (L)	+ ·1224 ± ·0240	T_2/T_3 (R), O_8/O_7 (R)	+ ·0350 ± ·0234
P_2/P_1 (R), T_2/T_3 (R)	− ·1077 ± ·0241	T_2/T_3 (L), O_8/O_7 (L)	
P_2/P_1 (L), T_2/T_3 (L)		P_3/P_4 (R), T_6/T_5 (R)	− ·0340 ± ·0246
F_2/F_1 (R), S_5/S_4 (R)	− ·1000 ± ·0242	P_3/P_4 (L), T_6/T_5 (L)	
F_2/F_1 (L), S_5/S_4 (L)	− ·0981 ± ·0239	S_5/S_4 (R), O_8/O_9 (R)	− ·0246 ± ·0248
F_2/F_1 (R), P_2/P_1 (R)	− ·0992 ± ·0239	S_5/S_4 (L), O_8/O_9 (L)	+ ·0075 ± ·0245
F_2/F_1 (L), P_2/P_1 (L)		T_6/T_5 (R), S_6/S_7 (R)	+ ·0205 ± ·0241
T_6/T_5 (R), O_8/O_9 (R)	+ ·0883 ± ·0234	T_6/T_5 (L), S_6/S_7 (L)	
T_6/T_5 (L), O_8/O_9 (L)		T_6/T_5 (R), S_5/S_4 (R)	·0160 ± ·0248
S_5/S_4 (R), Mx_3/Mx_1 (R)	− ·0520 ± ·0301	T_6/T_5 (L), S_5/S_4 (L)	
S_5/S_4 (L), Mx_3/Mx_1 (L)	− ·0692 ± ·0297	T_2/T_3 (R), S_6/S_7 (R)	− ·0117 ± ·0240
F_2/F_1 (R), P_3/P_4 (R)	+ ·0689 ± ·0241	T_2/T_3 (L), S_6/S_7 (L)	
F_2/F_1 (L), P_3/P_4 (L)	+ ·0032 ± ·0243	P_2/P_1 (R), T_6/T_5 (R)	+ ·0070 ± ·0246
S_6/S_7 (R), Mx_3/Mx_1 (R)	− ·0657 ± ·0309	P_2/P_1 (L), T_6/T_5 (L)	
S_6/S_7 (L), Mx_3/Mx_1 (L)		S_6/S_7 (R), Ml_2/Ml_1 (R)	·0057 ± ·0250
T_2/T_3 (R), O_8/O_9 (R)	+ ·0654 ± ·0234	S_6/S_7 (L), Ml_2/Ml_1 (L)	
T_2/T_3 (L), O_8/O_9 (L)	+ ·0604 ± ·0233	F_2/F_1 (R), S_6/S_7 (R)	+ ·0014 ± ·0238
S_5/S_4 (R), O_8/O_7 (R)	− ·0620 ± ·0247	F_2/F_1 (L), S_6/S_7 (L)	
S_5/S_4 (L), O_8/O_7 (L)		S_6/S_7 (R), O_8/O_7 (R)	− ·0012 ± ·0241
S_6/S_7 (R), Mx_3/Mx_1 (R)	− ·0617 ± ·0287	S_6/S_7 (L), O_8/O_7 (L)	

− ·0164. Working this out by the P_{λ_n} test, we find Q_{λ_n} = ·000,0012, or it is not possible to consider that we are dealing with random deviations from an average relationship measured by − ·0164. This is really clear from Table XI itself, for it would appear that the correlation between certain bones is much higher, significantly higher, than that between others.

To see whether it is feasible to throw any light on the matter we have drawn up Table XII, which gives the average correlation between the indices of bones on the same side, whether adjacent or not. First values are those regardless of sign, the second values in square brackets pay attention to sign.

Confining our attention here to the mean correlation values independent of sign we see that the shape of the parietal has the closest average relationship to the shapes of other cranial bones. The highest average correlation with an individual bone is that with the occipital, where all the correlations are negative; this might

TABLE XII. *Average Relationship of Bone Indices (on same side) having regard to Contiguity**.

Bone	Frontal	Parietal	Temporal	Sphenoid	Occipital	Maxillary	Malar
Frontal	—	Contiguous ·0571 [− ·0090]	Non-contiguous ·0234 [+ ·0187]	Contiguous ·0678 [− ·0649]	Non-contiguous ·0265 [+ ·0265]	Slightly contiguous ·0303 [+ ·0030]	Contiguous ·0257 [+ ·0257]
Parietal	Contiguous ·0571 [− ·0090]	—	Contiguous ·0799 [+ ·0229]	Slightly contiguous ·0680 [+ ·0163^5]	Contiguous ·1185 [− ·1185]	Non-contiguous ·0499 [+ ·0285]	Non-contiguous ·0397 [− ·0397]
Temporal	Non-contiguous ·0234 [+ ·0187]	Contiguous ·0799 [+ ·0229]	—	Contiguous ·0593 [+ ·0483]	Contiguous ·0582 [+ ·0582]	Non-contiguous ·0511 [+ ·0388]	Slightly contiguous ·0238 [+ ·0235]
Sphenoid	Contiguous ·0678 [− ·0649]	Slightly contiguous ·0680 [+ ·0163^5]	Contiguous ·0593 [+ ·0483]	—	Contiguous ·0278 [− ·0073]	Contiguous ·0576 [− ·0418]	Contiguous ·0168 [+ ·0130]
Occipital	Non-contiguous ·0265 [+ ·0265]	Contiguous ·1185 [− ·1185]	Contiguous ·0582 [+ ·0582]	Contiguous ·0278 [− ·0073]	—	Non-contiguous ·0495 [+ ·0400]	Non-contiguous ·0170 [− ·0070]
Maxillary	Slightly contiguous ·0303 [+ ·0030]	Non-contiguous ·0499 [+ ·0285]	Non-contiguous ·0511 [+ ·0388]	Contiguous ·0576 [− ·0418]	Non-contiguous ·0495 [+ ·0400]	—	Contiguous ·1328 [− ·1328]
Malar	Contiguous ·0257 [+ ·0257]	Non-contiguous ·0397 [− ·0397]	Slightly contiguous ·0238 [+ ·0235]	Contiguous ·0168 [+ ·0130]	Non-contiguous ·0170 [− ·0070]	Contiguous ·1328 [− ·1328]	—
Total	·0393 [− ·0015]	·0722 [− ·0166]	·0553 [+ ·0377]	·0500 [− ·0038]	·0539 [− ·0031]	·0588 [− ·0024]	·0436 [− ·0222]

* Based on Tables IX, X and XI.

be attributed to the length of contour in contact, and such a view might be considered as confirmed by the second highest parietal correlation being with the temporal, but it is hardly in keeping with its relationship to the only slightly contiguous sphenoid or with the far more contiguous frontal. Indeed, while extent of contiguity plays a part in these correlations, it is clearly not the only factor to be considered. Thus the highest individual average correlation is that between malar and maxillary, where all the separate correlations are negative, yet the mere extent of contact between the latter pair is hardly in excess of that between parietal and occipital, or between parietal and frontal. Clearly the correlation will depend not only on the extent of contours in contact, but also on the extent relative to the size of the bones in contact. Indeed there are many other factors which may take part in determining these correlations—not only the original nature of the bones themselves, but the order in which they are laid down, etc. What is the really important point to be regarded is this: that, while many of the correlations are insignificant, some are certainly significant, but they are in the latter case so small that any idea of predicting the shape of one bone from the shape of an adjacent bone is idle*. As illustration let us take the maxillary index Mx_2/Mx_1 and the malar index Ml_2/Ml_1 on the right side of the skull; with the relatively high correlation $-{\cdot}1217 \pm {\cdot}0290$, the prediction formula of the latter from the former is by the values in Tables IV and XI:

$$\text{Mean}\left(\frac{Ml_2}{Ml_1}\right) \text{for array} = \text{Mean}\left(\frac{Ml_2}{Ml_1}\right) \text{for population} - {\cdot}1217 \times \frac{5{\cdot}8490}{5{\cdot}3937}\left(\frac{Mx_2}{Mx_1} - 83{\cdot}1522\right)$$

$$= \quad " \qquad\qquad " \qquad\qquad - {\cdot}1320\left(\frac{Mx_2}{Mx_1} - 83{\cdot}1522\right).$$

In other words, if the maxillary index be raised by a unit above its mean value, the malar index would be lowered only by ·1320 of a unit, while the standard error of a determination from the above formula is 5·8055, or is only some ·045 less than the standard deviation of the entire population!

Our general conclusion must be that contiguity does influence to a small extent the indices of shape of the cranial bones. But the influence is so small and the factors determining it so little at present investigated, that very little error would be introduced by supposing for the time being that all these indices whether of contiguous or non-contiguous bones are independent.

The striking feature of the investigation up to this point is the intensity of the homology factor as compared with the adjacency factor.

If, notwithstanding these general considerations, the reader desires a reasonable scheme to represent the whole distribution of contiguous bone indices, we may

* The reader must bear in mind a certain apparent paradox: Two adjacent bones will normally always "fit," and it might be considered that this indicated a high correlation of shapes. But this is not so, the shapes of the two bones follow no general law. Thus, if we take the correlation of the parietal and occipital indices, P_2/P_1 and O_2/O_1, $= -{\cdot}0540 \pm {\cdot}0241$, which cannot be considered individually as certainly significant, the mean of the group of occipital indices corresponding to a given parietal index will scarcely differ at all from the mean occipital index value; thus there is no sensible association of one index with a second.

consider whether treating all the correlations as of one sign, but with their individual standard errors, they could be reasonably represented by a random sample from a population with correlation equal to the average numerical value ·0655. The computing labour is somewhat lengthy, and since the standard errors are nearly equal might reasonably have been lightened by treating them as the same, and equal to their mean, but to avoid a not unusual type of criticism, they have been kept distinct. The numbers on which the correlations are based (see Tables I and III) are very considerable, and the correlations themselves are small and many insignificant; we have thus no hesitation in treating their distribution as normal. The P_{λ_n} test being used, the probability integrals and their logarithms were obtained for the two opposite directions, with the following results:

Probability Integrals: From left to right	From right to left
$\log_{10}\lambda_n = -22{\cdot}683{,}6667,$	$\log_{10}\lambda'_n = -22{\cdot}639{,}2892,$
$\sqrt{n}\log_e 10 = \sqrt{44}\log_e 10 = 2{\cdot}880{,}7838,$	$= 2{\cdot}880{,}7838,$
$u = 7{\cdot}87413,$	$u' = 7{\cdot}85873,$
$P_{\lambda_n} = I(43, 7{\cdot}87413),$	$P_{\lambda'_n} = I(43, 7{\cdot}85873).$

Since n is greater than 30, Miss F. N. David's Table* cannot be used, and resort must be had to the *Tables of the Incomplete Γ-Function*†. Hence we find:

$$P_{\lambda_n} = {\cdot}8888, \qquad P_{\lambda'_n} = {\cdot}8863,$$
$$Q_{\lambda_n} = {\cdot}1112, \qquad Q_{\lambda'_n} = {\cdot}1127.$$

There is scarcely any difference between the two sets, or if the parent populations consisted of a series of surfaces of correlation $\rho = {\cdot}0655$ we should obtain, once in nine random samples, a sample more improbable than the observed one. This does not justify us in asserting that the 44 index correlations of contiguous bones were so obtained, but it does justify us in stating that a moderately reasonable graduation of the observed results may be summed up in random sampling from a parent population of correlation coefficients with mean $\rho = {\cdot}0655$. And this gives far greater definition to the system than the simple statement that the mean of the correlations is ·0655.

(10) *Cross-Correlations.* Groups (iv) and (v).

Some remarkable results flow from these cross-correlations, indicating how great is the influence of homology and how small that of contiguity. We divide our indices into two classes, namely:

A. The correlation of an index of one bone with the index of a second bone on the opposite side contiguous with the homologue of the first bone. The values of such correlations are given in Table XIII, p. 448. Out of 66 possible values, owing to the great labour already spent in making correlation tables and working them out (see p. 433), only 22 values were determined.

* *Biometrika*, Vol. xxvi. 1934, pp. 1—11. Miss David gives P_{λ_n} for $\log_{10}\lambda_n$.

† Reissue *Biometrika* Office, University College, London, price 42*s.* net.

TABLE XIII.

Cross-Correlations. A. Index of a Bone with the Index of a Second Bone on the other side contiguous with the homologue of the First.*

Index Pair		Correlation and p.e.	Index Pair		Correlation and p.e.
$T_2/T_1\,(R)$,	$S_5/S_4\,(L)$	$+\cdot 1208 \pm \cdot 0240$	$P_2/P_1\,(L)$,	$T_2/T_3\,(R)$	—
$T_2/T_1\,(L)$,	$S_5/S_4\,(R)$	$+\cdot 1139 \pm \cdot 0243$	$P_2/P_1\,(R)$,	$O_8/O_9\,(L)$	—
$Mx_2/Mx_1\,(R)$,	$Ml_2/Ml_1\,(L)$	$-\cdot 0985 \pm \cdot 0290$	$P_3/P_1\,(L)$,	$O_8/O_9\,(R)$	—
$Mx_2/Mx_1\,(L)$,	$Ml_2/Ml_1\,(R)$	$\cdot 1185 \pm \cdot 0253$	$P_2/P_1\,(R)$,	$O_8/O_7\,(L)$	—
$P_3/P_1\,(R)$,	$O_8/O_9\,(L)$	$-\cdot 1142 \pm \cdot 0242$	$P_3/P_1\,(L)$,	$O_8/O_7\,(R)$	—
$P_3/P_1\,(L)$,	$O_8/O_9\,(R)$	$\cdot 0800 \pm \cdot 0245$	$P_3/P_1\,(R)$,	$O_8/O_7\,(L)$	—
$P_3/P_1\,(R)$,	$T_2/T_1\,(L)$	$+\cdot 0781 \pm \cdot 0243$	$P_3/P_1\,(L)$,	$O_8/O_7\,(R)$	—
$P_3/P_1\,(L)$,	$T_2/T_1\,(R)$	$+\cdot 0993 \pm \cdot 0245$	$T_2/T_3\,(R)$,	$O_8/O_7\,(L)$	—
$T_2/T_1\,(R)$,	$O_8/O_9\,(L)$	$+\cdot 0823 \pm \cdot 0233$	$T_2/T_3\,(L)$,	$O_8/O_7\,(R)$	—
$T_2/T_3\,(L)$,	$O_8/O_9\,(R)$	$+\cdot 0757 \pm \cdot 0233$	$T_6/T_5\,(R)$,	$O_8/O_7\,(L)$	—
$F_2/F_1\,(R)$,	$S_5/S_4\,(L)$	$\cdot 0708 \pm \cdot 0240$	$T_6/T_5\,(L)$,	$O_8/O_7\,(R)$	—
$F_2/F_1\,(L)$,	$S_5/S_4\,(R)$	$-\cdot 0695 \pm \cdot 0243$	$T_6/T_5\,(R)$,	$O_8/O_9\,(L)$	—
$S_5/S_4\,(R)$,	$Mx_2/Mx_1\,(L)$	$-\cdot 0309 \pm \cdot 0301$	$T_6/T_5\,(L)$,	$O_8/O_9\,(R)$	—
$S_5/S_4\,(L)$,	$Mx_2/Mx_1\,(R)$		$T_2/T_3\,(R)$,	$S_6/S_7\,(L)$	—
$F_2/F_1\,(R)$,	$P_2/P_1\,(L)$	$+\cdot 0549 \pm \cdot 0242$	$T_2/T_3\,(L)$,	$S_6/S_7\,(R)$	—
$F_2/F_1\,(L)$,	$P_2/P_1\,(R)$	—	$T_6/T_7\,(R)$,	$S_6/S_7\,(L)$	
$S_6/S_7\,(R)$,	$Mx_3/Mx_1\,(L)$	—	$T_6/T_7\,(L)$,	$S_6/S_7\,(R)$	
$S_6/S_7\,(L)$,	$Mx_3/Mx_1\,(R)$	$\cdot 0127 \pm \cdot 0298$	$T_6/T_5\,(R)$,	$S_5/S_4\,(L)$	
$S_6/S_4\,(R)$,	$Ml_2/Ml_1\,(L)$	$+\cdot 0211 \pm \cdot 0260$	$T_6/T_5\,(L)$,	$S_5/S_4\,(R)$	
$S_6/S_4\,(L)$,	$Ml_2/Ml_1\,(R)$	$+\cdot 0125 \pm \cdot 0258$	$S_6/S_7\,(R)$,	$O_8/O_9\,(L)$	
$S_6/S_4\,(R)$,	$O_8/O_9\,(L)$	$\cdot 0276 \pm \cdot 0248$	$S_6/S_7\,(L)$,	$O_8/O_9\,(R)$	
$S_6/S_4\,(L)$,	$O_8/O_9\,(R)$	$-\cdot 0075 \pm \cdot 0215$	$S_6/S_7\,(R)$,	$O_8/O_7\,(L)$	
$F_2/F_1\,(R)$,	$P_3/P_4\,(L)$	$+\cdot 0267 \pm \cdot 0242$	$S_6/S_7\,(L)$,	$O_8/O_7\,(R)$	
$F_2/F_1\,(L)$,	$P_3/P_4\,(R)$		$S_5/S_4\,(R)$,	$O_8/O_7\,(L)$	
$F_3/F_1\,(R)$,	$Ml_2/Ml_1\,(L)$	$+\cdot 0205 \pm \cdot 0243$	$S_5/S_4\,(L)$,	$O_8/O_7\,(R)$	
$F_3/F_1\,(L)$,	$Ml_2/Ml_1\,(R)$	$+\cdot 0235 \pm \cdot 0241$	$S_5/S_4\,(R)$,	$Mx_3/Mx_1\,(L)$	
$F_3/F_1\,(R)$,	$S_6/S_7\,(L)$	—	$S_5/S_4\,(L)$,	$Mx_3/Mx_1\,(R)$	
$F_3/F_1\,(L)$,	$S_6/S_7\,(R)$	—	$S_6/S_7\,(R)$,	$Ml_2/Ml_1\,(L)$	—
$P_2/P_1\,(R)$,	$T_6/T_7\,(L)$	—	$S_6/S_7\,(L)$,	$Ml_2/Ml_1\,(R)$	—
$P_2/P_1\,(L)$,	$T_6/T_7\,(R)$	—	$S_6/S_7\,(R)$,	$Mx_2/Mx_1\,(L)$	
$P_3/P_1\,(R)$,	$T_6/T_5\,(L)$	—	$S_6/S_7\,(L)$,	$Mx_2/Mx_1\,(R)$	
$P_3/P_1\,(L)$,	$T_6/T_5\,(R)$	—	$Mx_3/Mx_1\,(R)$,	$Ml_2/Ml_1\,(L)$	
$P_3/P_1\,(R)$,	$T_2/T_3\,(L)$	—	$Mx_3/Mx_1\,(L)$,	$Ml_2/Ml_1\,(R)$	

B. The correlation of an index on one bone with the index of a bone on the opposite side non-contiguous with the homologue of the first bone. The values of such correlations are given in Table XIV. In the same table are given the slightly contiguous values. In the first case there are 14 out of 40 correlations, and in the second case only 6 out of 16. If these be pooled there will be 20 out of 56. For the purpose of comparison we give some results in Table XV below.

These results are noteworthy, they indicate that for the purely numerical values, which are really the more important ones (see p. 443), the cross-correlations do not fall short of the direct correlations either for contiguous or non-contiguous pairs;

* As in previous tables the correlations which have not been worked out are indicated. Had time and energy sufficed for working out the whole series, we do not believe that the results would have differed much from those of our practically random sample.

TABLE XIV.

Cross-Correlations. B. Index of a Bone with the Index of a Second Bone on the other side non-contiguous or only slightly contiguous with the homologue of the First.

Index Pair		Correlation and p.e.	Index Pair		Correlation and p.e.
$O_8/O_9\ (R)$,	$Mx_2/Mx_1\ (L)$	$+\cdot 0910 \pm \cdot 0285$	$T_6/T_5\ (L)$,	$Mx_2/Mx_1\ (R)$	—
$O_8/O_9\ (L)$,	$Mx_2/Mx_1\ (R)$	$+\cdot 0900 \pm \cdot 0283$	$T_6/T_5\ (R)$,	$Mx_3/Mx_1\ (L)$	—
$P_3/P_4\ (R)$,	$Mx_2/Mx_1\ (L)$	$+\cdot 0669 \pm \cdot 0298$	$T_6/T_5\ (L)$,	$Mx_3/Mx_1\ (R)$	—
$P_3/P_4\ (L)$,	$Mx_2/Mx_1\ (R)$	$+\cdot 0801 \pm \cdot 0308$	$O_8/O_9\ (R)$,	$Mx_3/Mx_1\ (L)$	—
$P_3/P_4\ (R)$,	$Ml_2/Ml_1\ (L)$	$-\cdot 0246 \pm \cdot 0258$	$O_8/O_9\ (L)$,	$Mx_3/Mx_1\ (R)$	—
$P_3/P_4\ (L)$,	$Ml_2/Ml_1\ (R)$	$-\cdot 0635 \pm \cdot 0261$	$O_8/O_7\ (R)$,	$Mx_2/Mx_1\ (L)$	—
$T'_2/T'_3\ (R)$,	$Mx_2/Mx_1\ (L)$	$+\cdot 0593 \pm \cdot 0284$	$O_8/O_7\ (L)$,	$Mx_2/Mx_1\ (R)$	—
$T'_2/T'_3\ (L)$,	$Mx_2/Mx_1\ (R)$	$+\cdot 0544 \pm \cdot 0283$	$O_8/O_7\ (R)$,	$Mx_3/Mx_1\ (L)$	—
$F'_2/F'_1\ (R)$,	$T_2/T_3\ (L)$	$-\cdot 0515 \pm \cdot 0229$	$O_8/O_7\ (L)$,	$Mx_3/Mx_1\ (R)$	—
$F'_2/F'_1\ (L)$,	$T'_2/T_3\ (R)$	$-\cdot 0447 \pm \cdot 0231$	$O_8/O_7\ (R)$,	$Ml_2/Ml_1\ (L)$	—
$F'_2/F'_1\ (R)$,	$O_8/O_0\ (L)$	$+\cdot 0157 \pm \cdot 0231$	$O_8/O_7\ (L)$,	$Ml_2/Ml_1\ (R)$	—
$F'_2/F'_1\ (L)$,	$O_8/O_0\ (R)$	$+\cdot 0415 \pm \cdot 0232$			
$O_8/O_0\ (R)$,	$Ml_2/Ml_1\ (L)$	$-\cdot 0076 \pm$ 0248		Slightly	contiguous
$O_8/O_9\ (L)$,	$Ml_2/Ml_1\ (R)$	$-\cdot 0382 \pm \cdot 0248$	$P_3/P_4\ (R)$,	$S_5/S_4\ (L)$	$+\cdot 0722 \pm \cdot 0250$
$F'_2/F'_1\ (R)$,	$T_6/T_5\ (L)$	—	$P_3/P_4\ (L)$,	$S_5/S_4\ (R)$	$+\cdot 0631 \pm \cdot 0256$
$F'_2/F'_1\ (L)$,	$T_6/T_5\ (R)$	—	$T'_2/T_3\ (R)$,	$Ml_2/Ml_1\ (L)$	$+\cdot 0063 \pm \cdot 0243$
$F'_2/F'_1\ (R)$,	$O_8/O_7\ (L)$	—	$T'_2/T_3\ (L)$,	$Ml_2/Ml_1\ (R)$	$+\cdot 0328 \pm \cdot 0246$
$F'_2/F'_1\ (L)$,	$O_8/O_7\ (R)$	—	$F_2/F_1\ (R)$,	$Mx_2/Mx_1\ (L)$	$+\cdot 0196 \pm \cdot 0282$
$P_2/P_1\ (R)$,	$Mx_2/Mx_1\ (L)$	—	$F_2/F_1\ (L)$,	$Mx_2/Mx_1\ (R)$	$+\cdot 0323 \pm \cdot 0287$
$P_2/P_1\ (L)$,	$Mx_2/Mx_1\ (R)$	—	$F_2/F_1\ (R)$,	$Mx_3/Mx_1\ (L)$	—
$P_2/P_1\ (R)$,	$Mx_3/Mx_1\ (L)$	—	$F_2/F_1\ (L)$,	$Mx_3/Mx_1\ (R)$	—
$P_2/P_1\ (L)$,	$Mx_3/Mx_1\ (R)$	—	$P_2/P_1\ (R)$,	$S_6/S_7\ (L)$	—
$P_2/P_1\ (R)$,	$Ml_2/Ml_1\ (L)$	—	$P_2/P_1\ (L)$,	$S_6/S_7\ (R)$	—
$P_2/P_1\ (L)$,	$Ml_2/Ml_1\ (R)$	—	$P_3/P_4\ (R)$,	$S_6/S_7\ (L)$	—
$P_3/P_4\ (R)$,	$Mx_3/Mx_1\ (L)$	—	$P_3/P_4\ (L)$,	$S_6/S_7\ (R)$	—
$P_3/P_4\ (L)$,	$Mx_3/Mx_1\ (R)$	—	$P_2/P_1\ (R)$,	$S_5/S_4\ (L)$	—
$T'_2/T'_3\ (R)$,	$Mx_3/Mx_1\ (L)$	—	$P_2/P_1\ (L)$,	$S_5/S_4\ (R)$	—
$T'_2/T'_3\ (L)$,	$Mx_3/Mx_1\ (R)$	—	$T_6/T_5\ (R)$,	$Ml_2/Ml_1\ (L)$	—
$T'_6/T'_5\ (R)$,	$Mx_2/Mx_1\ (L)$	—	$T_6/T_5\ (L)$,	$Ml_2/Ml_1\ (R)$	—

TABLE XV. *Means of Direct and Cross-Correlations of Indices.*

	Direct Correlations Means		Cross-Correlations Means	
	Arithmetical	Algebraical	Arithmetical	Algebraical
Contiguous	·0656	$-\cdot 0164$	·0656	$+\cdot 0035$
Non-contiguous	·0397	$+\cdot 0197$	·0478*	$+\cdot 0248$†

the only real difference occurs in the algebraical values of the contiguous pairs; even here the average algebraic correlation of the cross pairs is higher than for the direct pairs. The correlations are all small but many of them must be considered as

* This is ·0521 without the "slightly contiguous."

† $+\cdot 0192$ without the "slightly contiguous."

significant. The result is a striking instance of the power of homology, the correlation of the index of a bone with that of a second bone on its own side is on the average not more intense than it is with the homologue of the second bone on the opposite side. A comparison of the individual correlations in Table XIII with the corresponding correlations in Table XI shows to what extent this is true for individual pairs of contiguous bones, and a like comparison of Tables IX and XIV indicates the character of the relationship for non-contiguous individual bones. The reader will find some discrepancies where the difference is very considerable compared to the probable error of the difference. Thus compare

$$F_2/F_1\,(R),\quad P_2/P_1\,(R) = -\cdot 0992 \;\text{ with }\; F_2/F_1\,(R),\quad P_2/P_1\,(L) = +\cdot 0549.$$

Further compare

$$T_2/T_3\,(R),\quad O_8/O_9\,(R) = +\cdot 0654 \;\text{ and }\; T_2/T_3\,(L),\quad O_8/O_9\,(L) = +\cdot 0604$$

with $T_2/T_3\,(L),\quad O_8/O_9\,(R) = +\cdot 0757$ and $T_2/T_3\,(R),\quad O_8/O_9\,(L) = +\cdot 0823,$

both cross-correlations being greater than both direct correlations. Again we may compare

$$F_2/F_1\,(R),\quad P_3/P_4\,(R) = +\cdot 0689 \;\text{ and }\; F_2/F_1\,(L),\quad P_3/P_4\,(L) = +\cdot 0032$$

with $F_2/F_1\,(R),\quad P_3/P_4\,(L) = +\cdot 0267$;

and once more

$$F_2/F_1\,(R),\quad Ml_2/Ml_1\,(R) = +\cdot 0368 \;\text{ and }\; F_2/F_1\,(L),\quad Ml_2/Ml_1\,(L) = +\cdot 0146$$

with $F_2/F_1\,(R),\quad Ml_2/Ml_1\,(L) = +\cdot 0205$ and $F_2/F_1\,(L),\quad Ml_2/Ml_1\,(R) = +\cdot 0235.$

Many of these results can be attributed to the random sampling, for the correlations are so small as to be of the same order as their probable errors. While it is possible that if the correlations had been worked out for the same pairs of bones in each case, the bones on the same side contiguous and non-contiguous might have shown on the average higher correlations than with the corresponding bones on the opposite side, the differences if significant are so small that we cannot assert that they exist when a random selection of indices has been worked out for each category and we base our conclusions on these. We can, however, assert that the factor of homology is so influential that with the small correlations which exist between the indices of non-homologous bones, we cannot prove on our data that the correlation of the index of a bone A on one side with the index of a bone B, contiguous or non-contiguous, on the same side is definitely greater than the correlation of the index of A with the index of the bone B' on the opposite side, where B' is the homologue of B.

(11) *Correlation of Indices with Laterality and without Laterality.*

We define an index with laterality as one based on the ratio of two lengths both taken on the same bone, that bone being situated on the right or left side of the skull. An index without laterality is based on the ratio of two lengths neither of which is peculiar to one side of the skull, but both are measured on the same bone. A list of these non-bilateral lengths is given on p. 427. We have confined our

attention to five such non-lateral indices, namely two on the frontal, F_3/F_4, F_5/F_4, and three on the occipital, O_4/O_1, O_6/O_3 and $O_5/(O_2+O_3)$. We will deal first with the non-contiguous, or only slightly contiguous values, our Group (x). The correlations are given in Table XVI. The mean value independent of sign for the non-contiguous correlations is ·0672, and for the slightly contiguous group ·0467 we have no hesitation in pooling the two series, for we should anticipate that the latter would have been the higher. The two series combined give a mean numerical value of ·0604. This is somewhat greater than the mean numerical value of the correlations for the contiguous bones in Table XVII (p. 452), i.e. ·0538. No clearer confirmation could be found of the statement that contiguity has little to do with the mean

TABLE XVI.

Correlation of Indices on non-contiguous Bones, one Index of the pair having no Laterality.

Index Pair	Correlation and p.e.	Index Pair	Correlation and p.e.
F_5/F_4, $T_2/T_3\,(R)$	$+\cdot0893\pm\cdot0229$	F_5/F_4, $O_8/O_9\,(L)$	—
F_5/F_4, $T_2/T_3\,(L)$	—	F_3/F_4, $O_8/O_9\,(R)$	$+\cdot0591\pm\cdot0231$
$O_5/(O_2+O_3)$, $Mx_2/Mx_1\,(R)$	$+\cdot0754\pm\cdot0285$	F_3/F_4, $O_8/O_9\,(L)$	—
$O_5/(O_2+O_3)$, $Mx_2/Mx_1\,(L)$	—	Slightly contiguous	
F_5/F_4, $T_6/T_5\,(R)$	$+\cdot0644\pm\cdot0231$	F_3/F_4, $Mx_2/Mx_1\,(R)$	$+\cdot0571\pm\cdot0281$
F_5/F_4, $T_6/T_5\,(L)$	—	F_3/F_4, $Mx_2/Mx_1\,(L)$	—
F_3/F_4, $T_6/T_5\,(R)$	$+\cdot0635\pm\cdot0231$	F_3/F_4, $Mx_3/Mx_1\,(R)$	$-\cdot0526\pm\cdot0305$
F_3/F_4, $T_6/T_5\,(L)$	—	F_3/F_4, $Mx_3/Mx_1\,(L)$	—
F_3/F_4, $O_8/O_7\,(R)$	$+\cdot0629\pm\cdot0230$	F_5/F_4, $Mx_3/Mx_1\,(R)$	$-\cdot0401\pm\cdot0306$
F_3/F_4, $O_8/O_7\,(L)$	—	F_5/F_4, $Mx_3/Mx_1\,(L)$	—
O_4/O_1, $Mx_3/Mx_1\,(R)$	$-\cdot0619\pm\cdot0311$	F_5/F_4, $Mx_2/Mx_1\,(R)$	$+\cdot0368\pm\cdot0281$
O_4/O_1, $Mx_3/Mx_1\,(L)$	—	F_5/F_4, $Mx_2/Mx_1\,(L)$	—
F_5/F_4, $O_8/O_9\,(R)$	$+\cdot0612\pm\cdot0231$		

values found for the series in Tables XIII and XIV. We have to admit that our correlations are far from an exhaustive list of possible correlations, which might be worked out, but those chosen have not been in any way selected to give high or low values of the correlations. If there be really any sensible difference between the non-contiguous pairs in Table XVI, and the contiguous pairs in Table XVII, we can hardly discover it on the values worked out thus far.

(12) *Correlation of Indices on Contiguous Bones, one of the Indices correlated having no Laterality.*

Of the 28 indices falling into Table XVII, no less than 19 are negative. In this the table differs markedly from Table XVI where out of 12 correlations 9 are positive. The negative correlations have a mean of $-\cdot0541$ and the positive a mean of $+\cdot0533$. When we consider only the numerical value the mean correlation is, as we have stated in the last section, ·0538. It is difficult however to lay stress on the near equality of the mean positive and negative correlations for we can change the sign

by inverting the ratio. Thus if we had taken for our index 100 O_1/O_4, which would have made the mean of that index under, instead of over, 100, we should have found $P_3/P_4(R)$ increasing with O_1/O_4, instead of decreasing as it does with O_4/O_1; similarly if we had taken our index $100(O_2+O_3)/O_5$, which would again have given a mean value under instead of over 100, we should have got the opposite sign for the index correlation $S_5/S_4(R)$, $(O_2+O_3)/O_5$ to that of $S_5/S_4(R)$ and $O_5/(O_2+O_3)$. In other words the signs of index correlations, as we have before stated, are at our choice, depending on which constituent of the ratio we assign to the numerator, which to the denominator.

TABLE XVII.

Correlation of Indices on Contiguous Bones, one of the Indices correlated having no Laterality.

Index Pair		Correlation and p.e.	Index Pair		Correlation and p.e.
$P_3/P_4(R)$,	O_4/O_1	− ·1296 ± ·0241	F_5/F_4,	$Mt_2/Mt_1(R)$	− ·0607 ± ·0243
$P_3/P_4(L)$,	O_4/O_1		F_5/F_4,	$Mt_2/Mt_1(L)$	
$S_5/S_4(R)$,	$O_5/(O_2+O_3)$	− ·1154 ± ·0246	F_3/F_4,	$S_6/S_7(R)$	+ ·0439 ± ·0237
$S_5/S_4(L)$,	$O_5/(O_2+O_3)$		F_3/F_4,	$S_6/S_7(L)$	
F_5/F_4,	$P_3/P_4(R)$	+ ·1103 ± ·0239	$S_6/S_7(R)$,	O_4/O_1	·0381 ± ·0241
F_5/F_4,	$P_3/P_4(L)$		$S_6/S_7(L)$,	O_4/O_1	
$S_5/S_4(R)$,	O_6/O_3	− ·1053 ± ·0244	$T_6/T_5(R)$,	O_6/O_3	− ·0252 ± ·0235
$S_5/S_4(L)$,	O_6/O_3		$T_6/T_5(L)$,	O_6/O_3	
F_3/F_4,	$S_5/S_4(R)$	− ·0807 ± ·0243	$T_2/T_3(R)$,	O_6/O_3	·0247 ± ·0234
F_3/F_4,	$S_5/S_4(L)$		$T_2/T_3(L)$,	O_6/O_3	
$S_5/S_4(R)$,	O_4/O_1	− ·0780 ± ·0247	F_3/F_4,	$Mt_2/Mt_1(R)$	− ·0230 ± ·0241
$S_5/S_4(L)$,	O_4/O_1		F_3/F_4,	$Mt_2/Mt_1(L)$	
F_5/F_4,	$S_5/S_4(R)$	− ·0764 ± ·0243	$T_2/T_3(R)$,	O_4/O_1	+ ·0231 ± ·0235
F_5/F_4,	$S_5/S_4(L)$		$T_2/T_3(L)$,	O_4/O_1	
F_5/F_4,	$P_2/P_1(R)$	− ·0733 ± ·0242	$P_2/P_1(R)$,	O_6/O_3	+ ·0227 ± ·0245
F_5/F_4,	$P_2/P_1(L)$		$P_2/P_1(L)$,	O_6/O_3	
F_3/F_4,	$P_2/P_1(R)$	− ·0726 ± ·0240	$T_6/T_5(R)$,	$O_5/(O_2+O_3)$	− ·0220 ± ·0236
F_3/F_4,	$P_2/P_1(L)$		$T_6/T_5(L)$,	$O_5/(O_2+O_3)$	
$P_3/P_4(R)$,	$O_5/(O_2+O_3)$	+ ·0707 ± ·0246	$T_2/T_3(R)$,	$O_5/(O_2+O_3)$	− ·0121 ± ·0236
$P_3/P_4(L)$,	$O_5/(O_2+O_3)$		$T_2/T_3(L)$,	$O_5/(O_2+O_3)$	
F_3/F_4,	$P_3/P_4(R)$	+ ·0704 ± ·0241	F_5/F_4,	$S_6/S_7(R)$	+ ·0113 ± ·0237
F_3/F_4,	$P_3/P_4(L)$		F_5/F_4,	$S_6/S_7(L)$	
$P_2/P_1(R)$,	O_4/O_1	− ·0663 ± ·0244	$P_2/P_1(R)$,	$O_5/(O_2+O_3)$	− ·0089 ± ·0246
$P_2/P_1(L)$,	O_4/O_1		$P_2/P_1(L)$,	$O_5/(O_2+O_3)$	
$P_3/P_4(R)$,	O_6/O_3	+ ·0662 ± ·0244	$S_6/S_7(R)$,	$O_5/(O_2+O_3)$	·0057 ± ·0239
$P_3/P_4(L)$,	O_6/O_3		$S_6/S_7(L)$,	$O_5/(O_2+O_3)$	
$T_6/T_5(R)$,	O_4/O_1	+ ·0609 ± ·0235	$S_6/S_7(R)$,	O_6/O_3	·0038 ± ·0238
$T_6/T_5(L)$,	O_4/O_1		$S_6/S_7(L)$,	O_6/O_3	

As it does not seem possible to lay any stress on contiguity as differentiating the correlations given in Tables XVI and XVII, we will pool the 40 correlations, which then provide a mean numerical correlation of ·0558, and simply ask whether these 40 correlations of lateral with non-lateral indices may be taken as a random sample from parent populations with a correlation coefficient of $r = ·0558$. This inquiry, if very laborious, will enable us at any rate to say whether it is reasonable to

represent by a single value of r the influence of non-lateral indices on lateral indices. The results are as follows:

Probability Integrals: From left to right / From right to left

$$\log \lambda_n = -15{\cdot}894{,}7938, \qquad \log \lambda'_n = -16{\cdot}017{,}2577,$$

$$\sqrt{n}\log_{10} e = 2{\cdot}746{,}71958, \qquad \sqrt{n}\log_{10} e = 2{\cdot}746{,}71958,$$

$$u = -\frac{\log_{10}\lambda_n}{\sqrt{n}\log_{10} e} = 5{\cdot}786{,}828, \qquad u' = -\frac{\log_{10}\lambda'_n}{\sqrt{n}\log_{10} e} = 5{\cdot}831{,}4135,$$

$$P_{\lambda_n} = I(n-1,\, u) = I(39,\, 5{\cdot}786{,}828), \qquad P_{\lambda'_n} = I(n-1,\, u') = I(39,\, 5{\cdot}831{,}4135).$$

Hence from the *Incomplete* Γ-*Function Tables*:

$$P_{\lambda_n} = {\cdot}3083, \qquad P_{\lambda'_n} = {\cdot}3251,$$

$$Q_{\lambda_n} = {\cdot}6917, \qquad Q_{\lambda'_n} = {\cdot}6749.$$

The latter is somewhat the more stringent. It indicates that in 100 random samples from the population of correlation coefficients with mean ·0558, 67 would give a more improbable result than our system of correlations of indices of lateral and non-lateral bones. We do not assert that our system *is* such a selection, but we do assert that it is a *highly* reasonable description of what has been observed, *that is to say a* successful graduation. In other words the relation of lateral and non-lateral indices can be expressed by the single number ·0558.

(18) *Correlation of two Indices of the same Bone.*

This is our Group (vi) and the correlations falling under this heading are given in Table XVIII below:

TABLE XVIII. *Correlation of two Indices of the same Bone.*

Index Pair	Correlation and p.e.	Index Pair	Correlation and p.e.
(a_1) O_4/O_1, $O_8/O_7(R)$	$+{\cdot}6672 \pm {\cdot}0129$	(i_1) $S_8/S_7(R)$, $S_5/S_4(R)$	$+{\cdot}1633 \pm {\cdot}0247$
(a_2) O_4/O_1, $O_8/O_7(L)$	$+{\cdot}6042 \pm {\cdot}0147$	(j_1) $P_3/P_4(R)$, $P_2/P_1(R)$	$-{\cdot}1133 \pm {\cdot}0239$
(b) F_5/F_4, F_3/F_4	$+{\cdot}5006 \pm {\cdot}0168$	(k) $O_5/(O_2+O_3)$, O_4/O_1	$-{\cdot}1109 \pm {\cdot}0231$
(c) $O_5/(O_2+O_3)$, O_6/O_3	$+{\cdot}4764 \pm {\cdot}0180$	(l) O_4/O_1, O_6/O_3	$-{\cdot}0894 \pm {\cdot}0233$
(d_1) O_4/O_1, $O_8/O_9(R)$	$+{\cdot}3744 \pm {\cdot}0201$	(m_1) $O_8/O_9(R)$, O_6/O_3	$-{\cdot}0865 \pm {\cdot}0232$
(d_2) O_4/O_1, $O_8/O_9(L)$	$+{\cdot}3130 \pm {\cdot}0211$	(j_2) $P_3/P_4(L)$, $P_2/P_1(L)$	$-{\cdot}0753 \pm {\cdot}0241$
(e_1) $O_8/O_9(R)$, $O_8/O_7(R)$	$+{\cdot}3094 \pm {\cdot}0210$	(n_1) $T_6/T_5(R)$, $T_2/T_3(R)$	$+{\cdot}0679 \pm {\cdot}0233$
(f_1) $F_2/F_1(R)$, F_3/F_4	$+{\cdot}2677 \pm {\cdot}0210$	(o_1) $O_5/(O_2+O_3)$, $O_8/O_9(R)$	$-{\cdot}0667 \pm {\cdot}0233$
(e_2) $O_8/O_9(L)$, $O_8/O_7(L)$	$+{\cdot}2573 \pm {\cdot}0216$	(m_2) $O_8/O_9(L)$, O_6/O_3	$-{\cdot}0584 \pm {\cdot}0238$
(f_2) $F_2/F_1(L)$, F_3/F_4	$+{\cdot}2566 \pm {\cdot}0212$	(p_2) O_6/O_3, $O_8/O_7(L)$	$-{\cdot}0549 \pm {\cdot}0232$
(g_2) $Mx_2/Mx_1(L)$, $Mx_3/Mx_1(L)$	$+{\cdot}2284 \pm {\cdot}0303$	(n_2) $T_6/T_5(L)$, $T_2/T_3(L)$	$+{\cdot}0460 \pm {\cdot}0233$
(g_1) $Mx_2/Mx_1(R)$, $Mx_3/Mx_1(R)$	$+{\cdot}2103 \pm {\cdot}0311$	(q_2) $O_5/(O_2+O_3)$, $O_8/O_7(L)$	$+{\cdot}0416 \pm {\cdot}0233$
(h_2) F_5/F_4, $F_2/F_1(L)$	$+{\cdot}2062 \pm {\cdot}0218$	(q_1) $O_5/(O_2+O_3)$, $O_8/O_7(R)$	$+{\cdot}0327 \pm {\cdot}0233$
(i_2) $S_8/S_7(L)$, $S_5/S_4(L)$	$+{\cdot}1909 \pm {\cdot}0242$	(p_1) O_6/O_3, $O_8/O_7(R)$	$-{\cdot}0135 \pm {\cdot}0234$
(h_1) F_5/F_4, $F_2/F_1(R)$	$+{\cdot}1786 \pm {\cdot}0220$	(o_2) $O_5/(O_2+O_3)$, $O_8/O_9(L)$	$-{\cdot}0032 \pm {\cdot}0234$

The small letters in the corners of Columns 1 and 3 indicate pairs where they exist, the subscript 1 for right and 2 for left.

It will be seen at once that indices on the same bone present a wide range of correlation values. But the source of these differences is fairly easy of analysis. Thus:

(*a*) $O_4 = O_8(R) + O_8(L)$ and there will be a high positive correlation between $O_8(R)$ or $O_8(L)$ and O_4 if there be little between the other occipital measurements involved.

(*b*) F_1 is the denominator of both indices, hence if there be little correlation between the other frontal lengths (*b*) should show a considerable positive correlation.

(*c*) O_3 is common to both index denominators and this should involve positive correlation in the indices.

(*d*) The same condition holds as in (*a*) and involves positive correlation.

(*e*) O_8 is the numerator of both indices and positive correlation follows.

(*f*) and (*h*) F_4 is nearly parallel to F_1, and so to speak covers it, so that F_1 can be almost looked upon as a part of F_4, and positive index correlation may be anticipated.

(*g*) Mx_1 is the common denominator of both fractions, and we may anticipate positive correlation.

(*i*) S_4 overlies S_7, or the latter may be looked upon as part of the former; positive correlation will result.

(*j*) P_3 and P_1 probably possess considerable correlation, so that a negative correlation would arise between the (*j*) indices.

(*k*) and (*l*) O_1 is likely to have a negative correlation with both O_2 and O_3, thus productive of a negative correlation in the (*k*) and (*l*) indices.

(*m*) to (*q*) In the remaining indices one does not see any special reason for their magnitude or sign.

From the slender material we have, we may reason that this system of correlations of indices of the same bone is more or less capable of interpretation, if a high correlation arises when the two indices have a common factor, a somewhat lower correlation when a factor of one overlaps a factor of the other, and little or no correlation when neither of these cases can be seen to hold. Until the correlations of absolute lengths on the same bone have been worked out, it is not possible to ascertain the limits of the truth of this statement. But we may consider whether the 10 correlations in the m—q group could be reasonably considered as a random selection from a parent population of zero correlation. The values of the correlations in these groups are:

$-\cdot 0865 \pm \cdot 0232$,	$+\cdot 0460 \pm \cdot 0233$,
$+\cdot 0679 \pm \cdot 0233$,	$+\cdot 0416 \pm \cdot 0233$,
$-\cdot 0667 \pm \cdot 0233$,	$+\cdot 0327 \pm \cdot 0233$,
$-\cdot 0584 \pm \cdot 0233$,	$-\cdot 0135 \pm \cdot 0234$,
$-\cdot 0549 \pm \cdot 0232$,	$-\cdot 0032 \pm \cdot 0234$.

Mean of the six negative correlations $= -\cdot 0472$,

Mean of the four positive correlations $= \quad \cdot 0471$.

The probable error can be taken as ·0233 for the system, or the standard error as ·034545. *Working out the probability integrals and their logarithms, we have:*

$$\log_{10}\lambda_n = -7·358,4735, \qquad \log_{10}\lambda'_n = -4·868,7097,$$

$$\sqrt{n}\,\log_{10} e = \sqrt{10} \times ·434,2945 = 1·373,3508,$$

$$n = 5·358,0085, \qquad n' = 3·545,1086,$$

$$P_{\lambda_n} = I(9,\ 5·358,0085) = ·9731, \qquad P_{\lambda'_n} = I(9,\ 3·545,1086) = ·6818,$$

$$Q_{\lambda_n} = ·0269, \qquad Q_{\lambda'_n} = ·3182$$

Notwithstanding the value of $Q_{\lambda'_n}$, *which would warrant the "reasonableness"* of the hypothesis, we feel bound to take the more stringent Q_{λ_n} for our guide, or more probable distributions of the correlation values would occur in 97 % of random samples than in the sample actually observed if they were taken from a population of zero correlation.

Now there are those who accept a < ·01 probability as justifying the rejection of an hypothesis; others demand a < ·05 probability. We hold that in each case the actual probability should be determined, and then judgment formed in relation to the material on which it is based. In our case, we do not think that $Q_{\lambda_n} = ·03$ about, justifies us in accepting the hypothesis that, if we exclude cases of common factors in the indices and "overlapping," there is little or no correlation of indices on the same bone. We prefer to wait until the correlations of absolute lengths on the same bone have been measured, and we can trace the contribution of each absolute length to the make-up of the index correlations.

What we feel confidence in asserting is that common factors, and, to a less extent, "overlaps," do increase the numerical magnitude of the correlation of indices of the same bone.

(14) *Influence of Homology on Indices taken on like Bones.*

This section will deal with our Group (viii). Unfortunately, the number of correlations falling under this heading and already worked out are few. They are given in Table XIX below.

In this table, beside the correlations of heterologous indices on homologous bones, we have the correlations of heterologous indices on the same bone, and further the homologous indices of homologous bones. It would be of great value were we able to deduce approximate values for the correlations in Column 2 of Table XIX from the data given in Columns 4, 6 and 8, that is to say if we *could deduce the cross-correlations from our knowledge of direct correlations and* the correlations for homology.

Formulae, having at least some theoretical basis, have been suggested with this end in view. Let A and B be two characters, and the subscripts R and L refer to the right and left sides of the skull, then we will test the following relations:

$$\left.\begin{aligned} r_{A_R B_L} &= \tfrac{1}{2}\left(r_{A_R B_R} \times r_{B_R B_L} + r_{B_L A_L} \times r_{A_L A_R}\right) \\ r_{A_L B_R} &= \tfrac{1}{2}\left(r_{A_L B_L} \times r_{B_L B_R} + r_{A_R B_R} \times r_{A_R A_L}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(\epsilon).$$

TABLE XIX.

Correlation of Heterologous Indices on a pair of Homologous Bones.

Heterologous Indices on *Homologous Bones*			Heterologous Indices on same Bone			Homologous Bones, *same Index*			Homologous Bones, same Index		
Index Pair		Correlation and p.e.	Index Pair		Correlation and p.e.	Index Pair		Correlation and p.e.	Index Pair		Correlation and p.e.
$S_6/S_7(R)$,	$S_5/S_4(L)$	$+\cdot1787\pm\cdot0244$	$S_6/S_7(R)$,	$S_5/S_4(R)$	$+\cdot1633\pm\cdot0247$	$S_6/S_7(R)$,	$S_6/S_7(L)$	$+\cdot9045\pm\cdot0043$	$S_5/S_4(R)$,	$S_5/S_4(L)$	$+\cdot8822\pm\cdot0056$
$S_6/S_7(L)$,	$S_5/S_4(R)$	$+\cdot1863\pm\cdot0245$	$S_6/S_7(L)$,	$S_5/S_4(L)$	$+\cdot1909\pm\cdot0242$	—		—	—		—
$T_6/T_5(R)$,	$T_2/T_3(L)$	$+\cdot0322\pm\cdot0234$	$T_6/T_5(R)$,	$T_2/T_3(R)$	$+\cdot0679\pm\cdot0233$	$T_6/T_5(R)$,	$T_6/T_5(L)$	$+\cdot6185\pm\cdot0145$	$T_2/T_3(R)$,	$T_2/T_3(L)$	$+\cdot7934\pm\cdot0086$
$T_6/T_5(L)$,	$T_2/T_3(R)$	$+\cdot0415\pm\cdot0234$	$T_6/T_5(L)$,	$T_2/T_3(L)$	$+\cdot0460\pm\cdot0233$	—		—	—		—
$P_3/P_4(R)$,	$P_2/P_1(L)$	$-\cdot1169\pm\cdot0245$	$P_3/P_4(R)$,	$P_2/P_1(R)$	$-\cdot1133\pm\cdot0239$	$P_3/P_4(R)$,	$P_3/P_4(L)$	$+\cdot4057\pm\cdot0207$	$P_2/P_1(R)$,	$P_2/P_1(L)$	$+\cdot7892\pm\cdot0093$
$P_3/P_4(L)$,	$P_2/P_1(R)$	$-\cdot1756\pm\cdot0240$	$P_3/P_4(L)$,	$P_2/P_1(L)$	$-\cdot0753\pm\cdot0241$	—		—	—		—
$O_8/O_9(R)$,	$O_6/O_7(L)$	$+\cdot0700\pm\cdot0231$	$O_8/O_9(R)$,	$O_6/O_7(R)$	$+\cdot3094\pm\cdot0210$	$O_8/O_9(R)$,	$O_8/O_9(L)$	$+\cdot4769\pm\cdot0179$	$O_6/O_7(R)$,	$O_6/O_7(L)$	$+\cdot6044\pm\cdot0146$
$O_8/O_9(L)$,	$O_6/O_7(R)$	$+\cdot0584\pm\cdot0231$	$O_8/O_9(L)$,	$O_6/O_7(L)$	$+\cdot2573\pm\cdot0216$	—		—	—		—
$Mx_2/Mx_1(R)$,	$Mx_3/Mx_1(L)$	$+\cdot1857\pm\cdot0316$	$Mx_2/Mx_1(R)$,	$Mx_3/Mx_1(R)$	$+\cdot2103\pm\cdot0311$	$Mx_2/Mx_1(R)$,	$Mx_2/Mx_1(L)$	$+\cdot9344\pm\cdot0038$	$Mx_3/Mx_1(R)$,	$Mx_3/Mx_1(L)$	$+\cdot7894\pm\cdot0125$
$Mx_2/Mx_1(L)$,	$Mx_3/Mx_1(R)$	$+\cdot1757\pm\cdot0324$	$Mx_2/Mx_1(L)$,	$Mx_3/Mx_1(L)$	$+\cdot2284\pm\cdot0303$	—		—	—		—

The observed values are set against the values deduced from the above formulae in the following table:

TABLE XX. *Observed Cross-Correlations compared with Values computed from* (ϵ).

A	B	Observed r_{AB}	Computed r_{AB}	Remarks
$S_6/S_7\,(R)$,	$S_5/S_4\,(L)$	$+\cdot1787 \pm \cdot0244$	$+\cdot1584$	Observed in excess, but compatible considering the probable errors
$S_6/S_7\,(L)$,	$S_5/S_4\,(R)$	$+\cdot1803 \pm \cdot0245$	$+\cdot1581$	
$T_6/T_5\,(R)$,	$T_2/T_3\,(L)$	$+\cdot0322 \pm \cdot0234$	$+\cdot0412$	*Observed and computed, quite* compatible
$T_6/T_5\,(L)$,	$T_2/T_3\,(R)$	$+\cdot0415 \pm \cdot0234$	$+\cdot0392$	
$P_3/P_4\,(R)$,	$P_2/P_1\,(L)$	$-\cdot1169 \pm \cdot0245$	$-\cdot0600$	Observed in excess of computed, and differences pointing towards significance
$P_3/P_4\,(L)$,	$P_2/P_1\,(R)$	$-\cdot1750 \pm \cdot0240$	$-\cdot0525$	
$O_8/O_9\,(R)$,	$O_8/O_7\,(L)$	$+\cdot0700 \pm \cdot0231$	$+\cdot1548^5$	Observed in defect of computed, and differences pointing towards significance
$O_8/O_9\,(L)$,	$O_8/O_7\,(R)$	$+\cdot0584 \pm \cdot0231$	$+\cdot1516$	
$Mx_2/Mx_1\,(R)$,	$Mx_3/Mx_1\,(L)$	$+\cdot1857 \pm \cdot0310$	$+\cdot1802$	Observed in defect, but differences so slight as to render the results compatible
$Mx_2/Mx_1\,(L)$,	$Mx_3/Mx_1\,(R)$	$+\cdot1757 \pm \cdot0324$	$+\cdot1884$	

The differences between the observed and computed correlations run first one way and then the other. The worst cases are those for the parietal and the occipital indices, but while for the former the observed are considerably in excess, in the latter the observed are considerably in defect. The results are on the whole not encouraging for the value of the formulae (ϵ).

The formulae cannot, however, be at once dismissed as not even approximate. It seemed worth while testing the formulae (ϵ) on a wider field. Table XXI is accordingly given here, although it belongs to another section of our work* on morphometric characters, namely, to that dealing with the correlations of heterologous absolute lengths.

Table XXI gives 120 correlations of absolute lengths, enough material to test formulae (ϵ) on 56 cases of absolute measurements. We do not intend in the present paper to analyse this material, but we may draw the reader's attention to one or two points having a bearing on our present inquiry.

Putting aside the Homologous Correlations, there are only four out of 28 double cases, where either direct or cross-correlation exceeds $\cdot35$. These are F_1, F_2 (about $\cdot64$), F_1, P_2 (about $\cdot54$), F_2, P_2 (about $\cdot46$) and T_2, S_5 (about $\cdot37$). On examination of our diagrams (pp. 425, 426) the reader will see some reason for these relatively high correlations. But he will probably be surprised to see how low the correlations on the whole are; out of 112 non-homologous correlations, 96 are below $\cdot25$ in value! It follows that the factor of size is not nearly as important as that of homology in the relationship of different parts of the skull. The last two columns of Table XXI

* From the 68 original measurements, nearly 2000 correlation tables could be formed—ample material for a further section of this paper on correlation of absolute lengths of individual bones of the skull. Only some 120 are given in Table XXI.

TABLE XXI. *Correlation Coefficients of Absolute Measurements.*

Characters		Uni-lateral Correlations	Uni-lateral Correlations	Cross-Correlations	Cross-Correlations	Homologous Correlations	Homologous Correlations
A	B	$r_{A_R B_R}$	$r_{A_L B_L}$	$r_{A_R B_L}$	$r_{A_L B_R}$	$r_{A_R A_L}$	$r_{B_R B_L}$
F_1	F_2	·64459 ± ·01324 (887)	·64490 ± ·01325 (884)	·63135 ± ·01363 (885)	·62401 ± ·01383 (887)	·9059 ± ·0041 (885)	·9384 ± ·0027 (887)
F_1	P_2	·54411 ± ·01682 (797)	·55331 ± ·01664 (791)	·50949 ± ·01777 (790)	·48849 ± ·01821 (796)	·9059 ± ·0041 (885)	·6738 ± ·0134 (754)
F_1	O_7	·16786 ± ·02227 (866)	·14901 ± ·02238 (868)	·17235 ± ·02221 (868)	·15460 ± ·02237 (866)	·9059 ± ·0041 (885)	·7954 ± ·0084 (864)
F_1	T_2	·20519 ± ·02199 (863)	·17773 ± ·02215 (870)	·18051 ± ·02213 (869)	·18295 ± ·02219 (863)	·9059 ± ·0041 (885)	·8145 ± ·0078 (856)
F_1	S_5	·21258 ± ·02325 (767)	·15791 ± ·02318 (788)	·19203 ± ·02313 (789)	·19635 ± ·02343 (766)	·9059 ± ·0041 (885)	·7962 ± ·0092 (722)
F_1	Ml_1	·14707 ± ·02377 (771)	·15323 ± ·02364 (776)	·16192 ± ·02358 (776)	·14376 ± ·02379 (771)	·9059 ± ·0041 (885)	·9399 ± ·0029 (718)
F_1	Mx_1	·18932 ± ·02407 (730)	·19628 ± ·02410 (724)	·20226 ± ·02405 (723)	·20191 ± ·02393 (731)	·9059 ± ·0041 (885)	·9766 ± ·0012 (692)
F_2	P_2	·44013 ± ·01860 (855)	·46383 ± ·01886 (788)	·46949 ± ·01872 (789)	·43063 ± ·01950 (794)	·9384 ± ·0027 (887)	·6738 ± ·0134 (754)
F_2	O_7	·09221 ± ·02277 (863)	·09242 ± ·02275 (864)	·11462 ± ·02259 (868)	·07784 ± ·02282 (863)	·9384 ± ·0027 (887)	·7954 ± ·0084 (864)
F_2	T_2	·14002 ± ·02252 (864)	·12279 ± ·02257 (866)	·11099 ± ·02263 (867)	·14882 ± ·02248 (861)	·9384 ± ·0027 (887)	·8145 ± ·0078 (856)
F_2	S_5	·03944 ± ·02430 (765)	·04764 ± ·02401 (786)	·04933 ± ·02397 (788)	·06783 ± ·02431 (763)	·9384 ± ·0027 (887)	·7962 ± ·0092 (722)
F_2	Ml_1	·04078 ± ·02430 (768)	·08513 ± ·02405 (774)	·08516 ± ·02405 (775)	·06631 ± ·02423 (768)	·9384 ± ·0027 (887)	·9399 ± ·0029 (718)
F_2	Mx_1	·09572 ± ·02473 (730)	·08287 ± ·02491 (723)	·12021 ± ·02472 (723)	·06292 ± ·02485 (728)	·9384 ± ·0027 (887)	·9766 ± ·0012 (692)
P_2	O_7	·21141 ± ·02310 (775)	·22452 ± ·02307 (771)	·22557 ± ·02295 (778)	·19277 ± ·02339 (771)	·6738 ± ·0134 (754)	·7954 ± ·0084 (864)
P_2	T_2	·01113 ± ·02410 (783)	− ·00913 ± ·02412 (782)	− ·00915 ± ·02407 (785)	·03160 ± ·02426 (772)	·6738 ± ·0134 (754)	·8145 ± ·0078 (856)
P_2	S_5	− ·05912 ± ·02470 (740)	− ·04321 ± ·02454 (752)	− ·01640 ± ·02484 (737)	− ·06682 ± ·02519 (711)	·6738 ± ·0134 (754)	·7962 ± ·0092 (722)
P_2	Ml_1	·07406 ± ·02539 (695)	·07297 ± ·02531 (703)	·06225 ± ·02530 (705)	·08769 ± ·02535 (697)	·6738 ± ·0134 (754)	·9399 ± ·0029 (718)
P_2	Mx_1	·10606 ± ·02601 (657)	·11425 ± ·02612 (650)	·12011 ± ·02608 (650)	·09814 ± ·02608 (656)	·6738 ± ·0134 (754)	·9766 ± ·0012 (692)
O_7	T_2	·08399 ± ·02306 (844)	·05754 ± ·02304 (851)	·06056 ± ·02308 (848)	·08433 ± ·02303 (846)	·7954 ± ·0084 (864)	·8145 ± ·0078 (856)
O_7	S_5	·02986 ± ·02464 (748)	·02113 ± ·02427 (772)	·04268 ± ·02430 (768)	·02957 ± ·02463 (749)	·7954 ± ·0084 (864)	·7962 ± ·0092 (722)
O_7	Ml_1	·05668 ± ·02450 (753)	·04960 ± ·02441 (760)	·05372 ± ·02443 (758)	·05470 ± ·02449 (754)	·7954 ± ·0084 (864)	·9399 ± ·0029 (718)
O_7	Mx_1	·14721 ± ·02464 (717)	·14950 ± ·02478 (708)	·14497 ± ·02484 (707)	·17333 ± ·02443 (717)	·7954 ± ·0084 (864)	·9766 ± ·0012 (692)
T_2	S_5	·37646 ± ·02105 (755)	·43502 ± ·01957 (781)	·35820 ± ·02110 (776)	·37283 ± ·02112 (736)	·8145 ± ·0078 (856)	·7962 ± ·0092 (722)
T_2	Ml_1	·08289 ± ·02431 (759)	·14093 ± ·02389 (766)	·09582 ± ·02425 (767)	·11448 ± ·02413 (761)	·8145 ± ·0078 (856)	·9399 ± ·0029 (718)
T_2	Mx_1	·13092 ± ·02472 (719)	·12254 ± ·02486 (714)	·09550 ± ·02505 (712)	·12626 ± ·02474 (720)	·8145 ± ·0078 (856)	·9766 ± ·0012 (692)
S_5	Ml_1	·16014 ± ·02545 (667)	·20571 ± ·02447 (697)	·15571 ± ·02527 (678)	·17720 ± ·02480 (894)	·7962 ± ·0092 (722)	·9399 ± ·0029 (718)
S_5	Mx_1	·20490 ± ·02568 (633)	·24992 ± ·02490 (645)	·21501 ± ·02570 (627)	·21501 ± ·02570 (627)	·7962 ± ·0092 (722)	·9766 ± ·0012 (692)
Ml_1	Mx_1	·18177 ± ·02541 (661)	·16939 ± ·02556 (657)	·17765 ± ·02554 (654)	·17765 ± ·02554 (654)	·9390 ± ·0029 (718)	·9766 ± ·0012 (692)
Numerical Mean		·17484	·17969	·17262	·17317	—	—

show us that the lowest homologous correlation is ·6738, and they fill the range from that value up to ·9766. But the highest absolute size non-homologous correlation is ·6449, and they range from this numerically down to ·0091. The only *certain negative correlation* is that between the parietal arc P_2 from bregma to sphenion, and the sphenoidal chord from krotaphion to the postreme point of the sphenoid as exposed on the base of the skull.

Thus it would seem that the factor of size as judged by the correlation of absolute lengths of the cranial bones, while greater than the factor of contiguity, is still very much less intense than the factor of homology.

While it may be shown from Columns 3 and 4 of Table XXI that correlations of absolute lengths on the left side, $r_{A_L B_L}$, do not as a whole differ significantly from those of absolute length on the right side, $r_{A_R B_R}$, yet the cross-correlations $r_{A_R B_L}$ and $r_{A_L B_R}$ are smaller, but while giving considerable differences (see Table XXII, Columns 3 and 4), they are yet again within the limits of random sampling. As the cross-correlations are less than the direct correlations, we are justified in seeing what values formulae (ϵ) will provide for the cross-correlations. The computed values are given in Column 5 of Table XXII. They show at once that the differences of $r_{A_R B_L}$ and $r_{A_L B_R}$ in Column 7 are wholly trivial and thus unlike the differences in Column 4. Thus, whether the calculated cross-correlations do or do not equal the observed values, the numerical values of

$$\tfrac{1}{2}(r_{A_R B_R} \times r_{B_R B_L} + r_{B_L A_L} \times r_{A_L A_R})$$

and

$$\tfrac{1}{2}(r_{A_L B_L} \times r_{B_L B_R} + r_{B_R A_R} \times r_{A_R A_L})$$

are for practical purposes equal.

On the other hand, if we compare the means of the observed cross-correlations $r_{A_R B_L}$ and $r_{A_L B_R}$ with the means of the corresponding values computed by (ϵ), the latter are in all the 28 cases smaller with the one exception of T_2 and Mx_1, where (ϵ) gives a value larger by ·0026.

It is clear, therefore, that whatever the index correlations may do, (ϵ) does not represent a general principle applying also to the correlations of absolute lengths. If O_m be an observed cross-correlation mean, i.e. $O_m = \frac{1}{2}(O_{R,L} + O_{L,R})$, and C_m be a computed cross-correlation mean, i.e. $C_m = \frac{1}{2}(C_{R,L} + C_{L,R})$, we have calculated the linear relation of O_m to C_m as given by Least Squares and we find the equation

$$O_m = 1{\cdot}098{,}138\, C_m + {\cdot}005{,}998 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\epsilon'),$$

where the constant term may be taken as ·0060.

The results are exhibited in Table XXIII.

It will be seen that the mean numerical difference between the observed and computed means is (·0094), and this is less than half the difference between the observed $r_{A_R B_L}$ and $r_{A_L B_R}$. Hence, if we suppose the difference between the R, L and L, R values of the cross-correlations to be merely due to random sampling, not to inherent right- and left-sidedness, it follows that formulae (ϵ') will provide, well

TABLE XXII.

Difference of Observed and Computed Cross-Correlations of Absolute Lengths.

Index Pair	Observed Correlations	Mean R, L and L, R	Difference $R, L - L, R$	Computed Correlations (c)	Mean R, L and L, R	Difference $R, L - L, R$
$F_1(R)$, $F_2(L)$	·0314 ± ·0136	·0277	+·0074	·5946	·5945	+·0001
$F_1(L)$, $F_2(R)$	·0240 ± ·0138			·5945		
$F_1(R)$, $P_2(L)$	·5095 ± ·0178	·4990	+·0210	·4339	·4334	+·0010
$F_1(L)$, $P_2(R)$	·4885 ± ·0182			·4329		
$F_1(R)$, $O_7(L)$	·1724 ± ·0222	·1635	+·0178	·1343	·1348	−·0010
$F_1(L)$, $O_7(R)$	·1546 ± ·0224			·1353		
$F_1(R)$, $T_2(L)$	·1808 ± ·0221	·1819	−·0022	·1641	·1647	−·0012
$F_1(L)$, $T_2(R)$	·1830 ± ·0222			·1653		
$F_1(R)$, $S_6(L)$	·1920 ± ·0231	·1942	−·0044	·1697	·1704	−·0014
$F_1(L)$, $S_6(R)$	·1964 ± ·0234			·1711		
$F_1(R)$, $Ml_1(L)$	·1619 ± ·0236	·1528	+·0181	·1385	·1385	−·0005
$F_1(L)$, $Ml_1(R)$	·1438 ± ·0238			·1380		
$F_1(R)$, $Mx_1(L)$	·2023 ± ·0241	·2021	+·0004	·1813	·1815	−·0003
$F_1(L)$, $Mx_1(R)$	·2019 ± ·0239			·1816		
$F_2(R)$, $P_2(L)$	·4695 ± ·0187	·4501	+·0389	·3059	·3043	+·0031
$F_2(L)$, $P_2(R)$	·4306 ± ·0195			·3028		
$F_2(R)$, $O_7(L)$	·1146 ± ·0226	·0962	+·0368	·0800	·0800	·0000
$F_2(L)$, $O_7(R)$	·0778 ± ·0228			·0800		
$F_2(R)$, $T_2(L)$	·1110 ± ·0229	·1299	−·0378	·1146	·1152	−·0011
$F_2(L)$, $T_2(R)$	·1488 ± ·0225			·1157		
$F_2(R)$, $S_6(L)$	·0493 ± ·0240	·0586	−·0185	·0400	·0405	·0009
$F_2(L)$, $S_6(R)$	·0678 ± ·0243			·0409		
$F_2(R)$, $Ml_1(L)$	·0852 ± ·0241	·0757	+·0189	·0605	·0605	−·0001
$F_2(L)$, $Ml_1(R)$	·0663 ± ·0242			·0606		
$F_2(R)$, $Mx_1(L)$	·1202 ± ·0247	·0916	+·0573	·0856	·0855	+·0002
$F_2(L)$, $Mx_1(R)$	·0629 ± ·0249			·0854		
$P_2(R)$, $O_7(L)$	·2256 ± ·0230	·2092	+·0328	·1597	·1601	−·0008
$P_2(L)$, $O_7(R)$	·1928 ± ·0234			·1605		
$P_2(R)$, $T_2(L)$	−·0092 ± ·0241	·0113	−·0409	·0015	·0007	+·0015
$P_2(L)$, $T_2(R)$	·0317 ± ·0243			·0000		
$P_2(R)$, $S_6(L)$	−·0164 ± ·0248	−·0416	−·0504	−·0401	−·0398	+·0006
$P_2(L)$, $S_6(R)$	−·0668 ± ·0252			−·0395		
$P_2(R)$, $Ml_1(L)$	·0623 ± ·0253	·0750	−·0254	·0594	·0593	+·0002
$P_2(L)$, $Ml_1(R)$	·0877 ± ·0254			·0592		
$P_2(R)$, $Mx_1(L)$	·1201 ± ·0261	·1091	+·0220	·0903	·0909	−·0012
$P_2(L)$, $Mx_1(R)$	·0981 ± ·0261			·0915		
$O_7(R)$, $T_2(L)$	·0606 ± ·0231	·0724	−·0237	·0571	·0574	−·0005
$O_7(L)$, $T_2(R)$	·0843 ± ·0230			·0576		
$O_7(R)$, $S_6(L)$	·0427 ± ·0243	·0361	+·0131	·0203	·0203	·0000
$O_7(L)$, $S_6(R)$	·0296 ± ·0246			·0203		
$O_7(R)$, $Ml_1(L)$	·0537 ± ·0244	·0542	−·0010	·0464	·0461	+·0005
$O_7(L)$, $Ml_1(R)$	·0547 ± ·0245			·0459		
$O_7(R)$, $Mx_1(L)$	·1450 ± ·0248	·1592	−·0283	·1313	·1314	·0002
$O_7(L)$, $Mx_1(R)$	·1733 ± ·0244			·1315		
$T_2(R)$, $S_6(L)$	·3583 ± ·0211	·3656	−·0145	·3270	·3267	+·0005
$T_2(L)$, $S_6(R)$	·3728 ± ·0211			·3265		
$T_2(R)$, $Ml_1(L)$	·0958 ± ·0243	·1052	−·0187	·0963	·0982	−·0037
$T_2(L)$, $Ml_1(R)$	·1145 ± ·0241			·1000		
$T_2(R)$, $Mx_1(L)$	·0955 ± ·0251	·1109	−·0308	·1138	·1135	+·0006
$T_2(L)$, $Mx_1(R)$	·1263 ± ·0247			·1132		
$S_6(R)$, $Ml_1(L)$	·1557 ± ·0253	·1665	−·0215	·1572	·1588	−·0032
$S_6(L)$, $Ml_1(R)$	·1772 ± ·0248			·1604		
$S_6(R)$, $Mx_1(L)$	·2150 ± ·0257	·2150	·0000	·1995	·2015	−·0041
$S_6(L)$, $Mx_1(R)$	·2150 ± ·0257			·2036		
$Ml_1(R)$, $Mx_1(L)$	·1776 ± ·0255	·1776	−·0001	·1684	·1683	+·0002
$Ml_1(L)$, $Mx_1(R)$	·1777 ± ·0255			·1681		

TABLE XXIII.

Comparison of Observed and Computed (by (ϵ')) Cross-Correlations.

Index Pair	Observed Mean O_m of R, L and L, R	Difference between observed R, L and L, R	Difference between observed Mean and Mean computed by (ϵ')	Computed Mean O_m R, L and L, R by (ϵ')
F_y, P_y	·6257	+ ·0074	− ·0311	·6588
F_y, P_y	·1969	+ ·0210	+ ·0121	·4869
F_y, O_y	·1635	+ ·0178	+ ·0095	·1540
F_y, T_y	·1819	− ·0022	− ·0049	·1868
F_y, S_y	·1942	− ·0044	+ ·0011	·1931
F_y, Ml_y	·1528	+ ·0181	− ·0053	·1581
F_y, Mx_y	·2021	+ ·0004	− ·0032	·2053
F_y, P_y	·4741	+ ·0389	+ ·0440	·4061
F_y, O_y	·0962	+ ·0308	+ ·0023	·0939
F_y, T_y	·1299	− ·0378	− ·0026	·1325
F_y, S_y	·0581	− ·0185	+ ·0010	·0571
F_y, Ml_y	·0757	+ ·0189	+ ·0033	·0724
F_y, Mx_y	·0916	+ ·0573	− ·0083	·0999
P_y, O_y	·2092	+ ·0328	+ ·0274	·1818
P_y, T_y	·0113	− ·0409	+ ·0045	·0068
P_y, S_y	·0416	− ·0601	+ ·0039	− ·0377
P_y, Ml_y	·0750	·0254	+ ·0039	·0711
P_y, Mx_y	·1091	+ ·0220	+ ·0033	·1058
O_y, T_y	·0724	·0237	+ ·0034	·0690
O_y, S_y	·0361	+ ·0131	+ ·0078	·0283
O_y, Ml_y	·0542	·0410	− ·0024	·0566
O_y, Mx_y	·1592	·0283	+ ·0089	·1503
T_y, S_y	·3056	− ·0145	+ ·0008	·3048
T_y, Ml_y	·1052	·0187	− ·0086	·1138
T_y, Mx_y	·1199	− ·0308	− ·0107	·1306
S_y, Ml_y	·1665	− ·0215	− ·0139	·1804
S_y, Mx_y	·2150	·0000	− ·0123	·2273
Ml_y, Mx_y	·1576	− ·0001	− ·0132	·1908
Mean Numerical Difference		·0215	·0094	—

within the limits of random sampling, the value of the cross-correlations in terms of direct and homologous correlations in the case of absolute lengths.

General Conclusions.

In considering the structure of the skull, we should *a priori* expect that the dimensions and shapes of the individual bones of which it is constructed would be controlled by certain factors, capable of being statistically investigated when appropriate measurements have been made on the individual bones.

In the first place the skull is approximately symmetrical, having mirror symmetry about the "median sagittal plane." This approximate symmetry would appear to involve a correspondence in size and shape of homologous bones. We term the source of this the *factor of homology*, and measure it by the correlations between corresponding lengths and indices taken on the homologous bones.

Secondly, we note not only the approximate symmetry of a skull, but that one skull is larger than a second. We should anticipate that the size of one bone would be highly related with the size of a second. This leads us to a *factor of size*, which may be measured by the correlation of lengths taken on different bones on the same side, or on heterologous bones on opposite sides.

Thirdly, we should consider that the bones of the skull fit one another or that there must be a relationship of shapes. This leads us to a *factor of shape* measured by the correlation between the indices of bones.

Fourthly, we might anticipate that the shape of a cranial bone would be more influenced by an adjacent bone than by one not in contact with it. This leads us to a *factor of contiguity*. This is measured by the correlations between adjacent and non-adjacent bones.

(i) It has been long a desire of statisticians to obtain statistical constants independent of size. For this purpose the coefficient of variation was introduced by one of the present writers as an attempt to get over the difficulty that a large length has usually a larger absolute variation. It is shown on the present data that the correlation between absolute size and absolute variation is on the whole not as large as might be anticipated, i.e. about $+ \cdot 28$*, while the correlation of absolute length and coefficient of variation is $- \cdot 12$. *In other words, we reduce the relation of variation to size by more than a half by using the coefficient of variation, but it over corrects. The coefficient of variation varies rather inversely than directly as the size of the organ.*

(ii) The fact that coefficients of variation are not equal even for two lengths measured on the same bone, led us to investigate whether variation was associated with the orientation of a measured length with regard to the fundamental planes of the skull. We divided up our data into broad classes, sagittal, transverse and vertical, with subclasses for lengths going in intermediate directions, and found that for coefficients of variation of absolute lengths there was least variation in the sagittal, most variation in the vertical, and an intermediate variation in the transverse direction. The mixed classes fitted well into this scheme. The same method was applied to the absolute variations (standard deviations), but the results were far less clear cut.

It will be seen from this result that the laying down of a given bone of an individual skull does not proceed equally in all directions: bone growth is not *isotropic*. This may be due to the existence of more than one centre of ossification or to other causes at present not studied.

(iii) Putting aside the like lengths on homologous bones, of the correlations of absolute lengths *only 16 correlations out of 112 exceed* $\cdot 35$, the remaining 96 lie below $\cdot 25$. Those exceeding $\cdot 35$ are F_1 with F_2 and P_2, F_2 with P_2 and T_2 with S_6. It is clear that the factor of size—except for these bones, frontal, parietal, temporal and sphenoid—is not an all-important factor in regulating the relationship

* Cf. *Biometrika*, Vol. XXII. p. 339, footnote †.

between individual bones of the skull. The mean numerical correlation of the whole 112 pairs is only ·1751*. If this be compared with the *mean correlation of the like absolute lengths on homologous bones*, i.e. ·8198†, we see that the latter is 4·7 times the former, or the factor of homology is almost five times as intense as the factor of size.

(iv) While lateral influence is found in absolute lengths (see p. 424, (ii)), *it has no appreciable influence on either the magnitude or variability of the indices of either heterologous or homologous bones* (see pp. 426, (iii); 435—436).

(v) In the case of like characters of homologous bones, the factor of homology *shows greater intensity on the facial bones*, whether judged by indices or absolute lengths. The malar, maxillary and frontal bones have the highest correlation coefficients, or we may say that the face is more symmetrical than the brain-box. The sphenoid is exceptional.

The mean index correlation for homologous bones is ·7414, while that for *absolute measurements of homologous bones* is ·8198. Thus the *factor of homology* is somewhat more powerful in the case of absolute lengths—in directing size—than in that of indices, i.e. governing shape. But in both cases the factor of homology *is far more intense than that of any other of the factors we have considered.*

(vi) *Confining ourselves to the relationship of indices of bones both on the same side*, we endeavoured to measure the power of proximity in determining the correlation of indices. We divided our bones into non-contiguous, slightly contiguous, and contiguous groups. The numerical mean correlations were ·0397, ·0434 and ·0655 respectively, showing increasing mean correlation with a greater contiguity. If we pay regard to sign, we have *for* the three groups: ·0197, ·0145 and − ·0164. The former result indicates that we may pool the slightly contiguous with the non-contiguous, giving non-contiguous and contiguous groups as ·0407 against ·0655. The latter result is *more or less meaningless*, owing to the idle character of sign in index correlations. We have shown in the course of the text that the whole system of contiguous bone indices might be reasonably looked upon *as a sample from a population of correlation* ·0655. Thus the maximum influence of the factor of adjacency may be represented by ·0655, as against the factor of homology ·7414, which is more than eleven times as great.

(vii) As far as shape—not size—is concerned, the magnitude of the homology factor is indicated by the fact that the cross-index correlations give numerically a *mean of ·0056 for contiguous bones*, while the direct index correlations for the same bones give ·0655, while for the non-contiguous bones the values are ·0456

* The L, L correlations are somewhat greater than the R, R correlations, probably significantly, but *the difference is not of practical importance.* The *cross-correlations* R, L and L, R are sensibly equal but somewhat less than the direct correlations. The *differences are again* so small that we have a further illustration of the power of the factor of homology: see foot of Table XXI, p. 458.

† Cf. *Biometrika*, Vol. XXII. p. 330, Table I.

and ·0397 respectively. It is thus on the average indifferent whether we correlate the index of a bone A with that of a bone B, contiguous or not with it on the same side, or with the index of a bone B' homologous to B.

Finally we can sum up as follows:

Average intensity of factor of homology:

For size ·8198, for shape ·7414.

Average intensity of factor of size for non-homologous bones:

For size ·1751*.

Average intensity of factor of contiguity at a maximum:

For size† ——, for shape ·0655.

From these numbers it will be clear to the reader that the only factor controlling the shape and size of the skull as a whole worthy of real consideration is that of homology. Homologous bones must grow both in shape and size alike—we have no suggestion to make of why they do so, we only note the fact that they do so. On the other hand, the factor for size is *of small intensity*; we cannot predict, except in a very few cases, the absolute size of one bone from another. *Nor can we predict from the shape of an individual bone what will be the shape* of a second, whether it be adjacent to it or not. It is as if bones spreading from *their centres of ossification grow wholly individually, except that homologous bones* grow alike. There is no fitting of adjacent bones to a standard pattern, no general *control factor or factors (beyond that of homology)* by which the skull as a whole is organised. There is no system of hereditary determinants guiding the shape *and size of the cranial bones to predetermined forms.* The ossification of the brain-box is merely a process of covering the brain with a bony surface, much as *a plasterer covers a wall by dabs of plaster each independent of the size and shape* of other dabs. Even the factor of homology may flow from the symmetry of the *brain itself. It would not, we hold, be far from the truth to say that it is the brain* which is inherited, the inheritance of size and shape of the skull itself is only an *indirect cause of this more fundamental inheritance.*

If the statements made above be even approximately true, what bearing have they on the craniometrician's processes? He has been seeking for some time for characters of the skull which are

(*a*) practically uncorrelated—independent of one another;

(*b*) racially differentiated.

The usual ethnographic characters of the anthropologists do not satisfy (*a*), however much they may be of value from the standpoint of (*b*). Not even in the case

* Even this reduces to ·1208, if we exclude the four exceptional cases of F_1 with F_2, F_1, F_2 with P_2 and T_2 with S_2.

† Not yet investigated.

of the indices usually provided by anthropologists is there independence*. From the data provided in the present paper, it is possible to select a system of lengths and indices with negligible intercorrelations. Will such lengths and indices satisfy (*b*) as well as (*a*)?

In other words, while in the individual skull there is no correlation of the growth of bones, will the general method of growth of the skull in different races be peculiar to each race? Have morphometric characters racial value? This can only be ascertained when they have been measured for additional races. But should they prove to have—as we anticipate—racial value, then the position of the Coefficient of Racial Likeness as a means of discriminating between races will be markedly strengthened. It needs a system of characters actually or practically independent.

We wish to thank most heartily Dr G. M. Morant for continuous help during the preparation of this paper.

* We may cite the following data from the present paper and that of Pearson and Davin (*Biometrika*, Vol. XVI. (1924), p. 348, Table V):

Mean Correlation of Indices	Morphometrical Indices	Ethnographical Indices
r Number	♂: + ·0511 ± ·0025 119	♂: + ·1468 ± ·0180, ♀: + ·1315 ± ·0210 19 19

Thus in the case of the males the mean morphometric index correlation is only about one-third of the mean ethnographic (or anthropometric) index. The morphometric correlations (excluding homologous indices) include 61 cases of unilateral and 58 of cross correlations.

MISCELLANEA.

(1) The Distribution of the Difference between the Extreme Observation and the Sample Mean in Samples of n from a Normal Universe.

By A. T. McKAY, D.Sc.

The purpose of the present paper is to discuss certain points relating to statistics formed by arranging the observations of a random sample from a normal universe in order of magnitude and constructing linear functions thereof. In particular, a method of determining the significance of the difference between the highest observation and the sample mean is developed.

1. Let $x_1, x_2, \ldots x_n$, arranged in order of ascending magnitude, be a random sample of n from a normal universe with mean m and unit standard deviation. Consider the statistic X formed by taking a linear function of the first r observations, i.e.

$$X = \sum_1^r a_r x_r, \qquad a_r \gtrless 0 \quad \ldots\ldots(1).$$

Suppose α and β are the characteristic function parameters of the mean $\bar{x}$ of the n observations and of the function X respectively; ϕ is the complete distribution function of X in samples of n and $\sigma(n)$ is its standard deviation; $\exists$ denotes the term "characteristic function of..."; and $c = \sum_1^r a_k$. Then if S denotes the distribution surface of x and X,

$$\exists(S) = e^{\alpha^2/2n + \alpha\beta c/n + m\alpha + mc\beta}\,\exists_\beta(\phi_{n-1}) \quad \ldots\ldots(2).$$

To prove this theorem consider the case in which $n=4$ and $r=2$. For this we have

$$\exists(S) = C\int_{-\infty}^{\infty} e^{\alpha z/n - \frac{1}{2}(z-m)^2}\,dz\int_{-\infty}^{z} e^{\alpha y/n - \frac{1}{2}(y-m)^2}\,dy\int_{-\infty}^{y} e^{(\alpha/n + a_2\beta)x - \frac{1}{2}(x-m)^2}\,dx$$
$$\int_{-\infty}^{x} e^{(\alpha/n + a_1\beta)u - \frac{1}{2}(u-m)^2}\,du \quad \ldots\ldots(3).$$

Changing the variable in the last integral on the right-hand side of (3) by writing $u - m - \frac{\alpha}{n} = t$ we find

$$\exists(S) = C e^{\alpha^2/2n^2 + \alpha m/n + a_1\alpha\beta/n}\int_{-\infty}^{\infty}(\ldots)\int_{-\infty}^{x-m-\alpha/n} e^{-t^2/2 + a_1\beta t}\,dt \quad \ldots\ldots(4).$$

Proceeding similarly by changing the variables of integration in turn we derive, in reverting to our original symbols,

$$\exists(S) = e^{\alpha^2/2n + \alpha\beta c/n + \alpha m + m\beta c}\int_{-\infty}^{\infty} e^{-z^2/2}\,dz\int_{-\infty}^{z} e^{-y^2/2}\,dy\int_{-\infty}^{y} e^{-x^2/2 + a_2\beta x}\,dx\int_{-\infty}^{x} e^{-u^2/2 + a_1\beta u}\,du \quad \ldots\ldots(5).$$

This latter expression is in agreement with the special case of (2) and the general theorem follows by induction.

Let us now seek the analytical expression for the surface S. Equation (2) may of course be written

$$\exists(S) = e^{\alpha^2/2n + m\alpha + \alpha\beta c/n}\,\exists_\beta\phi \quad \ldots\ldots(6),$$

so that interpreting first with respect to α and writing the corresponding variable x and the other variable y we have

$$\exists_\beta S(x, y) = \sqrt{\frac{n}{2\pi}}\, e^{-\frac{1}{2}n(x-m-\beta\epsilon/n)^2}\, \exists_\beta(\phi) \quad \text{.......(7)}$$

$$= \sqrt{\frac{n}{2\pi}}\, e^{-\frac{1}{2}n(x-m)^2-\beta^2\epsilon^2/2n}\, \exists_\beta \phi(y-\epsilon x+\epsilon m) \quad \text{.......(8),}$$

whence

$$S(x, y) = \sqrt{\frac{n}{2\pi}}\, e^{-\frac{1}{2}(x-m)^2} \times e^{-\frac{1}{2n}\epsilon^2\frac{d^2}{dy^2}}\, \phi\{y-\epsilon(x-m)\} \quad \text{.......(9).}$$

The correlation coefficient R is derived from (6) by taking logarithms, differentiating with respect to α, β and $\alpha\beta$, and writing $\alpha=\beta=0$. Whence

$$R = \frac{\epsilon}{\sqrt{n}\,\sigma(n)} \quad \text{.......(10).}$$

Further, if $E(y)$ denotes the expected value of y, then from (6) the regression curve of y on x is

$$\bar{y} = \frac{\left\{\frac{\alpha\epsilon}{n} + E(y)\right\}}{e^{\alpha^2/2n+m\alpha}}\, e^{\alpha^2/2n+m\alpha} \quad \text{.......(11),}$$

provided the arrays in the numerator and denominator are interpreted independently. Hence

$$\bar{y} = \frac{\left\{E(y) - \frac{\epsilon}{n}\frac{d}{dx}\right\}}{e^{-n/2(x-m)^2}}\, e^{-\frac{1}{2}n(x-m)^2} \quad \text{.......(12),}$$

$$\bar{y} = E(y) + \epsilon(x-m) \quad \text{.......(13).}$$

Similarly the regression curve of x on y is derived from

$$\bar{x} = \frac{\left(m+\frac{\beta\epsilon}{n}\right)\exists_\beta(\phi)}{\exists_\beta(\phi)} \quad \text{.......(14),}$$

$$\bar{x} = \left(m - \frac{\epsilon}{n}\frac{d}{dy}\log\phi\right) \quad \text{.......(15).}$$

II. Let us now consider the linear statistical function formed by subtracting the sample mean from the function X, i.e.

$$u = (X - \bar{x}) \quad \text{.......(16).}$$

The characteristic function for the distribution of u is to be found from (6) by writing $-\beta$ for α and then regarding β as referring to u. Thus

$$\exists_\beta f(u) = e^{\frac{1}{2n}(1-2\epsilon)\beta^2-m\beta}\, \exists_\beta(\phi) \quad \text{.......(17),}$$

whence

$$f(u) = e^{\frac{1}{2n}(1-2\epsilon)\frac{d^2}{du^2}}\, \phi(u+m) \quad \text{.......(18).}$$

We note also from (17) that if K_p denotes the pth semi-invariant of the distribution of u and K_p' refers similarly to the distribution of X, then

$$\left.\begin{array}{ll} \text{(Mean)} & K_1 = K_1' - m \\ \text{(Variance)} & K_2 = K_2' + \dfrac{(1-2\epsilon)}{n} \\ & K_p = K_p', \quad p = 2, 3, \ldots \end{array}\right\} \quad \text{.......(19).}$$

III. We shall now proceed to apply some of the foregoing results to the case when X = highest observation of the sample of n. Thus the ϵ associated with equation (1) is unity. Since no generality will be lost by so doing, we shall take $m=0$ in what follows. The means and standard

deviations of the distribution of X for various sizes of samples, which have already been given elsewhere*, enable us to construct the following table showing the standard deviation of $u=(X-\bar{x})$ and the correlation coefficient R for X and x.

n	s. d. of u	R
2	·8179	·850
5	·4976	·668
10	·4943	·539
20	·4751	·420
60	·4358	·284
100	·4176	·233
200	·3946	·176
500	·3677	·121
1000	·3500	·090

It is noteworthy that even for samples as large as 60 R is still quite moderate.

Let us now discuss the distribution of u more closely. It is obvious firstly that

$$\phi(x)=\frac{n}{\sqrt{2\pi}}e^{-x^2/2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-x^2/2}dx\right)^{n-1} \quad \text{..............(20)}.$$

So from (18), if $F_n(x)$ is the probability of a sample of n occurring in which u is less than x,

$$F_n(x)=e^{-\frac{1}{2n}\frac{d^2}{dx^2}}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-x^2/2}dx\right)^{n} \quad \text{..............(21)},$$

and

$$f(x)=F_n'(x)=\frac{n}{\sqrt{2\pi}}e^{-\frac{1}{2n}\frac{d^2}{dx^2}}e^{-x^2/2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}x^2}dx\right)^{n-1} \quad \text{..............(22)}.$$

Now suppose we proceed as though to expand the right-hand side of (22) by Leibnitz's Theorem, i.e.

$$F_n'(x)=\frac{n}{\sqrt{2\pi}}e^{-\frac{1}{2n}(D_1+D_2)^2}e^{-x_1^2/2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_2}e^{-x^2/2}dx\right)^{n-1} \quad \text{..............(23)},$$

where $D_1=\frac{\partial}{\partial x_1}$ and $D_2=\frac{\partial}{\partial x_2}$. Whence.

$$F_n'(x)=\frac{n}{\sqrt{2\pi}}e^{-\frac{1}{2n}(D_1^2+D_2^2)}\times e^{-\frac{1}{2}(x_1-D_2/n)^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_2}e^{-x^2/2}dx\right)^{n-1} \quad \text{.........(24)}.$$

Now

$$e^{-\frac{1}{2n}D_1^2}\times e^{-\frac{1}{2}(x_1-D_2/n)^2}=\sqrt{\frac{n}{n-1}}\,e^{-\frac{1}{2}\frac{n}{n-1}(x_1-D_2/n)^2} \quad \text{..............(25)}.$$

Substituting (25) in (24) and making one or two obvious steps we find

$$F_n'(x)=\frac{n}{\sqrt{2\pi}}\sqrt{\frac{n}{n-1}}\,e^{-\frac{1}{2}\frac{n}{n-1}x_1^2}\times e^{-\frac{1}{2}(n-1)D_2^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_2+x_1/(n-1)}e^{-x^2/2}dx\right)^{n-1} \quad \text{...(26)}.$$

Now we note

$$\left\{D_2^k\left(x_2+\frac{x_1}{n-1}\right)^m\right\}_{x_2=x_1=x}$$
$$=\left(\frac{n}{n-1}\right)^{m-k}D^k x^m$$
$$=\left\{\left(\frac{n-1}{n}\right)D\right\}^k\left(\frac{nx}{n-1}\right)^m.$$

* *Tables for Statisticians and Biometricians*, Part II. p. cxiv.

Thus, if ψ and χ are any two integral functions regular at the origin,

$$\left[\psi(D_2)\,\chi\left(x_2+\frac{x_1}{n-1}\right)\right]_{x_2=x_1=x}=\psi\left(\frac{n-1}{n}D\right)\chi\left(\frac{nx}{n-1}\right)\quad\text{.................(27)}.$$

Applying the latter theorem to (20),

$$F_n'(x)=\frac{n}{\sqrt{2\pi}}\sqrt{\frac{n}{n-1}}\,e^{-\frac{1}{2}\frac{n}{n-1}x^2}\times e^{-\frac{1}{2}\frac{n-1}{n^2}D^2}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x\frac{n}{n-1}}e^{-x^2/2}\,dx\right)^{n-1}\quad\text{......(28)},$$

$$F_n'(x)=\frac{n}{\sqrt{2\pi}}\,e^{-\frac{1}{2}\frac{n}{n-1}x^2}\sqrt{\frac{n}{n-1}}\,F_{n-1}\left(\frac{n}{n-1}x\right)\quad\text{.......................................(29)}.$$

We now consider the case for $n=2$. To find the distribution of $u=\frac{x_2-x_1}{2}$ (where x_2 is the larger) we must integrate

$$\frac{1}{2\pi}e^{-\frac{1}{2}(x_1^2+x_2^2)}\,dx_1\,dx_2\quad\text{.......................................(30)}$$

over the field for which

$$\left.\begin{aligned}&u\leqslant\frac{x_2-x_1}{2}<u+\delta u\\&0<(x_2-x_1)\end{aligned}\right\}\quad\text{.......................................}(30^{bis}).$$

Write $(x_2-x_1)=2w$ and $(x_2+x_1)=2v$. Whence (30) becomes

$$\frac{4e^{-(w^2+v^2)}}{\pi}\,dw\,dv\quad\text{...(31)},$$

and (30^{bis})

$$\left.\begin{aligned}&u\leqslant w\leqslant u+\delta u\\&0<w\end{aligned}\right\}\quad(31^{bis}).$$

Whence the distribution of u is

$$f_2(u)=F_2'(u)=\frac{2}{\sqrt{\pi}}e^{-u^2},\qquad u>0\quad\text{................................(32)}.$$

This represents one-half a normal curve. We note that the characteristic function of this distribution is

$$\exists_\beta f_2(u)=\frac{e^{\beta^2/4}}{\sqrt{\pi}}\int_{-\infty}^{\beta}e^{-t^2/4}\,dt\quad\text{...................................}(32^{bis}).$$

Further, using (29) and (32),

$$f_3(u)=\frac{3\sqrt{3}}{\pi}e^{-\frac{3}{4}u^2}\int_0^{\frac{1}{2}u}e^{-t^2}\,dt.$$

Other cases can be found similarly, but they are of no interest since the integrals cannot be evaluated simply.

Let us now return to equation (20). It is easy to see that when x is very small

$$F_n(x)=\frac{n}{2}n^{\frac{1}{2}}\frac{x^{n-1}}{(\sqrt{2\pi})^{n-1}},\qquad n>2\quad\text{................................(33)}.$$

Again,
$$P_n(x)=\int_x^\infty F'(x)\,dx\sim\frac{n}{\sqrt{2\pi}}\int_{x\sqrt{n/(n-1)}}^{\infty}e^{-t^2/2}\,dt\quad\text{.......................(34)},$$

provided $P_{n-1}\left(\frac{n}{n-1}x\right)$ is small compared with unity. But

$$P_{n-1}\left(\frac{n}{n-1}x\right)\leqslant\frac{(n-1)}{\sqrt{2\pi}}\int_{x\frac{n}{n-1}\sqrt{\frac{n-1}{n-2}}}^{\infty}e^{-t^2/2}\,dt\quad\text{.....................(35)}.$$

Whence $P_n(x)$ may be found approximately from equation (34) provided the expression on the

right-hand side of (35) is small compared with unity. The latter is always less than $P_n(x)$. So we may conclude that (34) can always be used for determining the low percentage points*.

In a given case the significance of departure of the highest observation from the mean of the sample is derived directly from equation (34).

Owing to the fact that $\frac{dP}{dx}$ is small when x is large it is clearly more accurate to determine $P_n(x)$ when x is given than to construct a table giving x for standard values of ·05, ·01, etc. of $P_n(x)$. We shall now consider one or two examples of the use of the statistical function u = highest observation minus the mean of the sample.

Example 1. K. Pearson† gives the capacities of 17 Moriori skulls, the highest value of which (1630) appears anomalous, and discusses the rejection of this observation on the basis of three different methods. The mean and standard deviation of the sample are respectively 1405·8 and 97·83. Thus $u=(1630-1405{\cdot}8)/\sigma$, where σ is the standard deviation of the parent universe. Let us assume that this latter is closely given by the sample estimate, and that the universe is normal. Then $u=(1630-1405{\cdot}8)/97{\cdot}83=2{\cdot}2917$. Also $u\sqrt{\frac{17}{16}}=2{\cdot}3622$. Whence from equation (34) $P_{17}(u)=17\times 0{\cdot}00908=0{\cdot}154$. *We should therefore conclude that 1630 was not an anomalous* value. Applied to the same data Irwin's test‡ gives a probability of 0·134. Using E. S. Pearson's table§ for the percentage limits of the range, a probability somewhat greater than 0·10 is obtained. The three tests show good agreement.

Example 2. In the course of routine testing of a standard leather product of a tannery five parallel tests yielded the following values for the hide substance content of the leather specimens:

32·44, 36·46, 39·04, 40·13, 41·00.

The first observation appears unduly low.

The question therefore facing the tannery chemist is whether it is worth while instituting inquiries as to why such an anomalous specimen has occurred. Long experience of the product in question has established a value of 2·226 for the standard deviation. Whence

$$u=(37{\cdot}95-32{\cdot}44)/2{\cdot}226=2{\cdot}476.$$

Also $$\sqrt{\tfrac{5}{4}}\,u=2{\cdot}767, \quad \text{whence } P_5(u)=5\times 0{\cdot}00283=0{\cdot}0142.$$

We should conclude from this that the observation is anomalous and inquiries could justifiably be instituted. Irwin's test applied to the same data gives a value of 0·08 for the probability, whilst *using the range as criterion* E. S. *Pearson's table gives a value of* 0·05. *The latter two* tests are just about *on* the usually accepted boundaries of significance. Accordingly on the basis of these one would hesitate to regard the observation as anomalous. It is of course difficult to decide which test gives the surest guidance, though the author's experience of the material in question leads him to believe that the value of 32·44 is definitely anomalous.

[NOTE. *Some readers may be interested in learning what information the* P_{λ_H} *criterion* provides with regard to the above *Example* 2. That criterion gives equal weight to every member

* The same kind of argument can always be applied in the case of the distribution of the highest observation of a sample from any parent universe. Let $f(x)$ be the parent universe, then the distribution of the highest observation in a sample of n is

$$F_n'(x)=nf(x)\left(\int_{-\infty}^{x} f(x)\,dx\right)^{n-1}.$$

$$\therefore P_n(x)=n\int_x^{\infty} f(x)\,dx,$$

provided $\frac{(n-1)}{n}P_n(x)$ is small compared with unity. This will always be the case in the vicinity of the levels of significance which are usually employed.

† *Loc. cit.* Tables, Part II. p. cviii.

‡ *Loc. cit.* Tables, Part II. p. 160.

§ *Biometrika*, Vol. xxiv. p. 416 (1932).

of the sample; this is also true of the β_2 test but not of Irwin's or McKay's test. The P_{λ_n} criterion gives the value ·086 to the probability, if we use the "established value" 2·226 for the standard deviation. Thus we infer that the outlier ought to be rejected, or that the sample has not been drawn at random from a normal population. On the other hand, if we adopt the standard deviation 3·165 given by the sample itself, we find the probability to be ·42, or no question of throwing out the outlier arises. But inquiring whether the standard deviation 3·165 was a reasonable random sample from a population of standard deviation 2·226, we find that the probability of a sample with its standard deviation $\geqslant$ 3·165 is only ·06, which agrees with the results of Irwin and E. S. Pearson's tests. It would seem accordingly that given the value 3·165 the sample is quite probable, but that such a standard deviation verges on the improbable if the established value of the parental population is 2·226. Ed.]

(ii) Heredity, mainly Human.

By ELDON MOORE.

Chapman and Hall, Ltd. London, 1934. Price 15s.

The author of this volume writes for "the educated and intelligent reader who knows nothing of the subject, but wishes to understand its main outlines without intending to pursue it much further." The book covers a wide field, including chapters on genes and the mechanism of heredity, on the endocrine glands, on physique and character, on hybrids, on inbreeding, on blood groups and on the inheritance of amentia and of ability; some investigations on inheritance in twins are described and an interesting section is concerned with intelligence and social class; the latter part of the volume provides a general review of such subjects as evolution, natural selection, mutation, Lamarckism, etc.

It requires courage to set out on so ambitious a programme and the author is to be congratulated on having produced a volume which clearly portrays the intense interest, academic and practical, attaching to the science of Genetics to-day; he has succeeded in suggesting the general trend of progress in this field and the volume will certainly stimulate interest in the subject.

Writing admittedly for the reader "who knows nothing of the subject" carries with it a special responsibility with regard to the accuracy of the facts given; a number of statements are made which will arrest the critical mind of a reader who has some knowledge and may perhaps shake his confidence. To give one instance on pp. 69—70: "The endocrine constitution...plays a part...in three of the commonest human ills with a genetic basis—asthma, cancer and tuberculosis, which are in some way related to one another....Cancerous patients are very seldom tubercular* while the tuberculous are but rarely afflicted by cancer....Tuberculosis is markedly rare both in asthmatic patients and in their families." All these statements are most challenging and are far from being proven facts; the genetic basis of cancer in man has not been established, nor that its sufferers, as such, carry any insurance against tuberculosis. The genetic basis of asthma must surely be regarded as standing upon precarious ground until we know more of the incidence of asthma in the general population and the incidence of all the other allergical diseases which may or may not replace it.

On what evidence does the author state that children who will later develop Huntington's chorea very frequently have unusually small heads? Other similar criticisms might be given but we can recommend a volume which contains so much interesting suggestion, and only regret the occasional lack of caution in the too definite presentation of unproven facts.

* [As a matter of fact cancer of the lungs is often found to develop on the site of healed tubercular lesions. See Karl Pearson, "Report on certain Cancer Statistics," *Archives of the Middlesex Hospital*, pp. 8—12. Nor can we yet assert that the tendency to cancer is certainly hereditary. See *ibid.* pp. 5—8; but compare Brenda Stoessiger, "Inheritance of Duration of Life and Cause of Death," *Annals of Eugenics*, Vol. v, "Cancer," pp. 148—150, especially p. 150. See also Percy Stocks, "A Cooperative Study of 450 Cancer Patients and an equal Number of Controls," *ibid.* Vol. v. pp. 274—278 and 280. Ed.]